Compressible Flow - TME085

Lecture Notes

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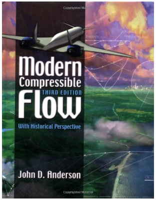
COMPRESSIBLE FLOW

Literature

This lecture series is based on the book *Modern Compressible Flow; With Historical Perspective* by John D. Anderson



Course Literature:

John D. Anderson
Modern Compressible Flow; With Historical Perspective
Third Edition (International Edition 2004)
McGraw-Hill, ISBN 007-124136-1

Literature

Content:

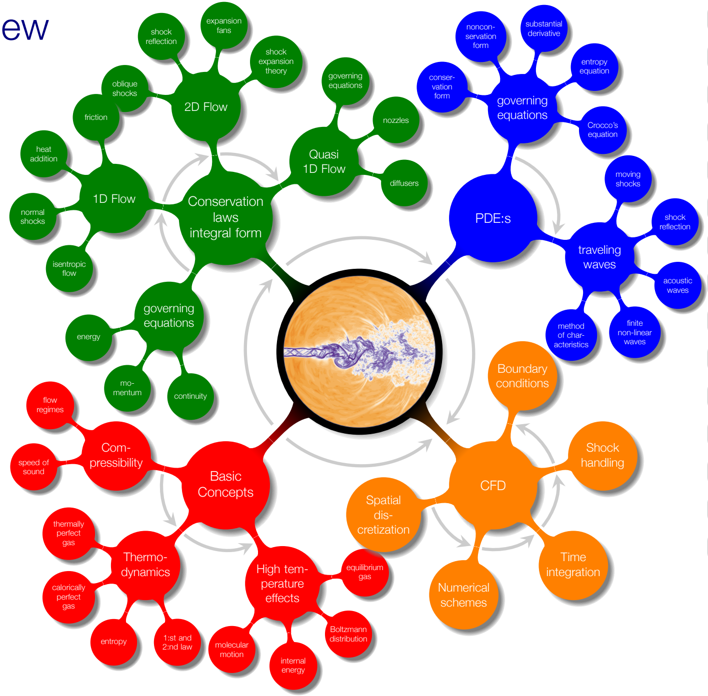
- Chapter 1-7: All
- Chapter 8-11: Excluded
- Chapter 12: Included, supplemented by lecture notes
- Chapter 13-15: Excluded
- Chapter 16-17: Some parts included (see lecture notes)

With the exception of the lecture notes supplementing chapter 12, all lecture notes are based on the book.

Learning Outcomes

1. Define the concept of compressibility for flows
2. Explain how to find out if a given flow is subject to significant compressibility effects
3. Describe typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)
4. Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on
5. Explain how thermodynamic relations enter into the flow equations
6. Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases
7. Explain why entropy is important for flow discontinuities
8. Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics
    a. 1D isentropic flow*
    b. normal shocks*
    c. 1D flow with heat addition*
    d. 1D flow with friction*
    e. oblique shocks in 2D*
    f. shock reflection at solid walls*
    g. contact discontinuities
    h. Prandtl-Meyer expansion fans in 2D
    i. detached blunt body shocks, nozzle flows
    j. unsteady waves and discontinuities in 1D
    k. basic acoustics
9. Solve engineering problems involving the above-mentioned phenomena (8a-8k)
10. Explain how the incompressible flow equations are derived as a limiting case of the compressible flow equations
11. Explain how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations
12. Explain the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions
13. Apply a given CFD code to a particular compressible flow problem
14. Analyze and verify the quality of the numerical solution
15. Explain the limitations in fluid flow simulation software
16. Report numerical analysis work in form of a technical report
    a. Describe a numerical analysis with details such that it is possible to redo the work based on the provided information
    b. Write a technical report (structure, language)
17. Search for literature relevant for a specific physical problem and summarize the main ideas and concepts found
18. Present engineering work in the form of oral presentations

Overview



Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

Lecture 9

Lecture 10

Lecture 11

Lecture 12

Lecture 13

Lecture 14

Lecture 15

LECTURE 1

Compressible Flow

"Compressible flow (gas dynamics) is a branch of fluid mechanics that deals with flows having significant changes in fluid density"

Wikipedia

Gas Dynamics

"... the study of motion of gases and its effects on physical systems ..."

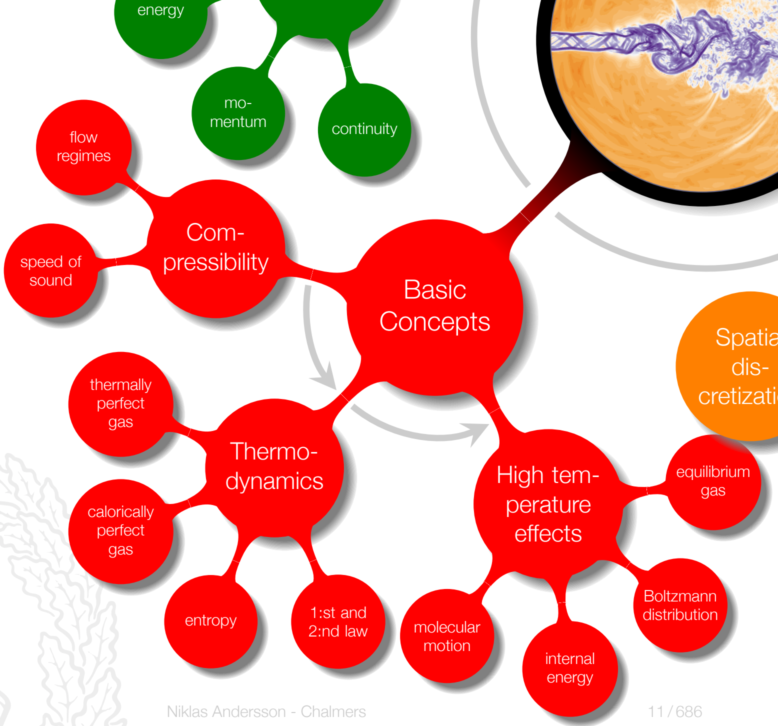
"... based on the principles of fluid mechanics and thermodynamics ..."

"... gases flowing around or within physical objects at speeds comparable to the speed of sound ..."

Wikipedia

Chapter 1
Compressible Flow

Overview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perfect
gas
entropy
1:st and
2:nd law
High tem-
perature
effects
equilibrium
gas
Boltzmann
distribution
molecular
motion
internal
energy

Addressed Learning Outcomes

1. Define the concept of compressibility for flows
2. Explain how to find out if a given flow is subject to significant compressibility effects
3. Describe typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)

6. Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases

in this lecture we will find out what compressibility means and do a brief review of thermodynamics

Roadmap - Introduction to Compressible Flow

- Introduction
- Compressible Flow: Applications
- Historical milestones
- Compressibility
- Flow regimes

- Review of thermodynamics
- Gas properties
- First and second law of thermodynamics
- Isentropic relations

Applications - Classical

- Treatment of calorically perfect gas
- Exact solutions of inviscid flow in 1D
- Shock-expansion theory for steady-state 2D flow
- Approximate closed form solutions to linearized equations in 2D and 3D
- Method of Characteristics (MOC) in 2D and axi-symmetric inviscid supersonic flows

Applications - Modern

- Computational Fluid Dynamics (CFD)
- Complex geometries (including moving boundaries)
- Complex flow features (compression shocks, expansion waves, contact discontinuities)
- Viscous effects
- Turbulence modeling
- High temperature effects (molecular vibration, dissociation, ionization)
- Chemically reacting flow (equilibrium & non-equilibrium reactions)

Applications - Examples

Turbo-machinery flows:

- Gas turbines, steam turbines, compressors
- Aero engines (turbojets, turbofans, turboprops)

Aeroacoustics:

- Flow induced noise (jets, wakes, moving surfaces)
- Sound propagation in high speed flows

External flows:

- Aircraft (airplanes, helicopters)
- Space launchers (rockets, re-entry vehicles)

Internall flows:

- Nozzle flows
- Inlet flows, diffusers
- Gas pipelines (natural gas, bio gas)

Free-shear flows:

- High speed jets

Combustion:

- Internal combustion engines (valve flow, in-cylinder flow, exhaust pipe flow, mufflers)
- Combustion induced noise (turbulent combustion)
- Combustion instabilities

Roadmap - Introduction to Compressible Flow

- Introduction
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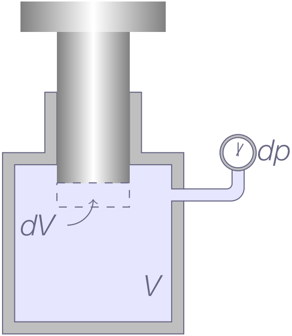
Chapter 1.2
Compressibility

Compressibility

$$\tau = -\frac{1}{\nu}\frac{\partial \nu}{\partial p},\ (\nu = \frac{1}{\rho})$$

Not really precise!

Is T held constant during the compression or not?



Compressibility

Two fundamental cases:

Constant temperature

- Heat is cooled off to keep T constant inside the cylinder
- The piston is moved slowly

Adiabatic process

- Thermal insulation prevents heat exchange
- The piston is moved fairly rapidly (*gives negligible flow losses*)

Compressibility

Isothermal process:

$$\tau_T = -\frac{1}{\nu}\left(\frac{\partial \nu}{\partial p}\right)_T$$

Adiabatic reversible (*isentropic*) process:

$$\tau_S = -\frac{1}{\nu}\left(\frac{\partial \nu}{\partial p}\right)_S$$

Air at normal conditions: $\tau_T \approx 1.0 \times 10^{-5}$ $[m^2/N]$
Water at normal conditions: $\tau_T \approx 5.0 \times 10^{-10}$ $[m^2/N]$

Compressibility

$$\tau = -\frac{1}{\nu}\frac{\partial \nu}{\partial p}$$

but

$$\nu = \frac{1}{\rho}$$

which gives

$$\tau = -\rho\frac{\partial}{\partial p}\left(\frac{1}{\rho}\right) = -\rho\left(-\frac{1}{\rho^2}\right)\frac{\partial \rho}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

Similarly:

$$\tau_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T, \quad \tau_S = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_S$$

Compressibility

Definition of compressible flow:

- If p changes with amount Δp over a characteristic length scale of the flow, such that the corresponding change in density, given by $\Delta\rho \sim \rho\tau\Delta$ p, is too large to be neglected, the flow is compressible (*typically, if* $\Delta\rho/\rho > 0.05$)

Important note:

- Bernoulli´s equation is restricted to incompressible flow, *i.e.* it is not valid for compressible flow!

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Chapter 1.3
Flow Regimes

Flow Regimes

The freestream Mach number is defined as

$$M_\infty = \frac{U_\infty}{a_\infty}$$

where U_∞ is the freestream flow speed and a_∞ is the speed of sound at freestream conditions

Flow Regimes

Assume first incompressible flow and estimate the max pressure difference using

$$\Delta p \approx \frac{1}{2}\rho_\infty U_\infty^2$$

For air at normal conditions we have

$$\tau_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T = \frac{1}{\rho R T} = \frac{1}{p}$$

(ideal gas law for perfect gas $p = \rho R T$)

Flow Regimes

Using the relations on previous slide we get

$$\frac{\Delta\rho}{\rho} \approx \tau_T \Delta p \approx \frac{1}{p_\infty}\frac{1}{2}\rho_\infty U_\infty^2 = \frac{\frac{1}{2}\rho_\infty U_\infty^2}{\rho_\infty R T_\infty}$$

for a calorically perfect gas we have

$$a = \sqrt{\gamma R T}$$

which gives us

$$\frac{\Delta\rho}{\rho} \approx \frac{\gamma U_\infty^2}{2a_\infty^2}$$

now, using the definition of Mach number we get

$$\frac{\Delta\rho}{\rho} \approx \frac{\gamma}{2} M_\infty^2$$

Flow Regimes

Incompressible	$M_\infty < 0.1$
Subsonic	$M_\infty < 1$ and $M < 1$ everywhere
Transonic	case 1: $M_\infty < 1$ and $M > 1$ locally case 2: $M_\infty > 1$ and $M < 1$ locally
Supersonic	$M_\infty > 1$ and $M > 1$ everywhere
Hypersonic	supersonic flow with high-temperature effects

Compressible

Local Mach number M is based on local flow speed, $U = |\mathbf{U}|$, and local speed of sound, a

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- Introduction ✓
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Chapter 1.4
Review of Thermodynamics

Thermodynamic Review

Compressible flow:

> *... strong interaction between flow and thermodynamics ...*

Perfect Gas

All intermolecular forces negligible

Only elastic collitions between molecules

$$p\nu = RT$$

or

$$\frac{p}{\rho} = RT$$

where R is the gas constant $[R] = J/kgK$

Also, $R = R_{univ}/M$ where M is the molecular weight of gas molecules (in $kg/kmol$) and $R_{univ} = 8314\,J/kmol\,K$

Internal Energy and Enthalpy

Internal energy e ($[e] = J/kg$)

Enthalpy h ($[h] = J/kg$)

$$h = e + p\nu = e + \frac{p}{\rho} \text{ (valid for all gases)}$$

For any gas in thermodynamic equilibrium, e and h are functions of only two thermodynamic variables (*any two variables may be selected*) *e.g.*

$$e = e(T, \rho)$$
$$h = h(T, p)$$

Internal Energy and Enthalpy

Special cases:

Thermally perfect gas:

$e = e(T)$ and $h = h(T)$

OK assumption for air at near atmospheric conditions and $100K < T < 2500K$

Calorically perfect gas:

$e = C_v T$ and $h = C_p T$ (C_v and C_p are constants)

OK assumption for air at near atmospheric pressure and $100K < T < 1000K$

Specific Heat

For thermally perfect (and calorically perfect) gas

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p, \quad C_v = \left(\frac{\partial e}{\partial T}\right)_v$$

since $h = e + p/\rho = e + RT$ we obtain:

$$C_p = C_v + R$$

The ratio of specific heats, γ, is defined as:

$$\gamma \equiv \frac{C_p}{C_v}$$

Specific Heat

Important!

calorically perfect gas:

C_v, C_p, and γ are constants

thermally perfect gas:

C_v, C_p, and γ will depend on temperature

Specific Heat

$$C_p - C_v = R$$

divide by C_v

$$\gamma - 1 = \frac{R}{C_v}$$

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p - C_v = R$$

divide by C_p

$$1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} = \frac{R}{C_p}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

Specific Heat

$$C_p - C_v = R$$

divide by C_v

$$\gamma - 1 = \frac{R}{C_v}$$

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p - C_v = R$$

divide by C_p

$$1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} = \frac{R}{C_p}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

valid for both thermally perfect and calorically perfect gas!

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First Law of Thermodynamics

A fixed mass of gas, separated from its surroundings by an imaginary flexible boundary, is defined as a ”system”. This system obeys the relation

$$de = \delta q - \delta w$$

where

de is a change in internal energy of system

δq is heat added to the system

δw is work done by the system (on its surroundings)

Note: *de* only depends on starting point and end point of the process while δq and δw depend on the actual process also

First Law of Thermodynamics

Examples:

Adiabatic process:

$\delta q = 0.$

Reversible process:

no dissipative phenomena (*no flow losses*)

Isentropic process:

a process which is both adiabatic and reversible

First Law of Thermodynamics

Reversible process:

$$\delta w = p d\nu = p d(1/\rho)$$

$$de = \delta q - p d(1/\rho)$$

Adiabatic & reversible process:

$$\delta q = 0.$$

$$de = -p d(1/\rho)$$

Entropy

Entropy s is a property of all gases, uniquely defined by any two thermodynamic variables, *e.g.*

$s = s(p, T)$ or $s = s(\rho, T)$ or $s = s(\rho, p)$ or $s = s(e, h)$ or ...

Second Law of Thermodynamics

Concept of entropy s:

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{ir}, \quad (ds_{ir} > 0.)$$

or

$$ds \geq \frac{\delta q}{T}$$

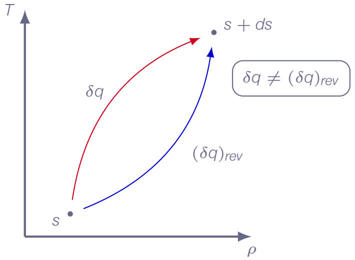
Second Law of Thermodynamics

Concept of entropy s:

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{ir}, \qquad (ds_{ir} > 0.)$$

or

$$ds \geq \frac{\delta q}{T}$$



Second Law of Thermodynamics

In general:

$$ds \geq \frac{\delta q}{T}$$

For adiabatic processes:

$$ds \geq 0.$$

Calculation of Entropy

For reversible processes ($\delta w = pd(1/\rho)$ and $\delta q = Tds$):

$$de = Tds - pd\left(\frac{1}{\rho}\right)$$

or

$$Tds = de + pd\left(\frac{1}{\rho}\right)$$

from before we have $h = e + p/\rho \Rightarrow$

$$dh = de + pd\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)dp \Leftrightarrow de = dh - pd\left(\frac{1}{\rho}\right) - \left(\frac{1}{\rho}\right)dp$$

Calculation of Entropy

For thermally perfect gases, $p = \rho RT$ and $dh = C_p dT \Rightarrow$

$$ds = C_p \frac{dT}{T} - R\frac{dp}{p}$$

Integration from starting point (1) to end point (2) gives:

$$s_2 - s_1 = \int_1^2 C_p \frac{dT}{T} - R \ln\left(\frac{p_2}{p_1}\right)$$

and for calorically perfect gases

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

Calculation of Entropy

If we instead use $de = C_v dT$ we get

for thermally perfect gases

$$s_2 - s_1 = \int_1^2 C_v \frac{dT}{T} - R \ln \left(\frac{\rho_2}{\rho_1} \right)$$

and for calorically perfect gases

$$s_2 - s_1 = C_v \ln \left(\frac{T_2}{T_1} \right) - R \ln \left(\frac{\rho_2}{\rho_1} \right)$$

Roadmap - Introduction to Compressible Flow

- Introduction ✓
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- Flow regimes ✓

- Review of thermodynamics
- Gas properties ✓
- First and second law of thermodynamics ✓
- Isentropic relations

Isentropic Relations

For calorically perfect gases, we have

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

For adiabatic reversible processes:

$$ds = 0. \Rightarrow s_1 = s_2 \Rightarrow C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = 0 \Rightarrow$$

$$\ln\left(\frac{p_2}{p_1}\right) = \frac{C_p}{R}\ln\left(\frac{T_2}{T_1}\right)$$

Isentropic Relations

$$\frac{C_p}{R} = \frac{C_p}{C_p - C_v} = \frac{\gamma}{\gamma - 1} \Rightarrow$$

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}}$$

Isentropic Relations

Alternatively

$$s_2 - s_1 = 0 = C_v \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right) \Rightarrow$$

$$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}}$$

Isentropic Relations - Summary

For an isentropic process and a calorically perfect gas we have

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

A.K.A. the isentropic relations

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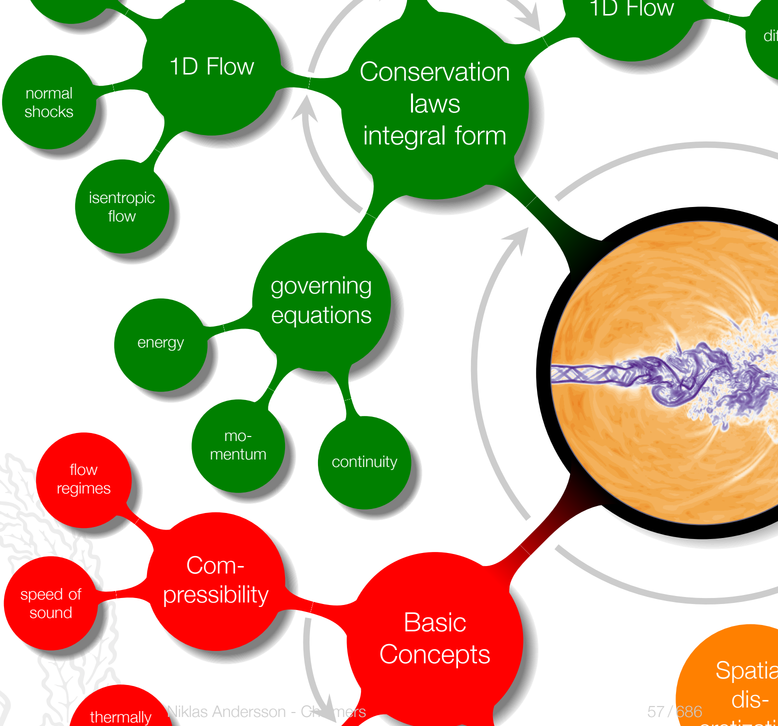
LECTURE 2

Chapter 2
Integral Forms of the Conservation Equations for Inviscid Flows

Overview


1D Flow
normal shocks
isentropic flow
Conservation laws integral form
1D Flow
governing equations
energy
mo- mentum
continuity
flow regimes
Com- pressibility
speed of sound
Basic Concepts
thermally
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Addressed Learning Outcomes

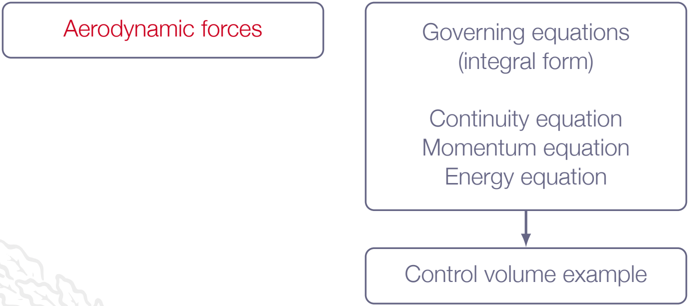
4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

5 Explain how thermodynamic relations enter into the flow equations

7 Explain why entropy is important for flow discontinuities

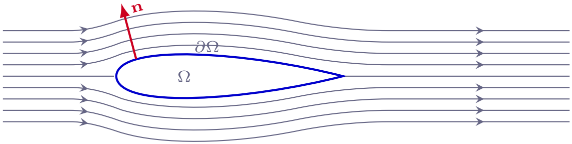
equations, equations and more equations

Roadmap - Integral Relations



Chapter 1.5
Aerodynamic Forces

Aerodynamic Forces



Ω	region occupied by body
$\partial\Omega$	surface of body
$\mathbf{n}$	outward facing unit normal vector

Aerodynamic Forces

Overall forces on the body du to the flow

$$\mathbf{F} = \oiint (-p\mathbf{n} + \tau \cdot \mathbf{n}) dS$$

where p is static pressure and τ is a stress tensor

Aerodynamic Forces

Drag is the component of $\mathbf{F}$ which is parallel with the freestream direction:

$$D = D_p + D_f$$

where D_p is drag due to pressure and D_f is drag due to friction

Lift is the component of $\mathbf{F}$ which is normal to the free stream direction:

$$L = L_p + L_f$$

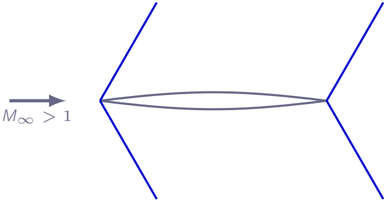
(L_f is usually negligible)

Aerodynamic Forces

Inviscid flow around slender body (*attached flow*)

- subsonic flow: $D = 0$
- transonic or supersonic flow: $D > 0$

Explanation: Wave drag



Aerodynamic Forces

- Wave drag is an inviscid phenomena, connected to the formation of compression shocks and entropy increase
- Viscous effects are present in all Mach regimes
- At transonic and supersonic conditions a particular phenomena named "shock/boundary-layer interaction" may appear
  - shocks trigger flow separation
  - usually leads to unsteady flow

Roadmap - Integral Relations

Aerodynamic forces

Governing equations
(integral form)

Continuity equation
Momentum equation
Energy equation

Control volume example

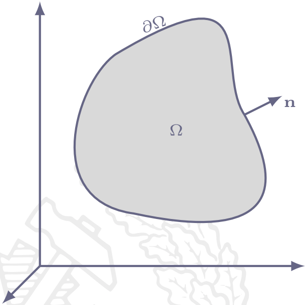
Integral Forms of the Conservation Equations

Conservation principles:

- conservation of mass
- conservation of momentum (*Newton's second law*)
- conservation of energy (*first law of thermodynamics*)

Integral Forms of the Conservation Equations

The control volume approach:



Notation:

Ω: fixed control volume

$\partial\Omega$: boundary of Ω

$\mathbf{n}$: outward facing unit normal vector

$\mathbf{v}$: fluid velocity

$v = |\mathbf{v}|$

Chapter 2.3
Continuity Equation

Continuity Equation

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{\substack{\text{rate of change of}\\ \text{total mass in } \Omega}} + \underbrace{\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{\substack{\text{net mass flow out}\\ \text{from } \Omega}} = 0$$

Note: notation in the text book $\mathbf{n} \cdot dS = d\mathbf{S}$

Chapter 2.4
Momentum Equation

Momentum Equation

Conservation of momentum:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}}_{\text{rate of change of total momentum in }\Omega} + \underbrace{\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right]dS}_{\text{net momentum flow out from }\Omega\text{ plus surface force on }\partial\Omega\text{ due to pressure}} = \underbrace{\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}}_{\text{rate of momentum generation due to forces inside }\Omega}$$

Note: friction forces due to viscosity are not included here. To account for these forces, the term $-(\tau\cdot\mathbf{n})$ must be added to the surface integral term.

Note: the body force, f, is force per unit mass.

Chapter 2.5
Energy Equation

Energy Equation

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V}}_{\substack{\text{rate of change of total}\\ \text{internal energy in } \Omega}} + \underbrace{\oiint\limits_{\partial\Omega} \left[\rho e_o(\mathbf{v}\cdot\mathbf{n}) + p\mathbf{v}\cdot\mathbf{n}\right] dS}_{\substack{\text{net flow of total internal energy}\\ \text{out from } \Omega \text{ plus work due to}\\ \text{surface pressure on } \partial\Omega}} = \underbrace{\iiint\limits_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}}_{\substack{\text{work due to forces}\\ \text{inside } \Omega}}$$

where

$$\rho e_o = \rho\left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \rho\left(e + \frac{1}{2}v^2\right)$$

is the total internal energy

Energy Equation

The surface integral term may be rewritten as follows:

$$\oiint_{\partial\Omega} \left[\rho \left(e + \frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n}) + p\mathbf{v}\cdot\mathbf{n}\right] dS$$

$$\Leftrightarrow$$

$$\oiint_{\partial\Omega} \left[\rho \left(e + \frac{p}{\rho} + \frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})\right] dS$$

$$\Leftrightarrow$$

$$\oiint_{\partial\Omega} \left[\rho \left(h + \frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})\right] dS$$

Energy Equation

Introducing total enthalpy

$$h_o = h + \frac{1}{2}v^2$$

we get

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n}]\, dS = \iiint\limits_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathscr{V}$$

Energy Equation

Note 1: to include friction work on $\partial\Omega$, the energy equation is extended as

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega}\left[\rho h_o \mathbf{v}\cdot\mathbf{n} - (\tau\cdot\mathbf{n})\cdot\mathbf{v}\right]dS = \iiint\limits_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

Note 2: to include heat transfer on $\partial\Omega$, the energy equation is further extended

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega}\left[\rho h_o \mathbf{v}\cdot\mathbf{n} - (\tau\cdot\mathbf{n})\cdot\mathbf{v} + \mathbf{q}\cdot\mathbf{n}\right]dS = \iiint\limits_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

where $\mathbf{q}$ is the heat flux vector

Energy Equation

Note 3: the force $\mathbf{f}$ inside Ω may be a distributed body force field

Examples:

- gravity
- Coriolis and centrifugal acceleration terms in a rotating frame of reference

Energy Equation

Note 4: there may be objects inside Ω which we choose to represent as sources of momentum and energy.

For example, there may be a solid object inside Ω which acts on the fluid with a force $\mathbf{F}$ and performs work $\dot{W}$ on the fluid

Momentum equation:

$$\frac{d}{dt}\iiint_{\Omega} \rho \mathbf{v} d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS = \iiint_{\Omega} \rho \mathbf{f} d\mathscr{V} + \mathbf{F}$$

Energy equation:

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n}\right] dS = \iiint_{\Omega} \rho \mathbf{f}\cdot\mathbf{v} d\mathscr{V} + \dot{W}$$

Roadmap - Integral Relations

Aerodynamic forces

Governing equations (integral form)

Continuity equation
Momentum equation
Energy equation

Control volume example

Integral Equations - Applications

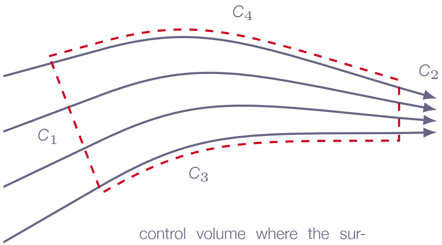
How can we use control volume formulations of conservation laws?

- Let $\Omega \to 0$: In the limit of vanishing volume the control volume formulations give the Partial Differential Equations (PDE:s) for mass, momentum and energy conservation (see Chapter 6)

- Apply in a "smart" way $\Rightarrow$ Analysis tool for many practical problems involving compressible flow (see Chapter 2, Section 2.8)

Integral Equations - Applications

Example: Steady-state adiabatic inviscid flow



control volume where the surfaces C_1 and C_2 are normal to the flow and C_3 and C_4 are parallel to the stream lines

Integral Equations - Applications

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{=0}+\underbrace{\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{-\rho_1 v_1 A_1 + \rho_2 v_2 A_2}=0$$

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}}_{=0}+\underbrace{\oiint\limits_{\partial\Omega}\left[\rho h_o\mathbf{v}\cdot\mathbf{n}\right]dS}_{-\rho_1 h_{o_1} v_1 A_1 + \rho_2 h_{o_2} v_2 A_2}=0$$

Integral Equations - Applications

Conservation of mass:

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

Conservation of energy:

$$\rho_1 h_{o_1} v_1 A_1 = \rho_2 h_{o_2} v_2 A_2$$

$$\Leftrightarrow$$

$$h_{o_1} = h_{o_2}$$

Total enthalpy h_o is conserved along streamlines in steady-state adiabatic inviscid flow

Roadmap - Integral Relations

Aerodynamic forces

Governing equations (integral form)

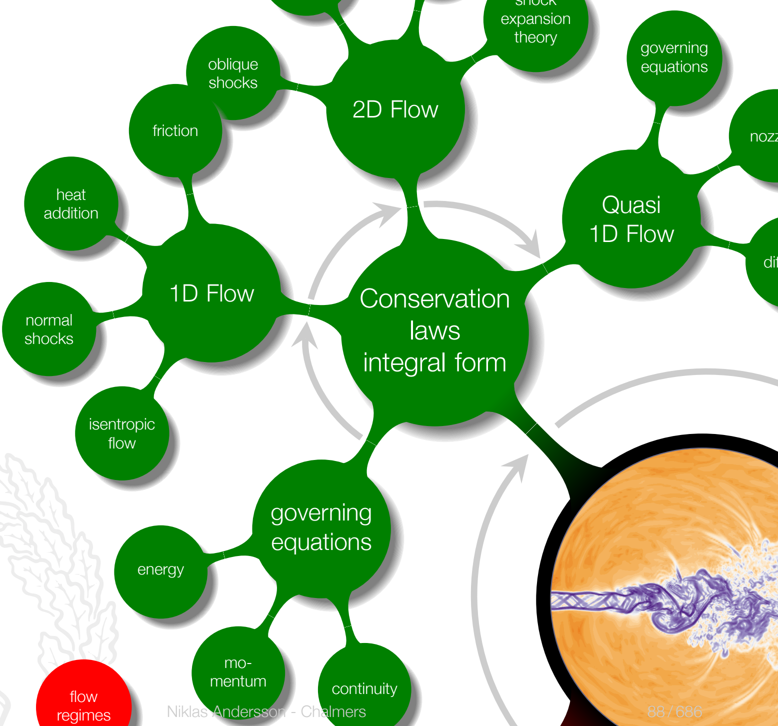
Continuity equation
Momentum equation
Energy equation

Control volume example

LECTURE 3

Chapter 3
One-Dimensional Flow

Overview


shock expansion theory
governing equations
oblique shocks
2D Flow
friction
heat addition
Quasi 1D Flow
1D Flow
normal shocks
Conservation laws integral form
isentropic flow
governing equations
energy
mo-mentum
continuity
flow regimes

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

5 Explain how thermodynamic relations enter into the flow equations

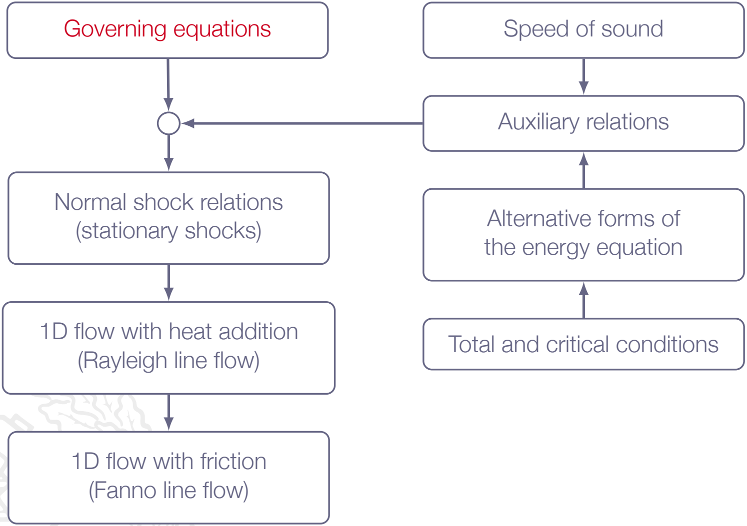
6 Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases

8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   a 1D isentropic flow*
   b normal shocks*

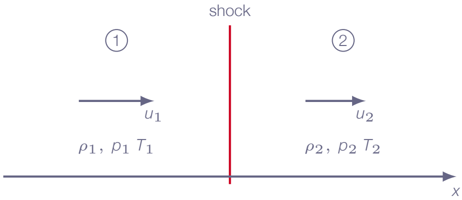
one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow



Chapter 3.2 One-Dimensional Flow Equations

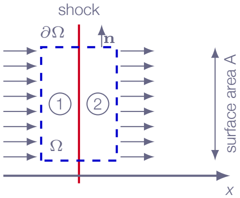
One-Dimensional Flow Equations



Assumptions:

- all flow variables only depend on x
- velocity aligned with x-axis

One-Dimensional Flow Equations



Control volume approach:

Define a rectangular control volume around shock, with upstream conditions denoted by 1 and downstream conditions by 2

One-Dimensional Flow Equations

Conservation of mass:

$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} = 0 \qquad \oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = \rho_2 u_2 A - \rho_1 u_1 A$$

$$\rho_1 u_1 = \rho_2 u_2$$

Conservation of momentum:

$$\frac{d}{dt}\iiint_{\Omega} \rho \mathbf{v} d\mathscr{V} = 0 \qquad \oiint_{\partial\Omega} \left[\rho(\mathbf{v} \cdot \mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS = (\rho_2 u_2^2 + p_2)A - (\rho_1 u_1^2 + p_1)A$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

One-Dimensional Flow Equations

Conservation of energy:

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} = 0 \qquad \oiint_{\partial\Omega} [\rho h_o \mathbf{v}\cdot\mathbf{n}]\, dS = \rho_2 h_{o_2} u_2 A - \rho_1 h_{o_1} u_1 A$$

$$\rho_1 u_1 h_{o_1} = \rho_2 u_2 h_{o_2}$$

Using the continuity equation this reduces to

$$h_{o_1} = h_{o_2}$$

or

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

One-Dimensional Flow Equations

Summary:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

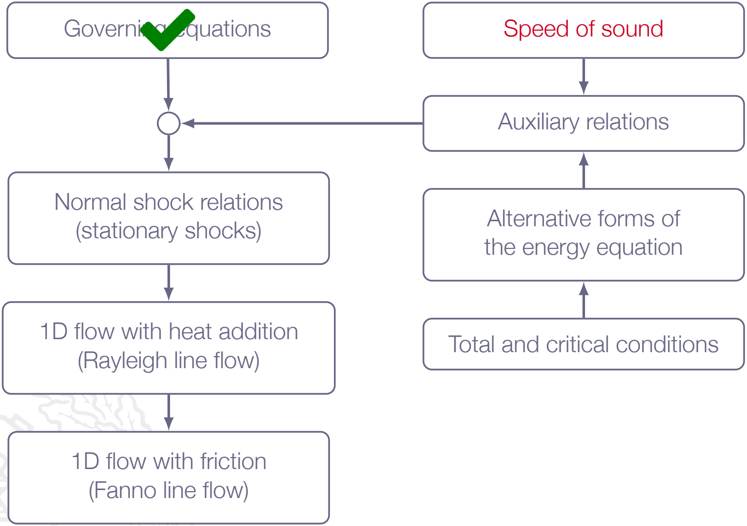
Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ analytical solution exists

Note: These equations are valid regardless of whether there is a shock or not inside the control volume

Roadmap - One-dimensional Flow



Chapter 3.3
Speed of Sound and Mach Number

Speed of Sound

Sound waves are small perturbations in ρ, $\mathbf{v}$, p, T (with constant entropy s) propagating through gas with speed a

It can be shown that sound waves propagate with a velocity given by

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

(valid for all gases)

Speed of Sound

Compressibility and speed of sound:

from before we have

$$\tau_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s$$

insert in relation for speed of sound

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{\rho \tau_s}$$

or

$$a = \sqrt{\frac{1}{\rho \tau_s}}$$

(valid for all gases)

Speed of Sound

Calorically perfect gas:

Isentropic process $\Rightarrow p = C\rho^\gamma$ (where C is a constant)

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma C \rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

which implies

$$a = \sqrt{\frac{\gamma p}{\rho}}$$

or

$$a = \sqrt{\gamma R T}$$

Mach Number

The mach number, M, is a local variable

$$M = \frac{v}{a}$$

where

$$v = |\mathbf{v}|$$

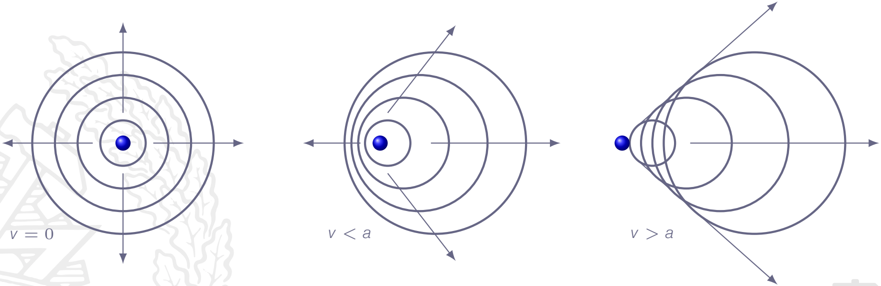
and a is the local speed of sound

In the free stream, far away from solid objects, the flow is undisturbed and denoted by subscript ∞

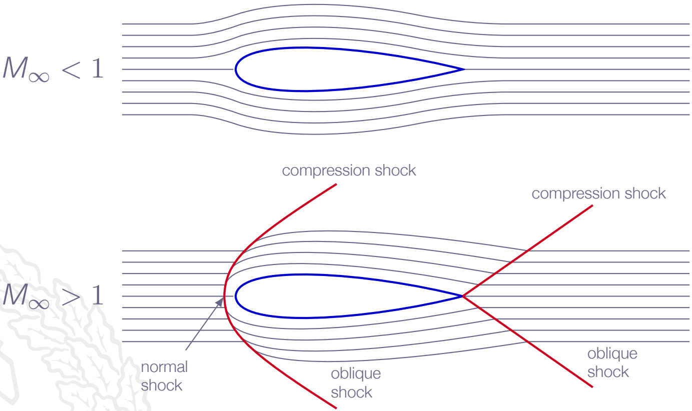
$$M_\infty = \frac{v_\infty}{a_\infty}$$

Physical Consequences of Speed of Sound

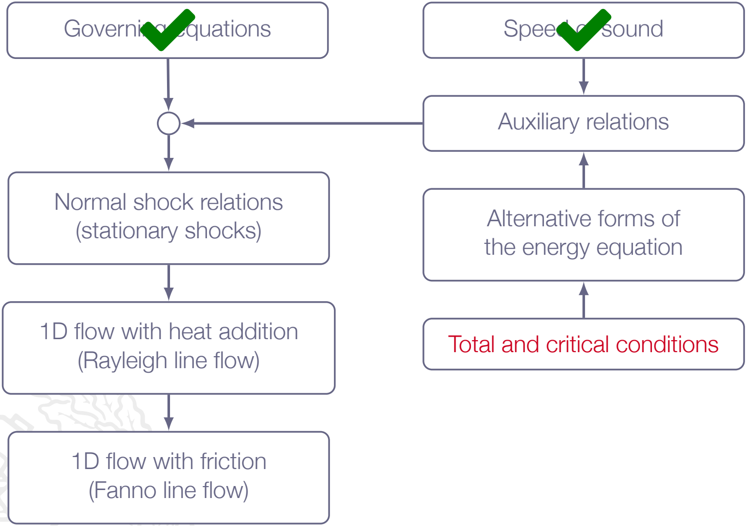
- Sound waves is the way gas molecules convey information about what is happening in the flow
- In subsonic flow, sound waves are able to travel upstream, since $v < a$
- In supersonic flow, sound waves are unable to travel upstream, since $v > a$



Physical Consequences of Speed of Sound



Roadmap - One-dimensional Flow



Chapter 3.4 Some Conveniently Defined Flow Parameters

Stagnation Flow Properties

Assumption: Steady inviscid flow

If the flow is slowed down isentropically (without flow losses) to zero velocity we get the so-called total conditions
(total pressure p_o, total temperature T_o, total density ρ_o)

Since the process is isentropic, we have (for calorically perfect gas)

$$\frac{p_o}{p}=\left(\frac{\rho_o}{\rho}\right)^{\gamma}=\left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Note that $v_o=0$ and $M_o=0$ by definition

Critical Conditions

If we accelerate the flow adiabatically to the sonic point, where $v = a$, we obtain the so-called critical conditions, *e.g.* p^*, T^*, ρ^*, a^*

where, by definition, $v^* = a^*$

As for the total conditions, if the process is also reversible (entropy is preserved) we obtain the relations (for calorically perfect gas)

$$\frac{p^*}{p} = \left(\frac{\rho^*}{\rho}\right)^{\gamma} = \left(\frac{T^*}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Total and Critical Conditions

For any given steady-state flow and location, we may think of an imaginary isentropic stagnation process or an imaginary isentropic sonic flow process

- We can compute total and critical conditions at any point
- They represent conditions realizable under an isentropic deceleration or acceleration of the flow
- Some variables like p_o and T_o will be conserved along streamlines if the flow is isentropic, but p_o is not conserved if entropy changes along the streamlines (due to viscous losses or shocks)

Total and Critical Conditions

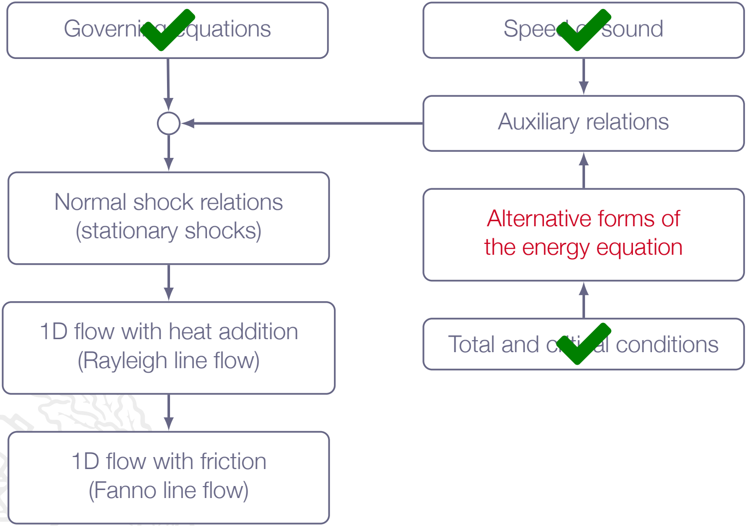
Note: The actual flow does not have to be adiabatic or isentropic from point to point, the total and critical conditions are results of an imaginary isentropic/adiabatic process at one point in the flow.

If the flow is not isentropic:

$$T_{o_A} \neq T_{o_B},\ p_{o_A} \neq p_{o_B},\ \ldots$$

However, with isentropic flow T_o, p_o, ρ_o, etc are constants

Roadmap - One-dimensional Flow



Chapter 3.5
Alternative Forms of the Energy Equation

Alternative Forms of the Energy Equation

For steady-state adiabatic flow, we have already shown that conservation of energy gives that total enthalpy, h_o, is constant along streamlines

For a calorically perfect gas we have $h = C_p T$ which implies

$$C_p T + \frac{1}{2}v^2 = C_p T_o$$

$$\frac{T_o}{T} = 1 + \frac{v^2}{2C_p T}$$

Inserting $C_p = \dfrac{\gamma R}{\gamma - 1}$ and $a^2 = \gamma R T$ we get

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

Alternative Forms of the Energy Equation

For calorically perfect gas (1D/2D/3D flows):

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\left(\frac{a^*}{a_o}\right)^2 = \frac{T^*}{T_o} = \frac{2}{\gamma + 1}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}$$

Note: tabulated values for these relations can be found in Appendix A.1

Alternative Forms of the Energy Equation

$$M^* \equiv \frac{v}{a^*}$$

For a calorically perfect gas (1D/2D/3D flows)

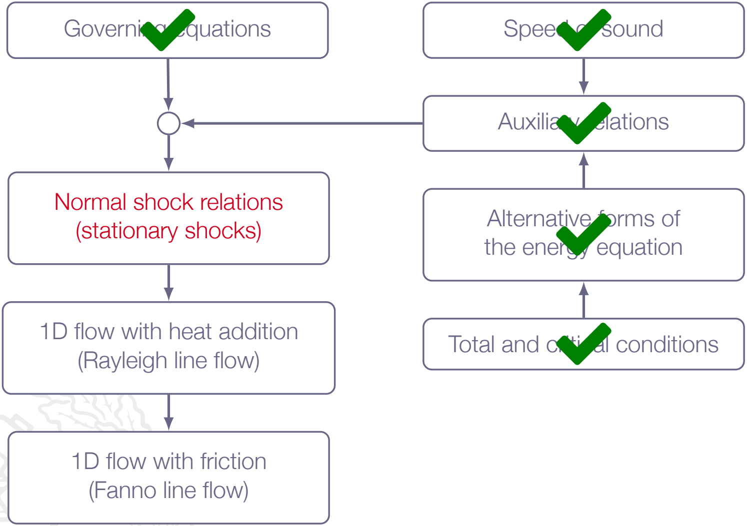
$$M^2 = \frac{2}{[(\gamma+1)/{M^*}^2] - (\gamma-1)}$$

This relation between M and M^* gives:

$$M^* = 0 \Leftrightarrow M = 0$$
$$M^* = 1 \Leftrightarrow M = 1$$
$$M^* < 1 \Leftrightarrow M < 1$$
$$M^* > 1 \Leftrightarrow M > 1$$

$$M^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}} \text{ when } M \rightarrow \infty$$

Roadmap - One-dimensional Flow



Chapter 3.6
Normal Shock Relations

One-Dimensional Flow Equations

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Normal Shock Relations

Calorically perfect gas

$$h = C_p T, \quad p = \rho R T$$

with constant C_p

Assuming that state 1 is known and state 2 is unknown

- 5 unknown variables: ρ_2, u_2, p_2, h_2, T_2
- 5 equations

$\Rightarrow$ solution can be found (see pages 88-90 for derivation)

Normal Shock Relations

Normal shock relations for calorically perfect gas:

$$T_{o_1} = T_{o_2}$$

$$a_{o_1} = a_{o_2}$$

$$a_1^* = a_2^* = a^*$$

$$u_1 u_2 = a^{*2}$$

(the Prandtl relation)

$$M_2^* = \frac{1}{M_1^*}$$

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

see table A.2 and figure 3.10 on p. 94

Normal Shock Relations

Normal shock $\Rightarrow M_1 > 1$

$$M_1 > 1 \Rightarrow M_1^* > 1$$

$$M_2^* = \frac{1}{M_1^*} \Rightarrow M_2^* < 1$$

$$M_2^* < 1 \Rightarrow M_2 < 1$$

After a normal shock the Mach number must be lower than 1.0

Normal Shock Relations

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$

- $M_1 = 1.0 \Rightarrow M_2 = 1.0$
- $M_1 > 1.0 \Rightarrow M_2 < 1.0$
- $M_1 \to \infty \Rightarrow M_2 \to \sqrt{(\gamma - 1)/(2\gamma)} = \{\gamma = 1.4\} = 0.378$

Normal Shock Relations

Are the normal shock relations valid for $M_1 < 1.0$?

Mathematically - yes!

Physically - ?

Normal Shock Relations

Let's have a look at the 2^{nd} law of thermodynamics

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

We get the ratios (T_2/T_1) and (p_2/p_1) from the normal shock relations

$$s_2 - s_1 = C_p \ln \left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right) \left(\frac{2 + (\gamma - 1)M_1^2}{(\gamma+1)M_1^2} \right) \right] + \\ - R \ln \left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right)$$

$M_1 = 1 \Rightarrow \Delta s = 0$ (Mach wave)

$M_1 < 1 \Rightarrow \Delta s < 0$ (not physical)

$M_1 > 1 \Rightarrow \Delta s > 0$

Normal Shock Relations

$M_1 > 1.0$ gives $M_2 < 1.0$, $\rho_2 > \rho_1$, $p_2 > p_1$, and $T_2 > T_1$

What about T_o and p_o?

Energy equation:

$$C_p T_1 + \frac{u_1^2}{2} = C_p T_2 + \frac{u_2^2}{2}$$

$$C_p T_{o_1} = C_p T_{o_2}$$

calorically perfect gas $\Rightarrow$

$$T_{o_1} = T_{o_2}$$

Normal Shock Relations

2^{nd} law of thermodynamics and isentropic deceleration to zero velocity (Δs unchanged since isentropic) gives

$$s_2 - s_1 = C_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{p_{o_2}}{p_{o_1}} = \{T_{o_1} = T_{o_2}\} = -R \ln \frac{p_{o_2}}{p_{o_1}}$$

$$\frac{p_{o_2}}{p_{o_1}} = e^{-(s_2 - s_1)/R}$$

i.e. the total pressure decreases over a normal shock

Normal Shock Relations

As the flow passes a stationary normal shock, the following changes will take place discontinuously across the shock:

ρ	increases
p	increases
u	decreases
M	decreases (from $M > 1$ to $M < 1$)
T	increases
p_o	decreases (due to shock loss)
s	increases (due to shock loss)
T_o	unaffected

Normal Shock Relations

The normal shock relations for calorically perfect gases are valid for $M_1 \leq 5$ (approximately) for air at standard conditions

Calorically perfect gas $\Rightarrow$ Shock depends on M_1 only

Thermally perfect gas $\Rightarrow$ Shock depends on M_1 and T_1

General real gas (non-perfect) $\Rightarrow$ Shock depends on M_1,p_1, and T_1

Chapter 3.7
Hugoniot Equation

Hugoniot Equation

Starting point: governing equations for normal shocks

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Eliminate u_1 and u_2 gives:

$$h_2 - h_1 = \frac{p_2 - p_1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Hugoniot Equation

Now, insert $h = e + p/\rho$ gives

$$e_2 - e_1 = \frac{p_2 + p_1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \frac{p_2 + p_1}{2}\left(\nu_1 - \nu_2\right)$$

which is the Hugoniot relation

Stationary Normal Shock in One-Dimensional Flow

Normal shock:

$$e_2 - e_1 = -\frac{p_2 + p_1}{2}(\nu_2 - \nu_1)$$

- More effective than isentropic process
- Gives entropy increase

Isentropic process:

$$de = -pd\nu$$

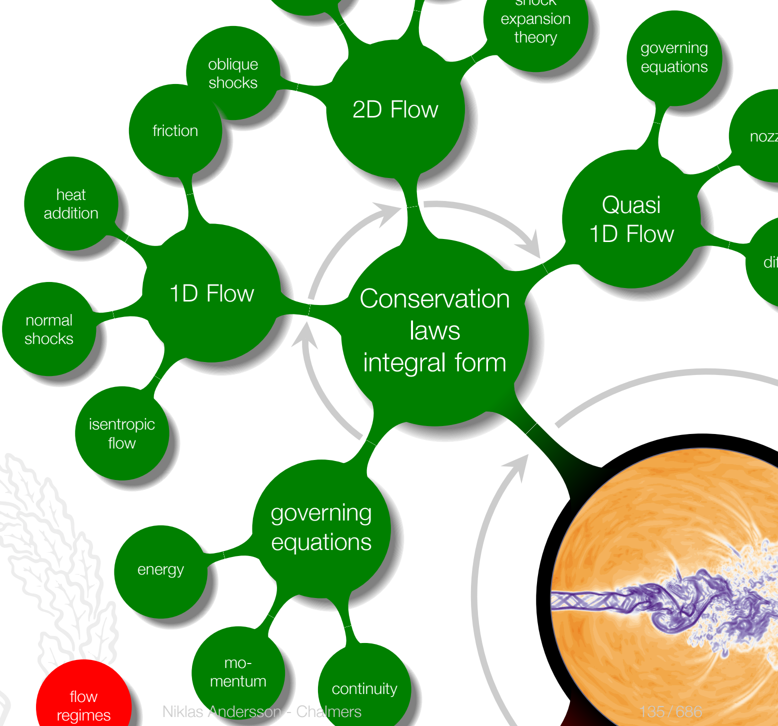
- More efficient than normal shock process

see figure 3.11 p. 100

LECTURE 4

Chapter 3
One-Dimensional Flow

Overview


expansion theory
oblique shocks
2D Flow
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friction
heat addition
Quasi 1D Flow
1D Flow
normal shocks
Conservation laws integral form
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energy
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flow regimes

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

5 Explain how thermodynamic relations enter into the flow equations

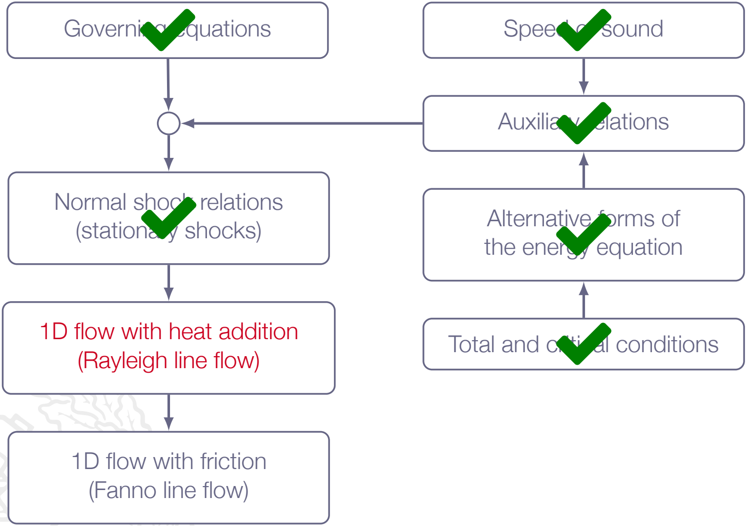
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   c 1D flow with heat addition*

   d 1D flow with friction*

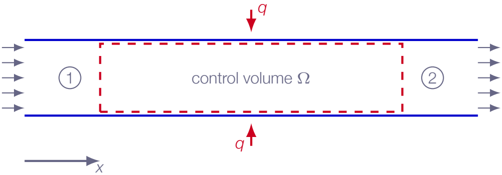
inviscid flow with friction?!

Roadmap - One-dimensional Flow



Chapter 3.8
One-Dimensional Flow with Heat Addition

One-Dimensional Flow with Heat Addition



Pipe flow:

- no friction
- 1D steady-state ⇒ all variables depend on x only
- q is the amount of heat per unit mass added between 1 and 2
- analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Heat Addition

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 + q = h_2 + \frac{1}{2}u_2^2$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ analytical solution exists

One-Dimensional Flow with Heat Addition

Calorically perfect gas ($h = C_p T$):

$$C_p T_1 + \frac{1}{2}u_1^2 + q = C_p T_2 + \frac{1}{2}u_2^2$$

$$q = \left(C_p T_2 + \frac{1}{2}u_2^2\right) - \left(C_p T_1 + \frac{1}{2}u_1^2\right)$$

$$C_p T_o = C_p T + \frac{1}{2}u^2 \Rightarrow$$

$$q = C_p(T_{o_2} - T_{o_1})$$

i.e. heat addition increases T_o downstream

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{T_2}{T_1} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]^2 \left(\frac{M_2}{M_1}\right)^2$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right] \left(\frac{M_1}{M_2}\right)^2$$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{p_{o_2}}{p_{o_1}} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]\left(\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{o_2}}{T_{o_1}} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]\left(\frac{M_2}{M_1}\right)^2\left(\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

One-Dimensional Flow with Heat Addition

Initially subsonic flow ($M < 1$)

- the Mach number, M, increases as more heat (per unit mass) is added to the gas
- for some limiting heat addition q^*, the flow will eventually become sonic $M = 1$

Initially supersonic flow ($M > 1$)

- the Mach number, M, decreases as more heat (per unit mass) is added to the gas
- for some limiting heat addition q^*, the flow will eventually become sonic $M = 1$

Note: The (*) condition in this context is not the same as the "critical" condition discussed for isentropic flow!!!

One-Dimensional Flow with Heat Addition

$$\frac{T}{T^*} = \left[\frac{1+\gamma}{1+\gamma M^2}\right]^2 M^2$$

$$\frac{\rho}{\rho^*} = \left[\frac{1+\gamma M^2}{1+\gamma}\right]\left(\frac{1}{M^2}\right)$$

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{p_o}{p_o^*} = \left[\frac{1+\gamma}{1+\gamma M^2}\right]\left(\frac{2+(\gamma-1)M^2}{(\gamma+1)}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T_o^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2}(2+(\gamma-1)M^2)$$

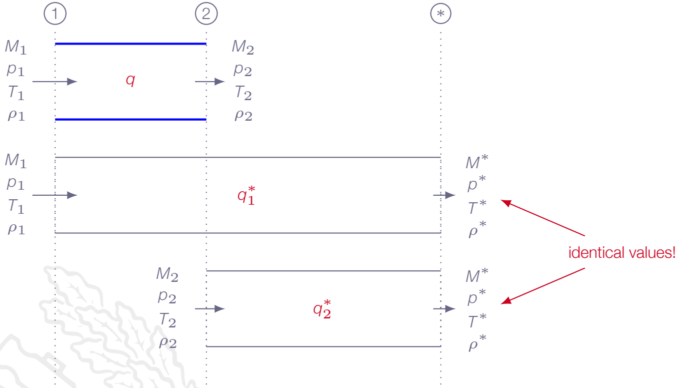
see Table A.3

One-Dimensional Flow with Heat Addition

Amount of heat per unit mass needed to choke the flow:

$$q^* = C_p(T_o^* - T_o) = C_p T_o \left(\frac{T_o^*}{T_o} - 1\right)$$

One-Dimensional Flow with Heat Addition



$$q_2^* = q_1^* - q$$

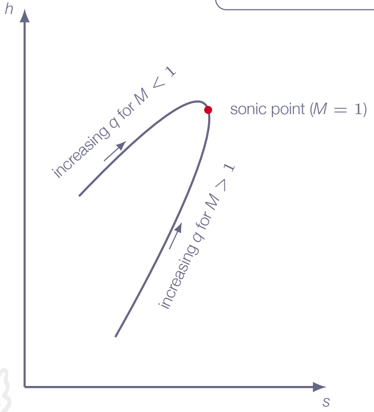
For a given flow, the starred quantities are constant values

One-Dimensional Flow with Heat Addition

Rayleigh curve

Lord Rayleigh 1842-1919
Nobel prize in physics 1904

Note: it is theoretically possible to heat an initially subsonic flow to reach sonic conditions and then continue to accelerate the flow by cooling



see Figure 3.13

One-Dimensional Flow with Heat Addition

$M < 1$: Adding heat will

- increase M
- decrease p
- increase T_o
- decrease p_o
- increase s
- increase u
- decrease ρ

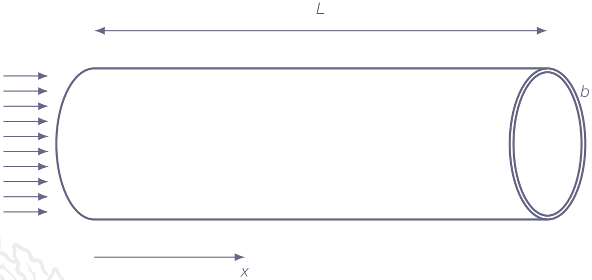
Flow loss - not isentropic process

$M > 1$: Adding heat will

- decrease M
- increase p
- increase T_o
- decrease p_o
- increase s
- decrease u
- increase ρ

One-Dimensional Flow with Heat Addition

Relation between added heat per unit mass (q) and heat per unit surface area and unit time ($\dot{q}_{wall}$)

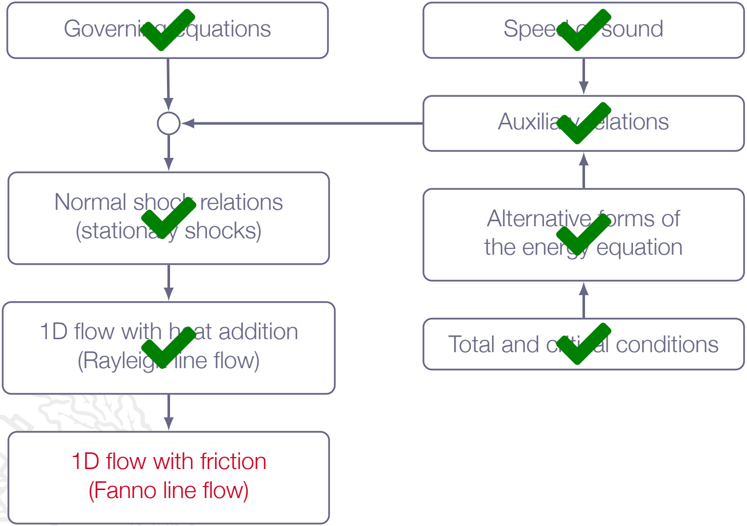


Pipe with arbitrary cross section (constant in x):

mass flow through pipe	$\dot{m}$
axial length of pipe	L
circumference of pipe	$b = 2\pi r$

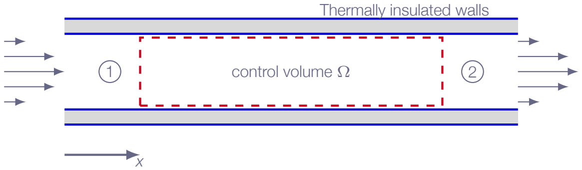
$$q = \frac{Lb\dot{q}_{wall}}{\dot{m}}$$

Roadmap - One-dimensional Flow



Chapter 3.9
One-Dimensional Flow with Friction

One-Dimensional Flow with Friction



Pipe flow:

- adiabatic ($q = 0$)
- cross section area A is constant
- average all variables in each cross-section ⇒ only x-dependence
- analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Friction

Wall-friction contribution in momentum equation

$$\oiint_{\partial\Omega} \tau_W dS = \bar{\tau}_W L b$$

where L is the tube length and b is the circumference

One-Dimensional Flow with Friction

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 - \bar{\tau}_w \frac{Lb}{A} = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

One-Dimensional Flow with Friction

If the tube has a round cross-section with diameter D:

$$\bar{\tau}_w \frac{Lb}{A} = \frac{4L}{D}\bar{\tau}_w$$

For small $L = \Delta x$, the momentum equation becomes

$$\rho_1 u_1^2 + p_1 - \bar{\tau}_w \frac{4}{D}\Delta x = \rho_2 u_2^2 + p_2$$

Now, let $\Delta x \to 0 \Rightarrow$

$$\frac{d}{dx}(\rho u^2 + p) = -\frac{4}{D}\tau_w$$

One-Dimensional Flow with Friction

Form mass conservation we get

$$\rho_1 u_1 = \rho_2 u_2 \Rightarrow \frac{d}{dx}(\rho u) = 0$$

Writing out all terms in the momentum equation gives

$$\frac{d}{dx}(\rho u^2 + p) = \rho u \frac{du}{dx} + u \underbrace{\frac{d}{dx}(\rho u)}_{=0} + \frac{dp}{dx} = -\frac{4}{D}\tau_w$$

and thus

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{4}{D}\tau_w$$

Common approximation for τ_w:

$$\tau_w = f\frac{1}{2}\rho u^2 \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D}\rho u^2 f$$

One-Dimensional Flow with Friction

Energy conservation:

$$h_{o_1} = h_{o_2} \Rightarrow \frac{d}{dx} h_o = 0$$

One-Dimensional Flow with Friction

Summary:

$$\frac{d}{dx}(\rho u) = 0$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D}\rho u^2 f$$

$$\frac{d}{dx}h_o = 0$$

Valid for all gases!

General gas ⇒ Numerical solution necessary

Calorically perfect gas ⇒ analytical solution exists (for constant f)

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\int_{x_1}^{x_2} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) \right]_{M_1}^{M_2}$$

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{1/2}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{-1/2}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

One-Dimensional Flow with Friction

Initially subsonic flow ($M_1 < 1$)

- M_2 will increase as L increases
- for a critical length L^*, the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Initially supersonic flow ($M_1 > 1$)

- M_2 will decrease as L increases
- for a critical length L^*, the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Note: The (*) condition in this context is not the same as the "critical" condition discussed for isentropic flow!!!

One-Dimensional Flow with Friction

$$\frac{T}{T^*} = \frac{(\gamma+1)}{2+(\gamma-1)M^2}$$

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{1/2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{1/2}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

see Table A.4

One-Dimensional Flow with Friction

and

$$\int_0^{L^*} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma} \ln\left(\frac{M^2}{1+\frac{\gamma-1}{2}M^2} \right) \right]_M^1$$

where L^* is the tube length needed to change current state to sonic conditions

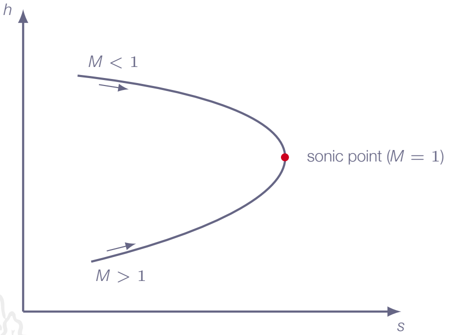
Let $\bar{f}$ be the average friction coefficient over the length L^* $\Rightarrow$

$$\frac{4\bar{f}L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma} \ln\left(\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \right)$$

Turbulent pipe flow $\rightarrow \bar{f} \sim 0.005$ ($Re > 10^5$, roughness $\sim 0.001D$)

One-Dimensional Flow with Friction

Fanno curve



see Figure 3.15

One-Dimensional Flow with Friction

$M < 1$: Friction will

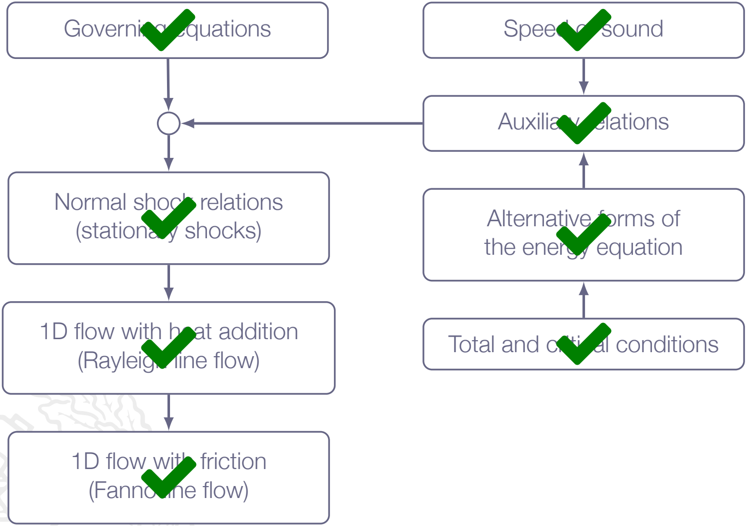
- increase M
- decrease p
- decrease T
- decrease p_o
- increase s
- increase u
- decrease ρ

Flow loss - non-isentropic flow

$M > 1$: Friction will

- decrease M
- increase p
- increase T
- decrease p_o
- increase s
- decrease u
- increase ρ

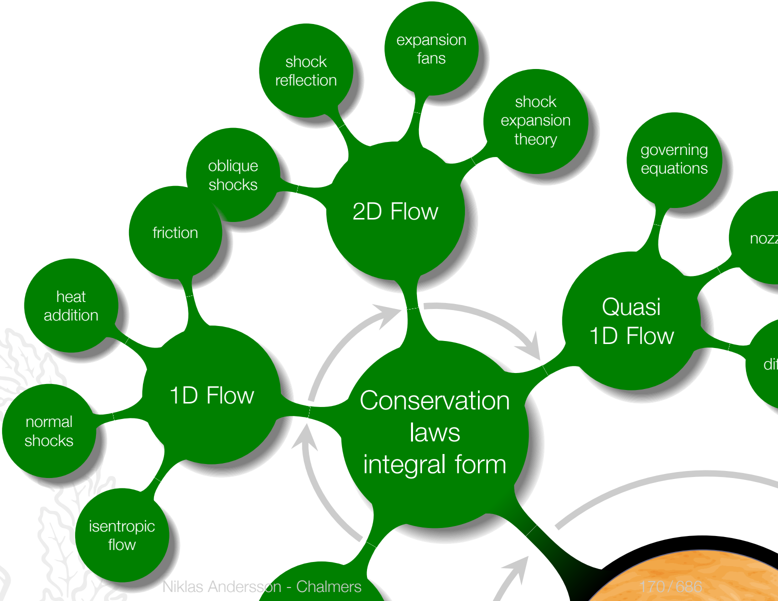
Roadmap - One-dimensional Flow



LECTURE 5

Chapter 4 Oblique Shocks and Expansion Waves

Overview


shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
governing equations
friction
heat addition
1D Flow
Quasi 1D Flow
Conservation laws integral form
normal shocks
isentropic flow

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

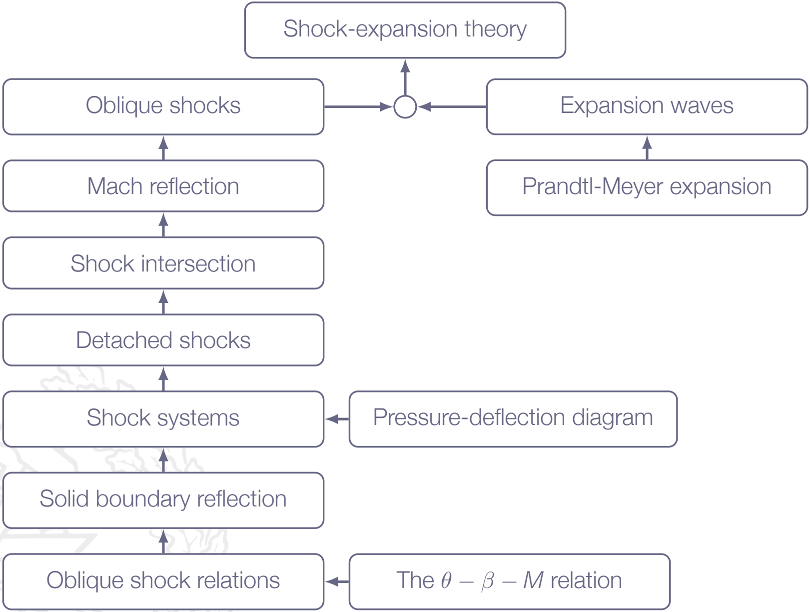
7 Explain why entropy is important for flow discontinuities

8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

- b normal shocks*
- e oblique shocks in 2D*
- f shock reflection at solid walls*
- g contact discontinuities
- i detached blunt body shocks, nozzle flows

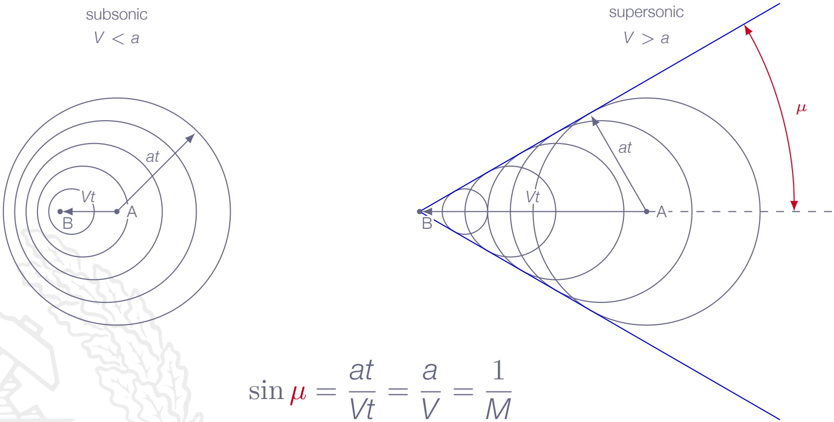
why do we get normal shocks in some cases and oblique shocks in other?

Roadmap - Oblique Shocks and Expansion Waves



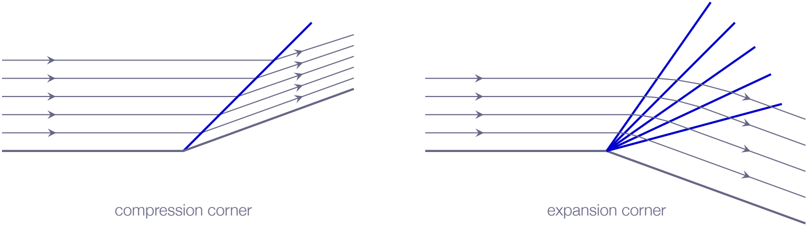
Mach Waves

A Mach wave is an infinitely weak oblique shock



$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Oblique Shocks and Expansion Waves



compression corner   expansion corner

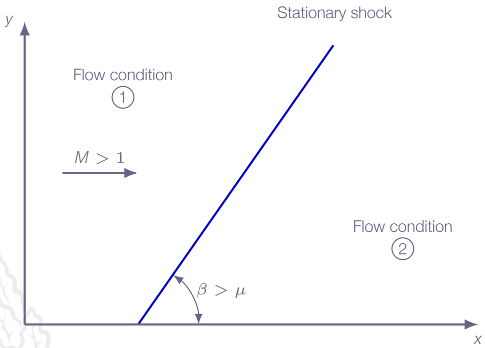
Supersonic two-dimensional steady-state inviscid flow (no wall friction)

In real flow, viscosity is non-zero $\Rightarrow$ boundary layers

For high-Reynolds-number flows, boundary layers are thin $\Rightarrow$ inviscid theory still relevant!

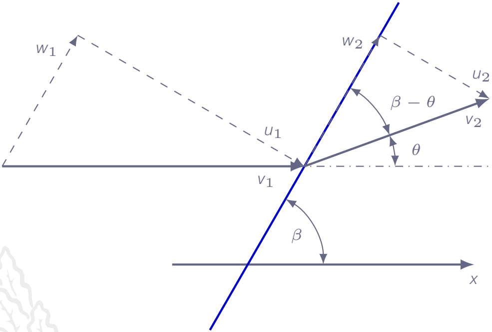
Oblique Shocks

Two-dimensional steady-state flow

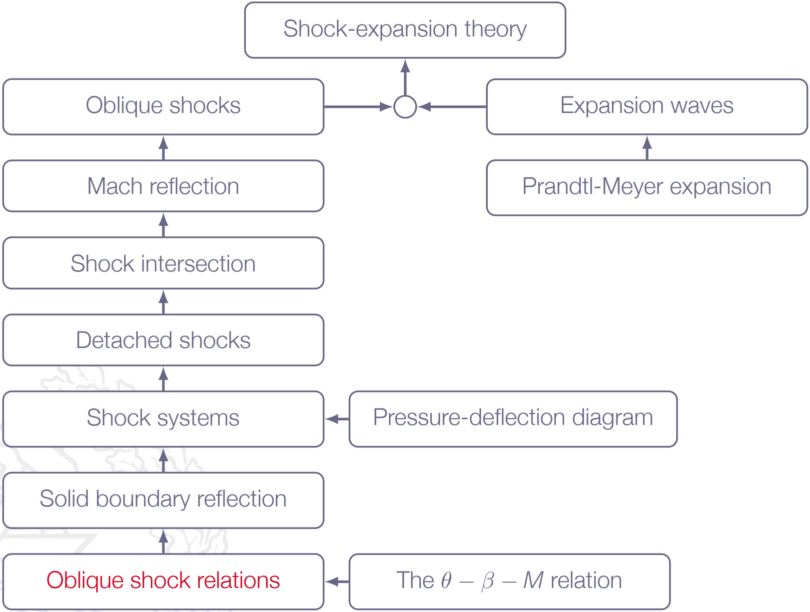


Oblique Shocks

Stationary oblique shock

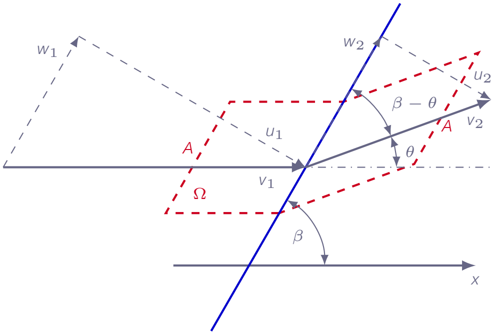


Roadmap - Oblique Shocks and Expansion Waves



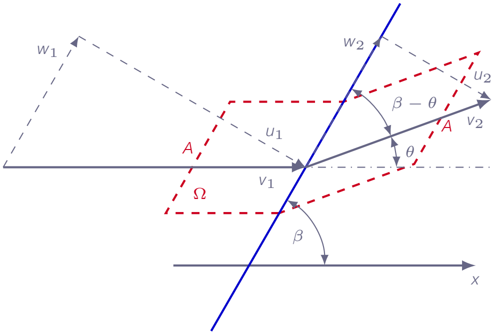
Chapter 4.3
Oblique Shock Relations

Oblique Shock Relations



- Two-dimensional steady-state flow
- Control volume aligned with flow stream lines

Oblique Shock Relations



Velocity notations:

$$M_{n_1} = \frac{u_1}{a_1} = M_1 \sin(\beta) \qquad M_1 = \frac{v_1}{a_1}$$

$$M_{n_2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta) \qquad M_2 = \frac{v_2}{a_2}$$

Oblique Shock Relations

Conservation of mass:

$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} + \oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0$$

Mass conservation for control volume Ω:

$$0 - \rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

Oblique Shock Relations

Conservation of momentum:

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}$$

Momentum in shock-normal direction:

$$0-(\rho_1u_1^2+p_1)A+(\rho_2u_2^2+p_2)A=0\Rightarrow$$

$$\boxed{\rho_1u_1^2+p_1=\rho_2u_2^2+p_2}$$

Oblique Shock Relations

Momentum in shock-tangential direction:

$$0 - \rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow$$

$$\boxed{w_1 = w_2}$$

Oblique Shock Relations

Conservation of energy:

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega} [\rho h_o \mathbf{v}\cdot\mathbf{n}]\, dS = \iiint\limits_{\Omega} \rho \mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

Energy equation applied to the control volume Ω:

$$0 - \rho_1 u_1 [h_1 + \frac{1}{2}(u_1^2 + w_1^2)]A + \rho_2 u_2 [h_2 + \frac{1}{2}(u_2^2 + w_2^2)]A = 0 \Rightarrow$$

$$\boxed{h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2}$$

Oblique Shock Relations

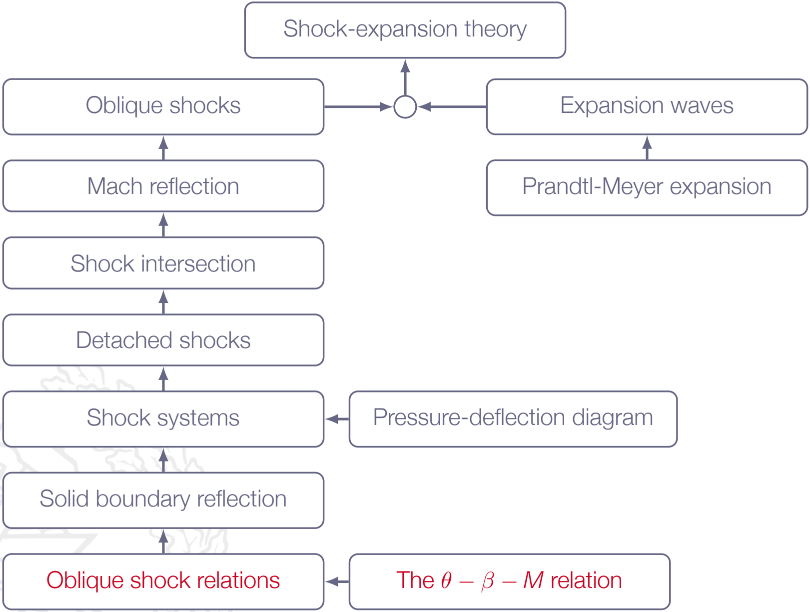
We can use the equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

Ratios such as ρ_2/ρ_1, p_2/p_1, and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

OBS! Do not not use ratios involving total quantities, *e.g.* p_{o_2}/p_{o_1}, T_{o_2}/T_{o_1}, obtained from formulas or tables for normal shock

Roadmap - Oblique Shocks and Expansion Waves



The θ-β-M Relation

It can be shown that

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}\right)$$

which is the θ-β-M relation

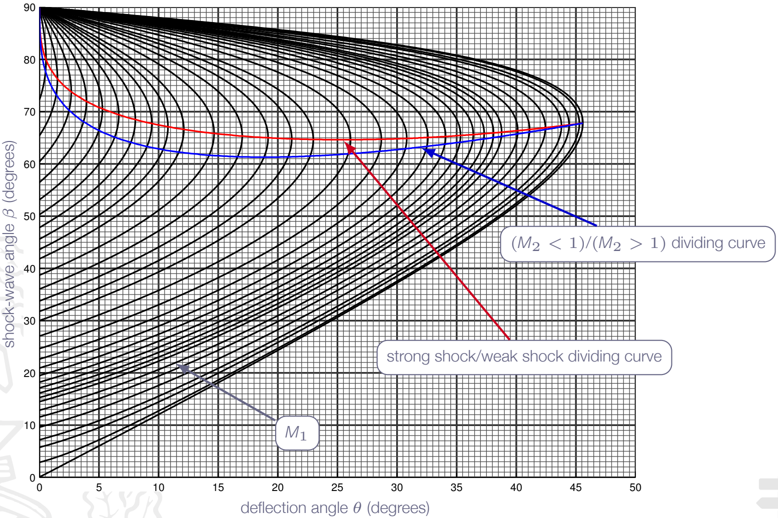
Does this give a complete specification of flow state 2 as function of flow state 1?

The θ-β-M Relation

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)$$

In general there are two solutions for a given M_1 (or none)

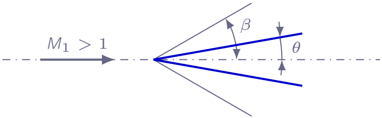
Oblique shock properties (the θ-β-M relation for $\gamma = 1.4$)



The θ-β-M Relation

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)$$

Example: Wedge flow



Two solution case:

Weak solution:

- smaller β, $M_2 > 1$ (except in some cases)

Strong solution:

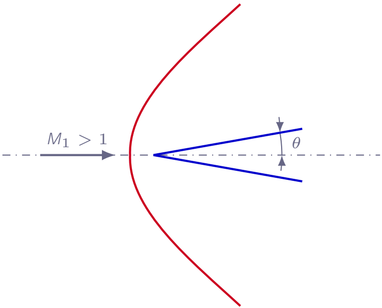
- larger β, $M_2 < 1$

Note: In Chapter 3 we learned that the mach number always reduces to subsonic values behind a shock. This is true for normal shocks (shocks that are normal to the flow direction). It is also true for oblique shocks if looking in the shock-normal direction.

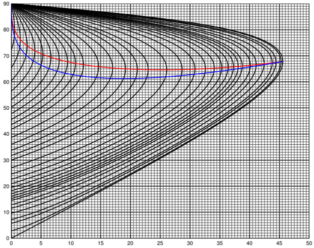
The θ-β-M Relation

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)$$

No solution case: Detached curved shock



The θ-β-M Relation - Skock Strength



- There is a small region where we may find weak shock solutions for which $M_2 < 1$
- In most cases weak shock solutions have $M_2 > 1$
- Strong shock solutions always have $M_2 < 1$
- In practical situations, weak shock solutions are most common
- Strong shock solution may appear in special situations due to high back pressure, which forces $M_2 < 1$

The θ-β-M Relation - Wedge Flow

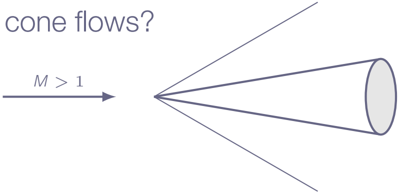
Summary for wedge flow:

1. θ-β-M relation $\Rightarrow \beta$ for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of $M_1 \Rightarrow$ M_{n_2} (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of $M_1 \Rightarrow$ ρ_2/ρ_1, p_2/p_1, etc
6. upstream conditions + ρ_2/ρ_1, p_2/p_1, etc $\Rightarrow$ downstream conditions

Chapter 4.4
Supersonic Flow over Wedges and Cones

Supersonic Flow over Wedges and Cones

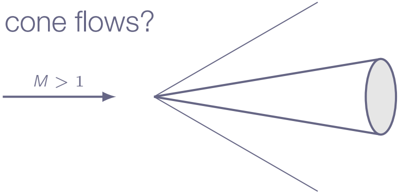
What about cone flows?



- Similar to wedge flow, we do get a constant-strength shock wave, attached at the cone tip (or else a detached curved shock)
- The attached shock is also cone-shaped

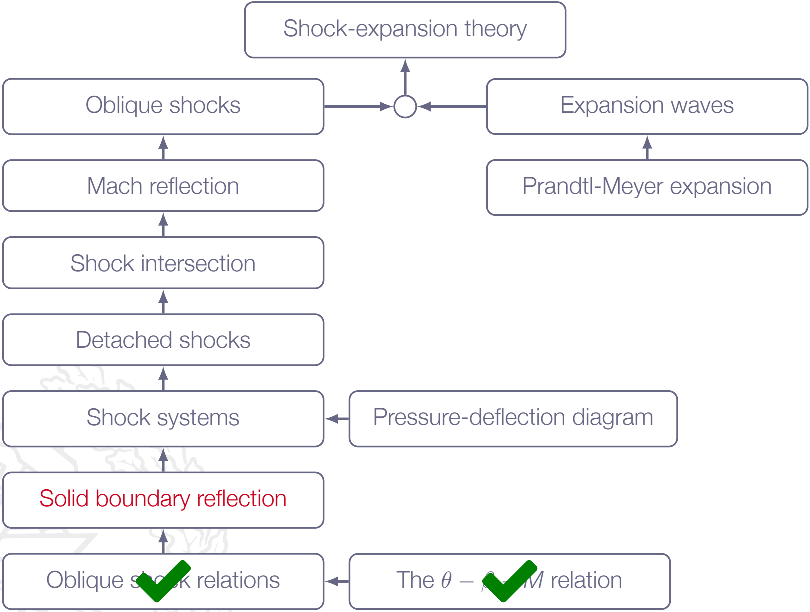
Supersonic Flow over Wedges and Cones

What about cone flows?



- The flow condition immediately downstream of the shock is uniform
- However, downstream of the shock the streamlines are curved and the flow varies in a more complex manner (3D relieving effect - as R increases there is more and more space around cone for the flow)
- β for cone shock is always smaller than that for wedge shock, if M_1 is the same

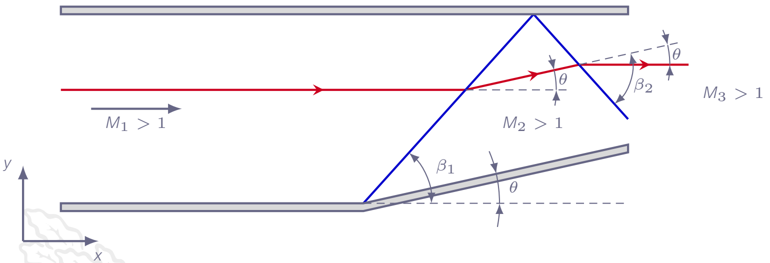
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.6
Regular Reflection from a Solid Boundary

Shock Reflection

Regular reflection of oblique shock at solid wall
(see example 4.10)



Assumptions:

- steady-state inviscid flow
- weak shocks

Shock Reflection

first shock:

- upstream condition:

  $M_1 > 1$, flow in x-direction

- downstream condition:

  weak shock $\Rightarrow M_2 > 1$
  deflection angle θ
  shock angle β_1

second shock:

- upstream condition:

  same as downstream condition of first shock

- downstream condition:

  weak shock $\Rightarrow M_3 > 1$
  deflection angle θ
  shock angle β_2

Shock Reflection

Solution:

first shock:

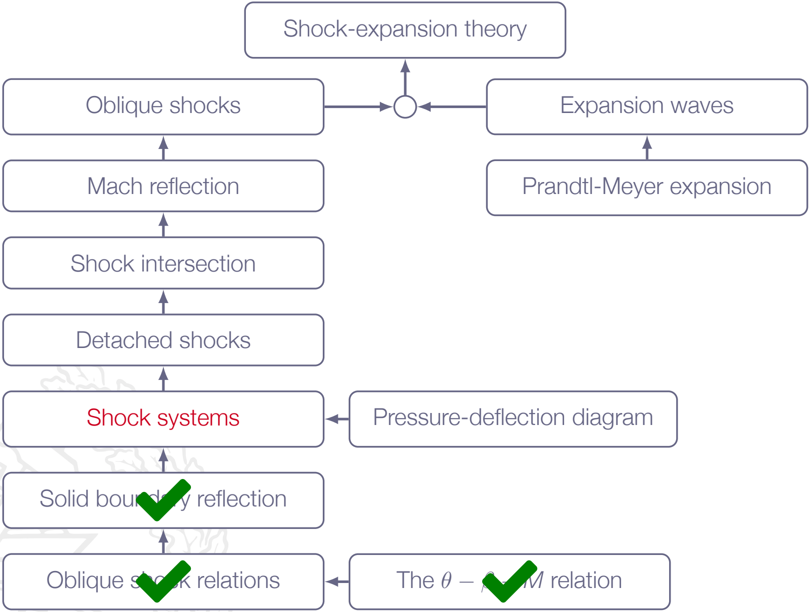
- β_1 calculated from θ-β-M relation for specified θ and M_1 (*weak solution*)
- flow condition 2 according to formulas for normal shocks ($M_{n_1} = M_1 \sin(\beta_1)$ and $M_{n_2} = M_2 \sin(\beta_1 - \theta)$)

second shock:

- β_2 calculated from θ-β-M relation for specified θ and M_2 (*weak solution*)
- flow condition 3 according to formulas for normal shocks ($M_{n_2} = M_2 \sin(\beta_2)$ and $M_{n_3} = M_3 \sin(\beta_2 - \theta)$)

$\Rightarrow$ complete description of flow and shock waves
(angle between upper wall and second shock: $\Phi = \beta_2 - \theta$)

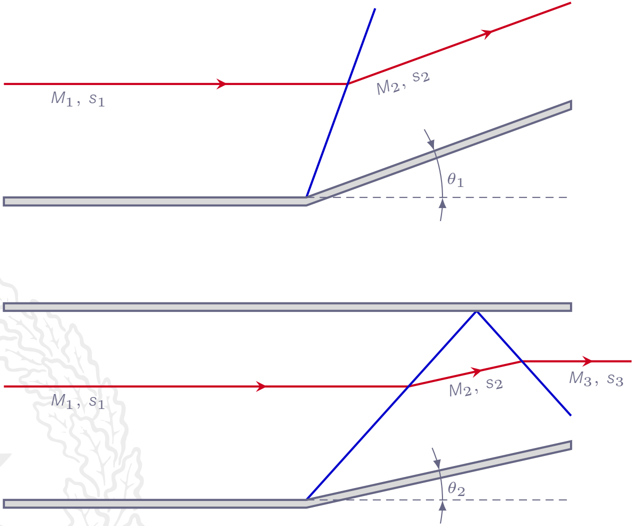
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.7
Comments on Flow Through Multiple Shock Systems

Flow Through Multiple Shock Systems

Single-shock compression versus multiple-shock compression:



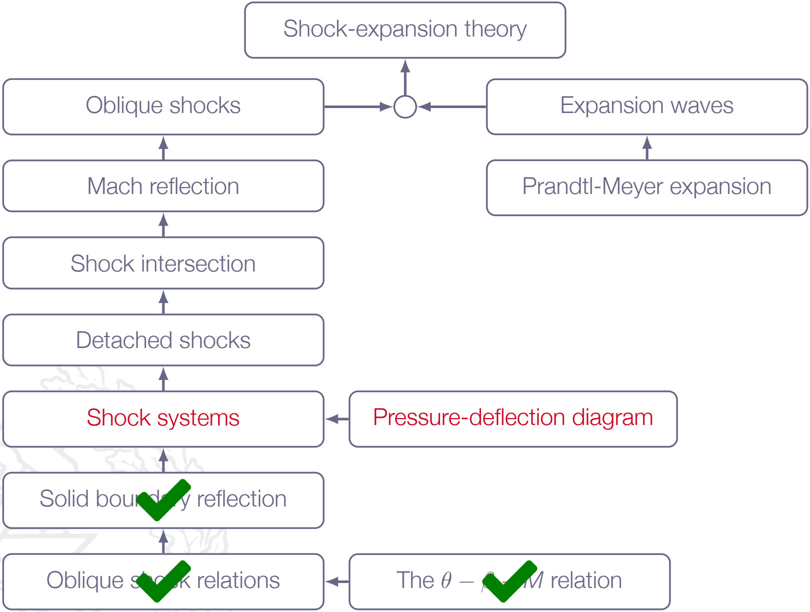
Flow Through Multiple Shock Systems

We may find θ_1 and θ_2 (for same M_1) which gives the same final Mach number

In such cases, the multiple shock flow has smaller losses

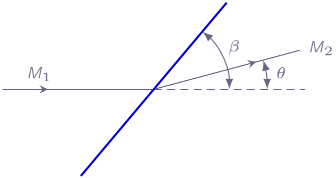
Explanation: entropy generation at a shock is a very non-linear function of shock strength

Roadmap - Oblique Shocks and Expansion Waves

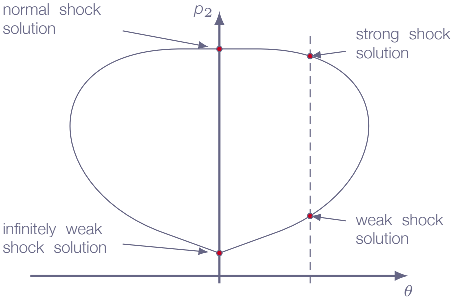


Chapter 4.8
Pressure Deflection Diagrams

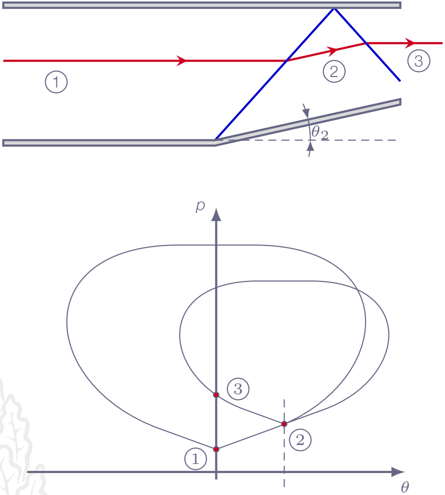
Pressure Deflection Diagrams



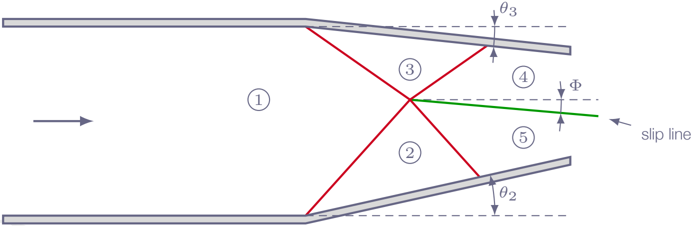
⇒ relation between p_2 and θ



Pressure Deflection Diagrams - Shock Reflection



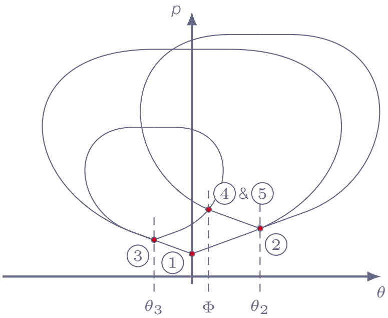
Pressure Deflection Diagrams - Shock Intersection



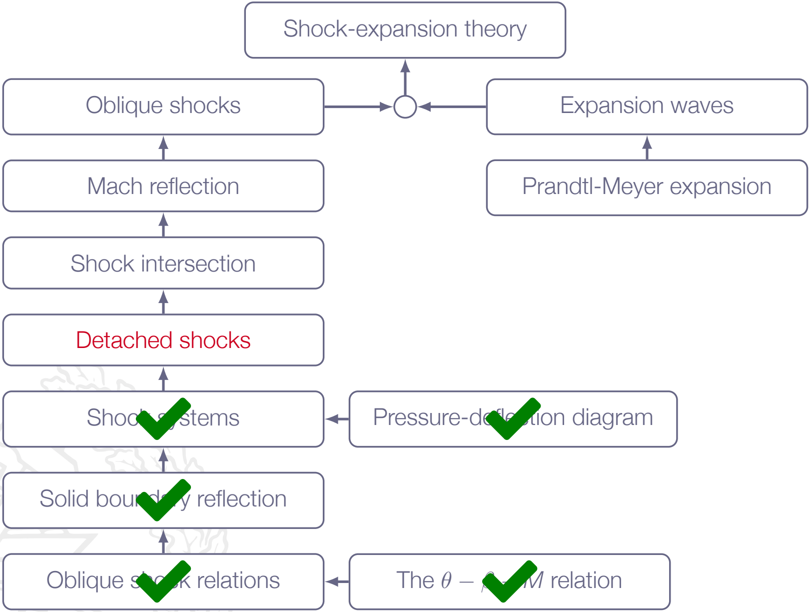
A slip line is a contact discontinuity

- discontinuity in ρ, T, s, v, and M
- continuous in p and flow angle

Pressure Deflection Diagrams - Shock Intersection

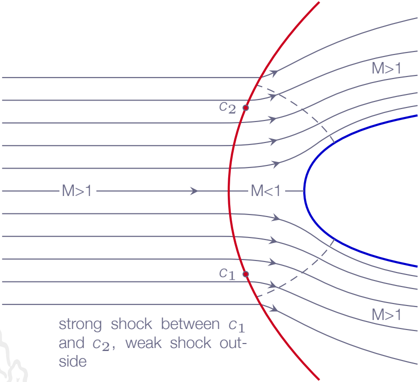


Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.12
Detached Shock Wave in Front of a Blunt Body

Detached Shocks



Detached Shocks

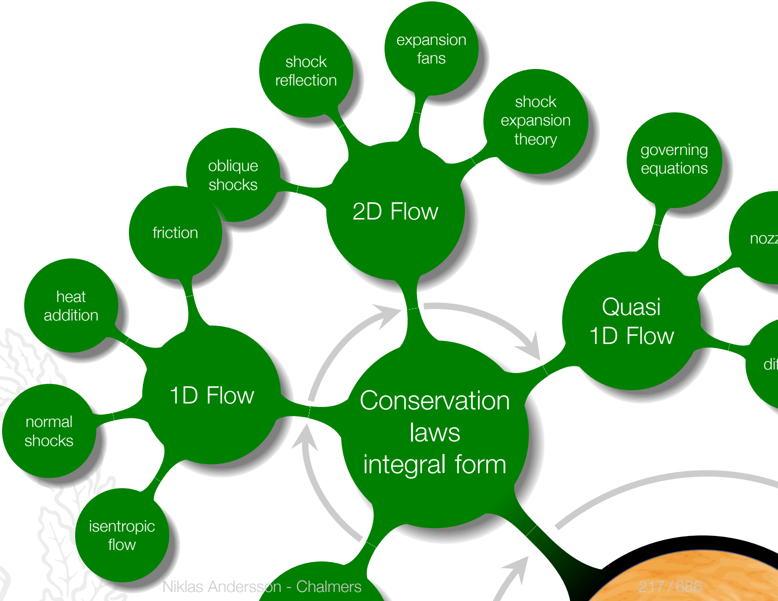
As we move along the detached shock form the centerline, the shock will change in nature as

- right in front of the body we will have a normal shock
- strong oblique shock
- weak oblique shock
- far away from the body it will approach a Mach wave, *i.e.* an infinitely weak oblique shock

LECTURE 6

Chapter 4
Oblique Shocks and Expansion Waves

Overview



shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
governing equations
friction
heat addition
1D Flow
Quasi 1D Flow
normal shocks
Conservation laws integral form
isentropic flow

Addressed Learning Outcomes

7 Explain why entropy is important for flow discontinuities

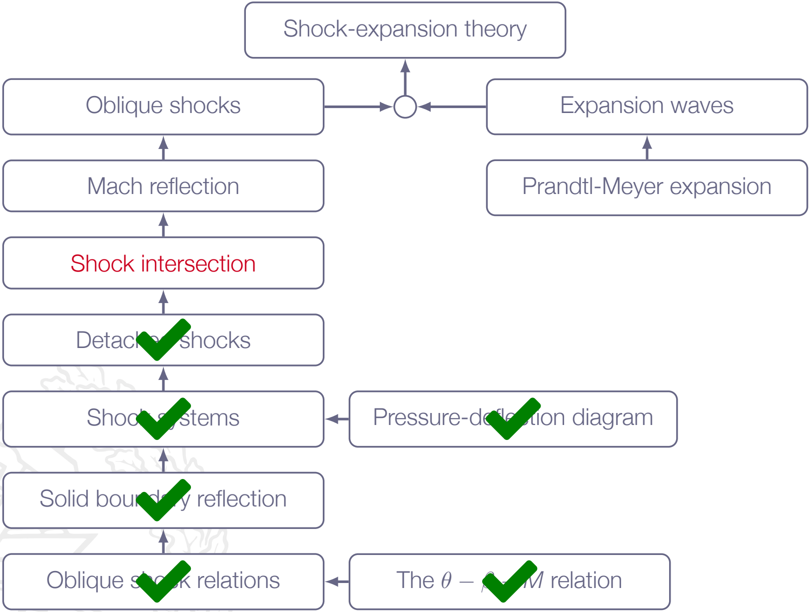
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   e oblique shocks in 2D*
   f shock reflection at solid walls*
   g contact discontinuities
   h Prandtl-Meyer expansion fans in 2D
   i detached blunt body shocks, nozzle flows

9 Solve engineering problems involving the above-mentioned phenomena (8a-8k)

what is the opposite of a shock?

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.10 Intersection of Shocks of the Same Family

Mach Waves

Oblique shock, angle β, flow deflection θ:

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

where

$$M_{n_1} = M_1 \sin(\beta)$$

and

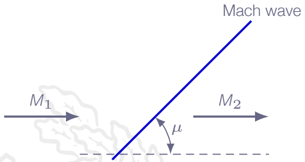
$$M_{n_2} = M_2 \sin(\beta - \theta)$$

Now, let $M_{n_1} \to 1$ and $M_{n_2} \to 1 \Rightarrow$ infinitely weak shock!

Such very weak shocks are called Mach waves

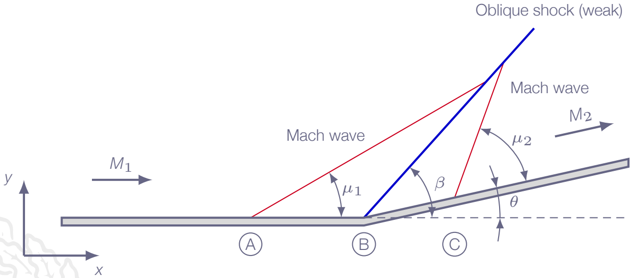
Mach Waves

$$M_{n_1} = 1 \Rightarrow M_1 \sin(\beta) = 1 \Rightarrow \beta = \arcsin(1/M_1)$$



- $M_2 \approx M_1$
- $\theta \approx 0$
- $\mu = \arcsin(1/M_1)$

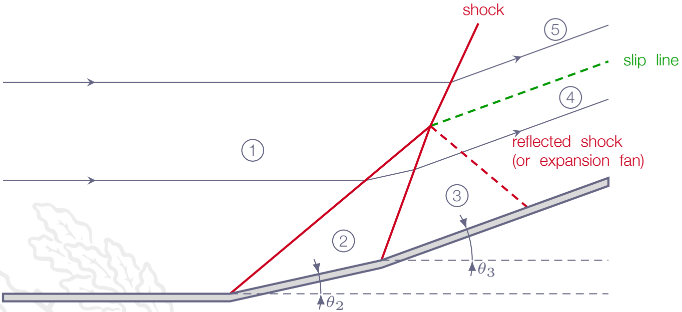
Mach Waves


Oblique shock (weak)
Mach wave
Mach wave
M_2
μ_2
M_1
μ_1
β
θ
y
x
A
B
C

Mach Waves

- Mach wave at A: $\sin(\mu_1) = 1/M_1$
- Mach wave at C: $\sin(\mu_2) = 1/M_2$
- Oblique shock at B: $M_{n_1} = M_1 \sin(\beta) \Rightarrow \sin(\beta) = M_{n_1}/M_1$
  - Existence of shock requires $M_{n_1} > 1 \Rightarrow \beta > \mu_1$
  - Mach wave intercepts shock!
- Also, $M_{n_2} = M_2 \sin(\beta - \theta) \Rightarrow \sin(\beta - \theta) = M_{n_2}/M_2$
  - For finite shock strength $M_{n_2} < 1 \Rightarrow (\beta - \theta) < \mu_2$
  - Again, Mach wave intercepts shock

Shock Intersection - Same Family



Shock Intersection - Same Family

Case 1: Streamline going through regions 1, 2, 3, and 4 (through two oblique (weak) shocks)

Case 2: Streamline going through regions 1 and 5 (through one oblique (weak) shock)

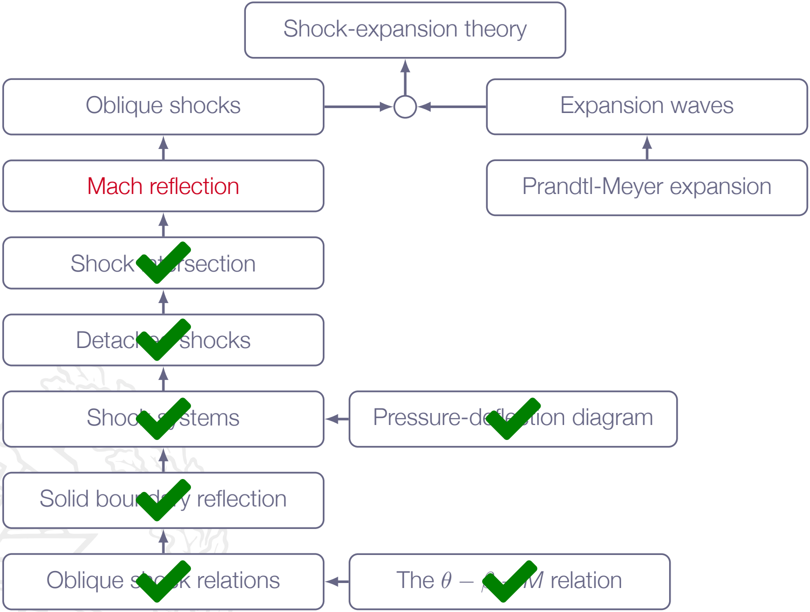
Problem: Find conditions 4 and 5 such that

a. $p_4 = p_5$
b. flow angle in 4 equals flow angle in 5

Solution may give either reflected shock or expansion fan, depending on actual conditions

A slip line usually appears, across which there is a discontinuity in all variables except p and flow angle

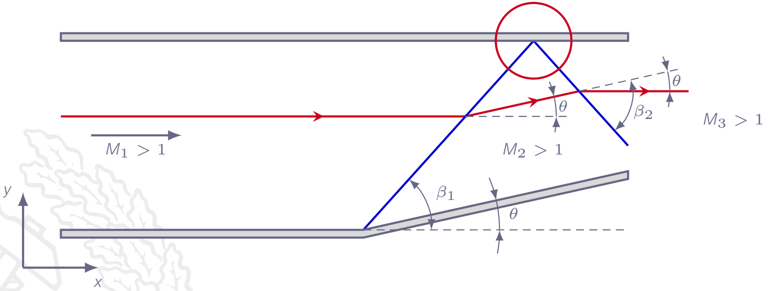
Roadmap - Oblique Shocks and Expansion Waves



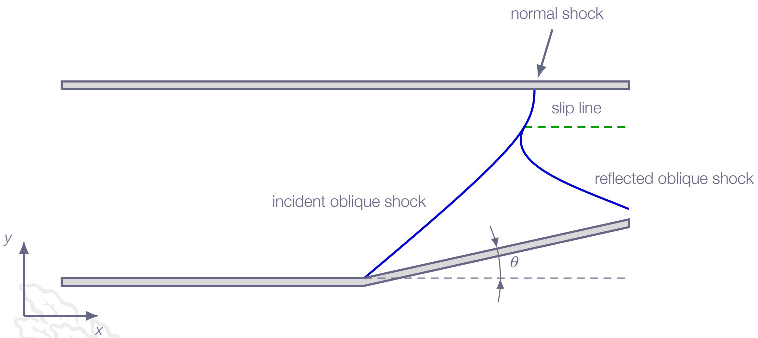
Chapter 4.11
Mach Reflection

Regular Shock Reflection

Regular reflection possible if both primary and reflected shocks are âweakâ (see θ-β-M relation)



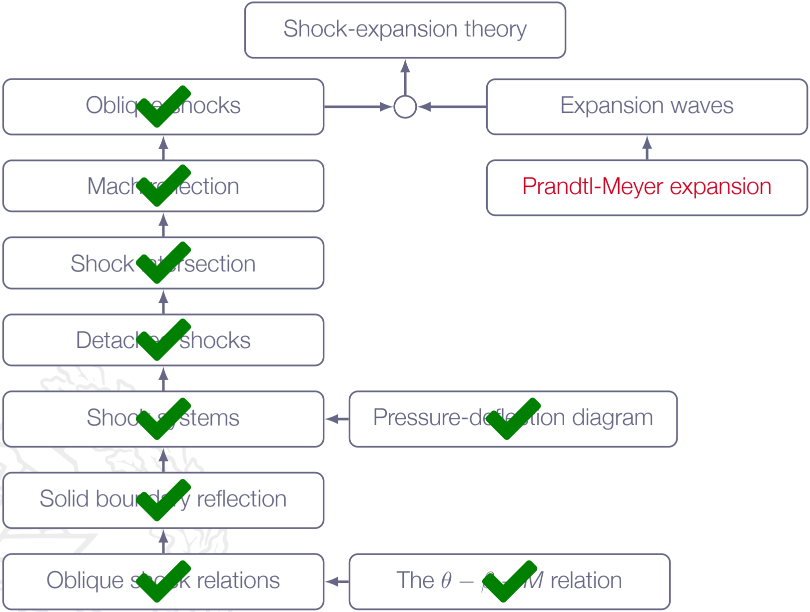
Mach Reflection



Mach reflection:

- appears when regular reflection is not possible
- more complex flow than for a regular reflection
- no analytic solution - numerical solution necessary

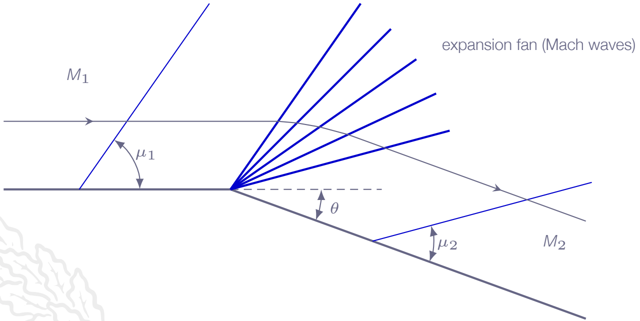
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.14
Prandtl-Meyer Expansion Waves

Prandtl-Meyer Expansion Waves

An expansion fan is a centered simple wave (also called Prandl-Meyer expansion)



- $M_2 > M_1$ (the flow accelerates through the expansion fan)
- $p_2 < p_1$, $\rho_2 < \rho_1$, $T_2 < T_1$

Prandtl-Meyer Expansion Waves

- Continuous expansion region
- Infinite number of weak Mach waves
- Streamlines through the expansion wave are smooth curved lines
- $ds = 0$ for each Mach wave $\Rightarrow$ the expansion process is ISENTROPIC!

Prandtl-Meyer Expansion Waves

- upstream of expansion $M_1 > 1$, $\sin(\mu_1) = 1/M_1$
- flow accelerates as it curves through the expansion fan
- downstream of expansion $M_2 > M_1$, $\sin(\mu_2) = 1/M_2$
- flow is isentropic $\Rightarrow s, p_o, T_o, \rho_o, a_o$, ... are constant along streamlines
- flow deflection: θ

Prandtl-Meyer Expansion Waves

It can be shown that $d\theta = \sqrt{M^2 - 1}\dfrac{dv}{v}$, where $v = |\mathbf{v}|$
(valid for all gases)

Integration gives

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dv}{v}$$

the term $\dfrac{dv}{v}$ needs to be expressed in terms of Mach number

$v = Ma \Rightarrow \ln v = \ln M + \ln a \Rightarrow$

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

Prandtl-Meyer Expansion Waves

Calorically perfect gas and adiabatic flow gives

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\left\{a = \sqrt{\gamma R T},\ a_o = \sqrt{\gamma R T_o}\right\} \Rightarrow \frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 \Rightarrow$$

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

or

$$a = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1/2}$$

Prandtl-Meyer Expansion Waves

Differentiation gives:

$$da = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-3/2} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

or

$$da = a \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

which gives

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a} = \frac{dM}{M} + \frac{-\frac{1}{2}(\gamma - 1)MdM}{1 + \frac{1}{2}(\gamma - 1)M^2} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2}\frac{dM}{M}$$

Prandtl-Meyer Expansion Waves

Thus,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M} = \nu(M_2) - \nu(M_1)$$

where

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M}$$

is the so-called Prandtl-Meyer function

Prandtl-Meyer Expansion Waves

Performing the integration gives:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

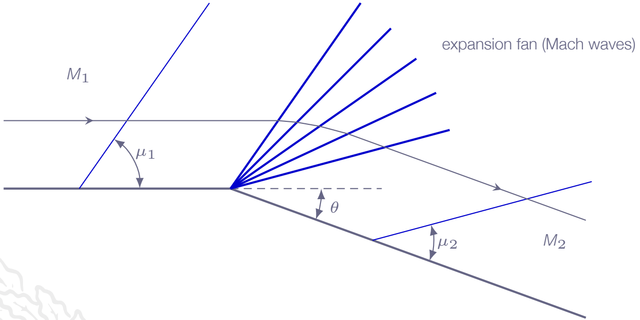
We can now calculate the deflection angle $\Delta\theta$ as:

$$\Delta\theta = \nu(M_2) - \nu(M_1)$$

$\nu(M)$ is tabulated in Table A.5 for a range of Mach numbers ($\gamma = 1.4$)

Prandtl-Meyer Expansion Waves

Example:



- $\theta_1 = 0$, $M_1 > 1$ is given
- θ_2 is given
- problem: find M_2 such that $\theta_2 = \nu(M_2) - \nu(M_1)$
- $\nu(M)$ for $\gamma = 1.4$ can be found in Table A.5

Prandtl-Meyer Expansion Waves

Since flow is isentropic, the usual isentropic relations apply:

(p_o and T_o are constant)

Calorically perfect gas:

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

Prandtl-Meyer Expansion Waves

since $p_{o_1} = p_{o_2}$ and $T_{o_1} = T_{o_2}$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right)\Big/\left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

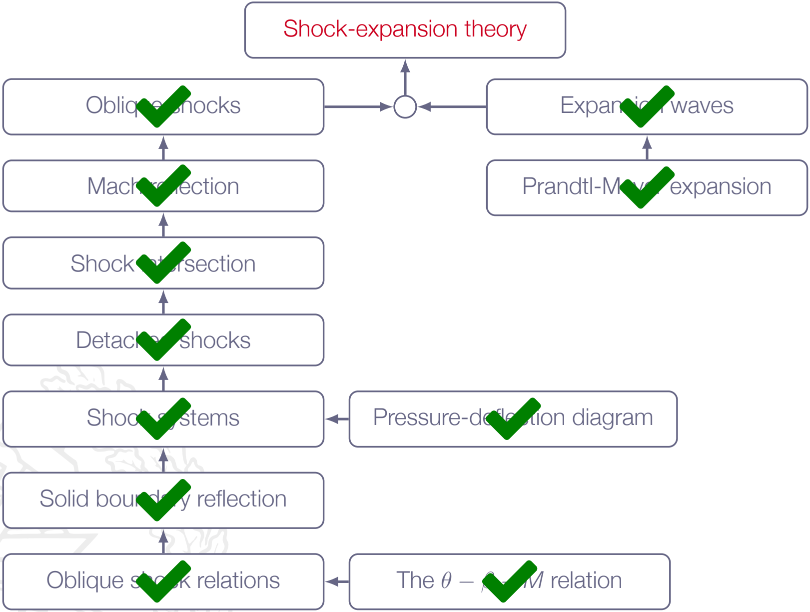
$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right)\Big/\left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

Prandtl-Meyer Expansion Waves

Alternative solution:

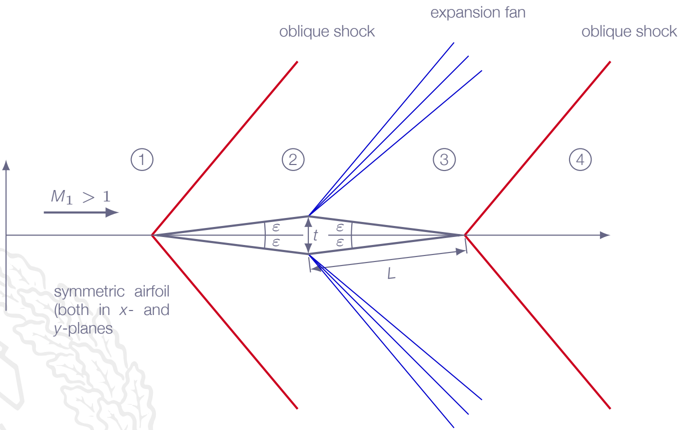
1. determine M_2 from $\theta_2 = \nu(M_2) - \nu(M_1)$
2. compute p_{o_1} and T_{o_1} from p_1, T_1, and M_1 (or use Table A.1)
3. set $p_{o_2} = p_{o_1}$ and $T_{o_2} = T_{o_1}$
4. compute p_2 and T_2 from p_{o_2}, T_{o_2}, and M_2 (or use Table A.1)

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.15
Shock Expansion Theory

Diamond-Wedge Airfoil



Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil

- symmetric airfoil
- zero incidence flow (freestream aligned with flow axis)

gives:

- symmetric flow field
- zero lift force on airfoil

Diamond-Wedge Airfoil

Drag force:

$$D = -\oiint_{\partial\Omega} p(\mathbf{n} \cdot \mathbf{e}_x)dS$$

$\partial\Omega$	airfoil surface
p	surface pressure
$\mathbf{n}$	outward facing unit normal vector
$\mathbf{e}_x$	unit vector in x-direction

Diamond-Wedge Airfoil

Since conditions 2 and 3 are constant in their respective regions, we obtain:

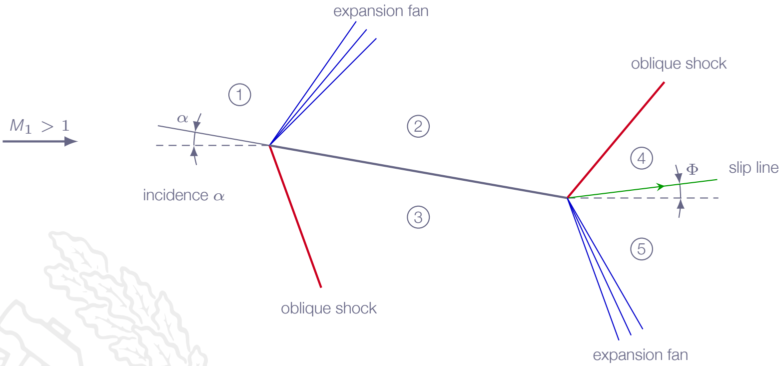
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

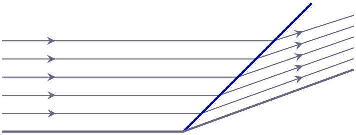
Flat-Plate Airfoil



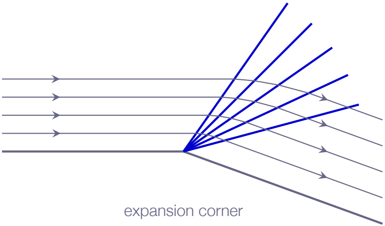
Flat-Plate Airfoil

- Flow states 4 and 5 must satisfy:
  - $p_4 = p_5$
  - flow direction 4 equals flow direction 5 (Φ)
- Shock between 2 and 4 as well as expansion fan between 3 and 5 will unjust themselves to comply with the requirements
- For calculation of lift and drag only states 2 and 3 are needed
- States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves



compression corner

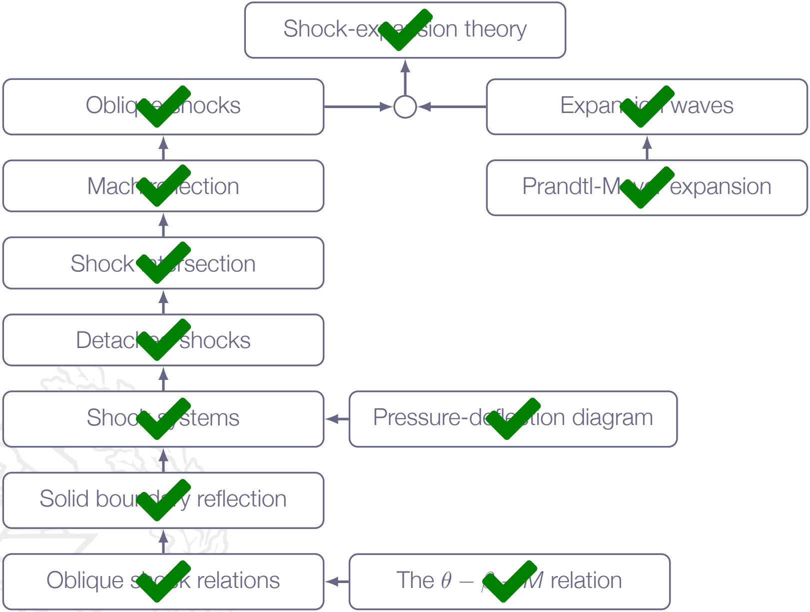


expansion corner

- M decrease
- $\|\mathbf{v}\|$ decrease
- p increase
- ρ increase
- T increase

- M increase
- $\|\mathbf{v}\|$ increase
- p decrease
- ρ decrease
- T decrease

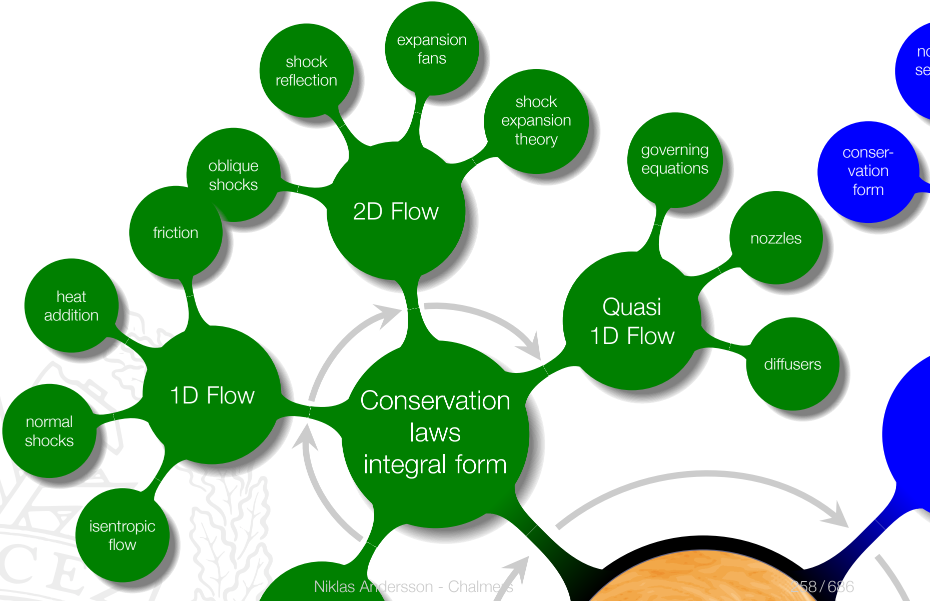
Roadmap - Oblique Shocks and Expansion Waves



LECTURE 7

Chapter 5
Quasi-One-Dimensional Flow

Overview


shock
reflection
expansion
fans
shock
expansion
theory
oblique
shocks
2D Flow
governing
equations
friction
nozzles
conser-
vation
form
heat
addition
Quasi
1D Flow
diffusers
1D Flow
Conservation
laws
integral form
normal
shocks
isentropic
flow

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

6 Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases

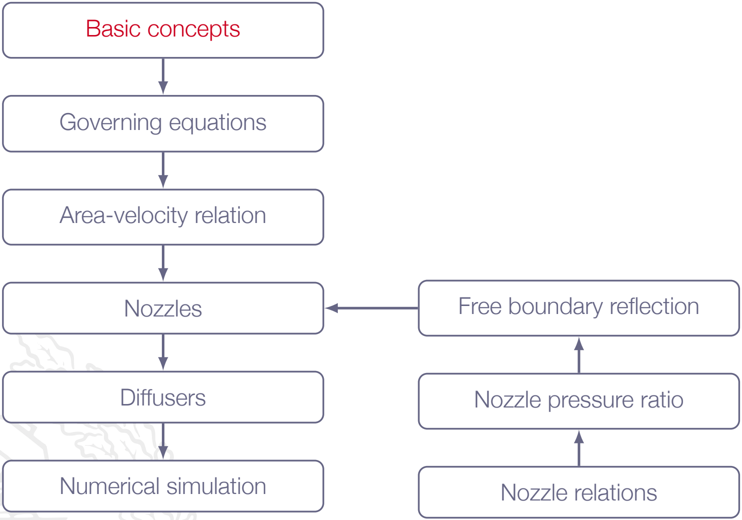
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   a 1D isentropic flow*

   i detached blunt body shocks, nozzle flows

what does quasi-1D mean? either the flow is 1D or not, or?

Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Flow

Chapter 3 - One-dimensional steady-state flow

- overall assumption:

  one-dimensional flow
  constant cross section area
- applications:

  normal shock
  one-dimensional flow with heat addition
  one-dimensional flow with friction

Chapter 4 - Two-dimensional steady-state flow

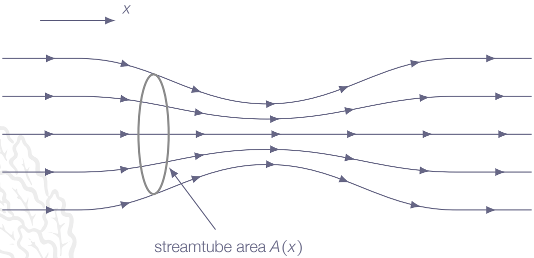
- overall assumption:

  two-dimensional flow
  uniform supersonic freestream
- applications:

  oblique shock
  expansion fan
  shock-expansion theory

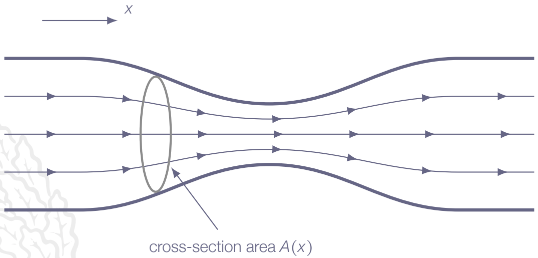
Quasi-One-Dimensional Flow

- Extension of one-dimensional flow to allow variations in streamtube area
- Steady-state flow assumption still applied

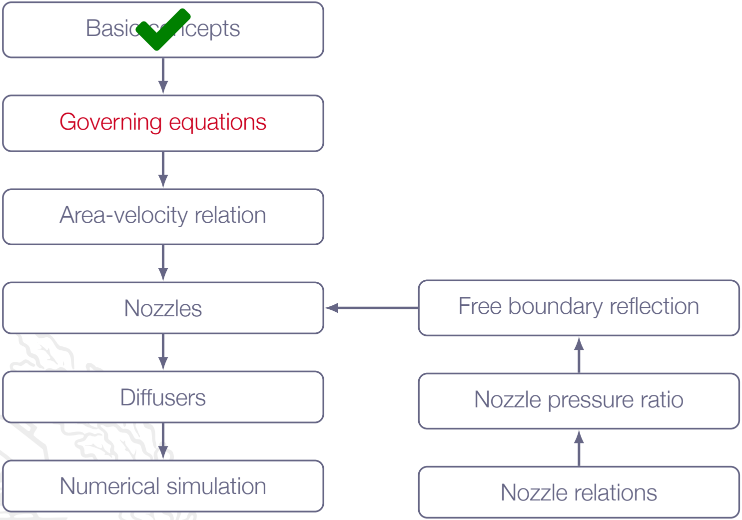


Quasi-One-Dimensional Flow

Example: tube with variable cross-section area



Roadmap - Quasi-One-Dimensional Flow

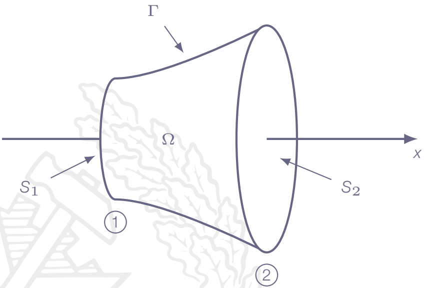


Chapter 5.2
Governing Equations

Governing Equations

Introduce cross-section-averaged flow quantities $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \ldots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Governing Equations - Mass Conservation

- steady-state
- no flow through Γ

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{-\rho_1 u_1 A_1+\rho_2 u_2 A_2} = 0$$

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

Governing Equations - Momentum Conservation

- steady-state
- no flow through Γ

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}}_{=0}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=0$$

$$\oiint\limits_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=-\rho_1u_1^2A_1+\rho_2u_2^2A_2$$

$$\oiint\limits_{\partial\Omega}p\mathbf{n}dS=-p_1A_1+p_2A_2-\int_{A_1}^{A_2}pdA$$

$$\boxed{(\rho_1u_1^2+p_1)A_1+\int_{A_1}^{A_2}pdA=(\rho_2u_2^2+p_2)A_2}$$

Governing Equations - Energy Conservation

- steady-state
- no flow through Γ

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}}_{=0}+\oiint\limits_{\partial\Omega}\left[\rho h_o(\mathbf{v}\cdot\mathbf{n})\right]dS=0$$

which gives

$$\rho_1 u_1 A_1 h_{o_1}=\rho_2 u_2 A_2 h_{o_2}$$

from continuity we have that $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow$

$$\boxed{h_{o_1}=h_{o_2}}$$

Governing Equations - Summary

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

$$h_{o_1} = h_{o_2}$$

Governing Equations - Differential Form

Continuity equation:

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

or

$$\rho u A = c$$

where c is a constant $\Rightarrow$

$$\boxed{d(\rho u A) = 0}$$

Governing Equations - Differential Form

Momentum equation:

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2 \Rightarrow$$

$$d\left[(\rho u^2 + p)A\right] = pdA \Rightarrow$$

$$d(\rho u^2 A) + d(pA) = pdA \Rightarrow$$

$$u \underbrace{d(\rho u A)}_{=0} + \rho u A du + A dp + p dA = p dA \Rightarrow$$

$$\rho u A du + A dp = 0 \Rightarrow$$

$$\boxed{dp = -\rho u du}$$

Euler's equation

Governing Equations - Differential Form

Energy equation:

$$h_{o_1} = h_{o_2} \Rightarrow$$

$$dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{dh + udu = 0}$$

Governing Equations - Differential Form

Summary (valid for all gases):

$$d(\rho u A) = 0$$

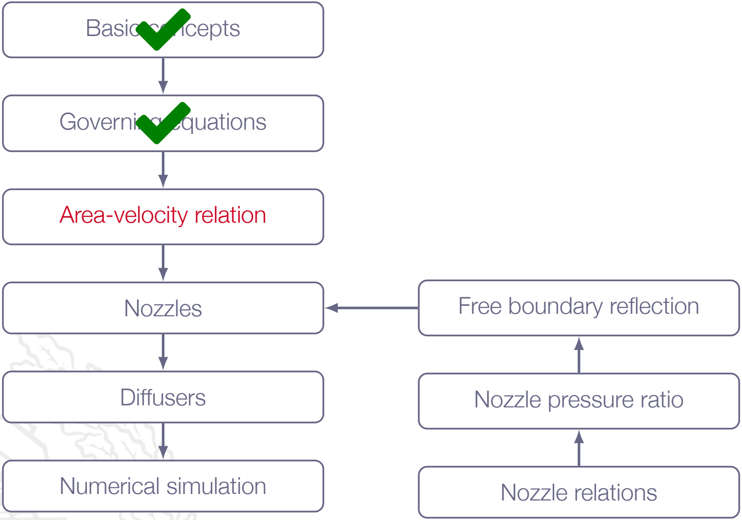
$$dp = -\rho u du$$

$$dh + u du = 0$$

Assumptions:

- quasi-one-dimensional flow
- inviscid flow
- steady-state flow

Roadmap - Quasi-One-Dimensional Flow



Chapter 5.3
Area-Velocity Relation

Area-Velocity Relation

$$d(\rho uA) = 0 \Rightarrow uAd\rho + \rho Adu + \rho udA = 0$$

divide by ρuA gives

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation:

$$dp = -\rho udu \Rightarrow \frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -udu$$

Assuming adiabatic, reversible (isentropic) process and the definition of speed of sound gives

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \Rightarrow a^2\frac{d\rho}{\rho} = -udu \Rightarrow \frac{d\rho}{\rho} = -M^2\frac{du}{u}$$

Area-Velocity Relation

Now, inserting the expression for $\frac{d\rho}{\rho}$ in the rewritten continuity equation gives

$$(1-M^2)\frac{du}{u}+\frac{dA}{A}=0$$

or

$$\frac{dA}{A}=(M^2-1)\frac{du}{u}$$

which is the area-velocity relation

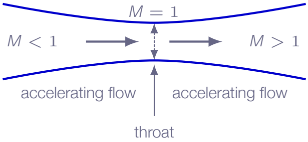
Area-Velocity Relation

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

$M < 1$: decreasing A correlated with increasing u

$M > 1$: increasing A correlated with increasing u

$M = 1$: $dA = 0$

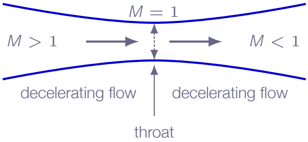


converging-diverging nozzle only possibility to obtain supersonic flow!

Area-Velocity Relation

Alternative:

Slowing down from supersonic to subsonic flow (supersonic diffuser)



in practice:
difficult to obtain completely shock-free flow in this case

Area-Velocity Relation

$$M \to 0 \Rightarrow \frac{dA}{A} = -\frac{du}{u}$$

$$\frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow$$

$$\frac{1}{Au}\left[udA + Adu\right] = 0 \Rightarrow$$

$$d(uA) = 0 \Rightarrow Au = c$$

where c is a constant

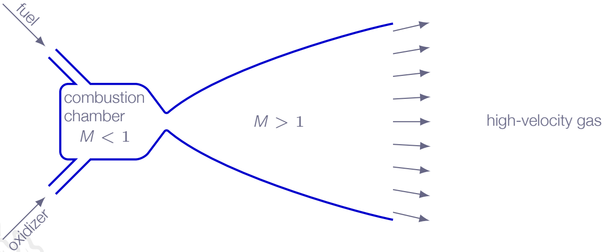
Area-Velocity Relation

Note 1: The area-velocity relation is only valid for isentropic flow

- not valid across a compression shock
  (due to entropy increase)

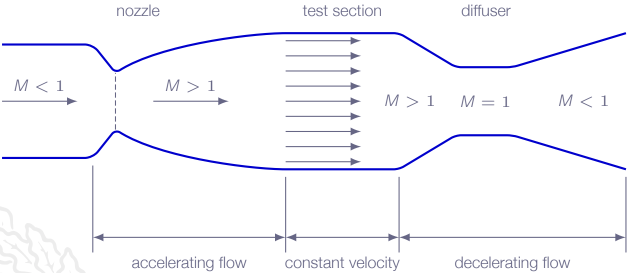
Note 2: The area-velocity relation is valid for all gases

Area-Velocity Relation Examples - Rocket Engine

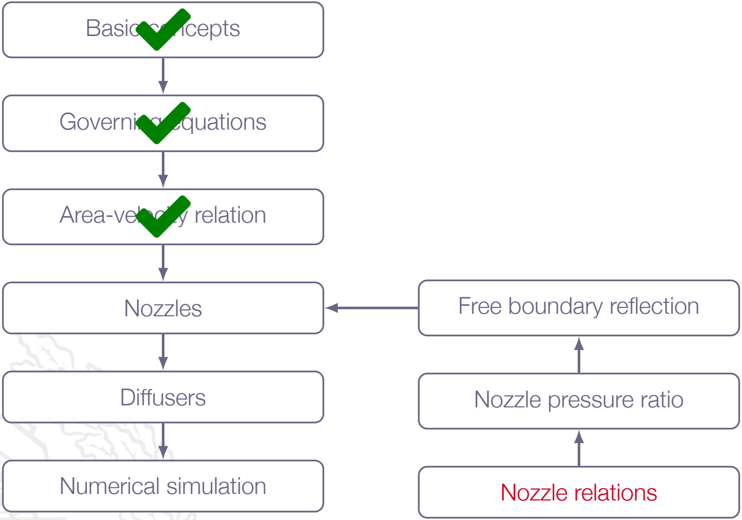


High-temperature, high-pressure gas in combustion chamber expand through the nozzle to very high velocities. Typical figures for a LH^2/LOx rocket engine: $p_o \sim 120$ [bar], $T_o \sim 3600$ [K], exit velocity ~ 4000 [m/s]

Area-Velocity Relation Examples - Wind Tunnel



Roadmap - Quasi-One-Dimensional Flow



Chapter 5.4
Nozzles

Nozzle Flow - Relations

Calorically perfect gas assumed:

From Chapter 3:

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

Critical conditions:

$$\frac{T_o}{T^*} = \left(\frac{a_o}{a^*}\right)^2 = \frac{1}{2}(\gamma + 1)$$

$$\frac{p_o}{p^*} = \left(\frac{T_o}{T^*}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

$$M^{*^2} = \frac{u^2}{a^{*^2}} = \frac{u^2}{a^2}\frac{a^2}{a^{*^2}} = \frac{u^2}{a^2}\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*^2}} \Rightarrow$$

$$M^{*^2} = M^2\frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}$$

Nozzle Flow - Relations

For nozzle flow we have

$$\rho u A = c$$

where c is a constant and therefore

$$\rho^* u^* A^* = \rho u A$$

or, since at critical conditions $u^* = a^*$

$$\rho^* a^* A^* = \rho u A$$

which gives

$$\frac{A}{A*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

Nozzle Flow - Relations

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

$$\left.\begin{aligned}
\frac{\rho_o}{\rho} &= \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}} \\
\frac{\rho^*}{\rho_o} &= \left(\frac{T_o}{T^*}\right)^{\frac{-1}{\gamma-1}} \\
\frac{a^*}{u} &= \frac{1}{M^*}
\end{aligned}\right\} \Rightarrow \frac{A}{A^*} = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{1}{\gamma-1}}M^*}$$

Nozzle Flow - Relations

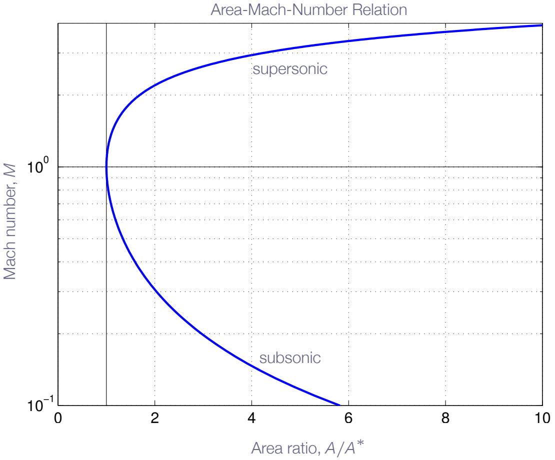
$$\left.\begin{aligned} \left(\frac{A}{A^*}\right)^2 &= \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{2}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{2}{\gamma-1}} {M^*}^2} \\ {M^*}^2 &= M^2\frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2} \end{aligned}\right\} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$

which is the area-Mach-number relation

Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$



Area-Mach-Number Relation

Note 1: Critical conditions used here are those corresponding to isentropic flow. Do not confuse these with the conditions in the cases of one-dimensional flow with heat addition and friction

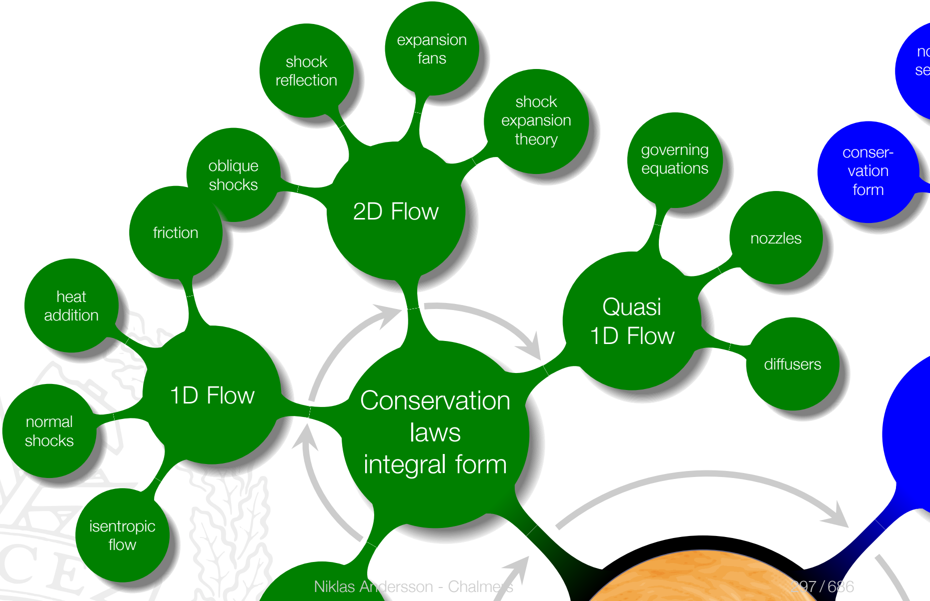
Note 2: For quasi-one-dimensional flow, assuming inviscid steady-state flow, both total and critical conditions are constant along streamlines unless shocks are present (then the flow is no longer isentropic)

Note 3: The derived area-Mach-number relation is only valid for calorically perfect gas and for isentropic flow. It is not valid across a compression shock

LECTURE 8

Chapter 5 Quasi-One-Dimensional Flow

Overview



shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
conser-vation form
heat addition
1D Flow
Quasi 1D Flow
diffusers
normal shocks
Conservation laws integral form
isentropic flow

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

7 Explain why entropy is important for flow discontinuities

8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   a 1D isentropic flow*
   b normal shocks*
   i detached blunt body shocks, nozzle flows

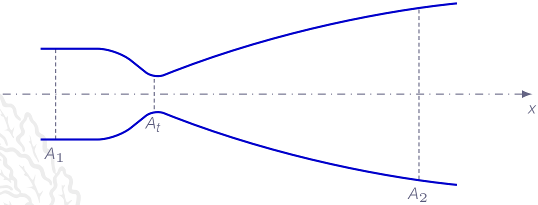
12 Explain the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions

time for rocket science!

Nozzle Flow

Assumptions:

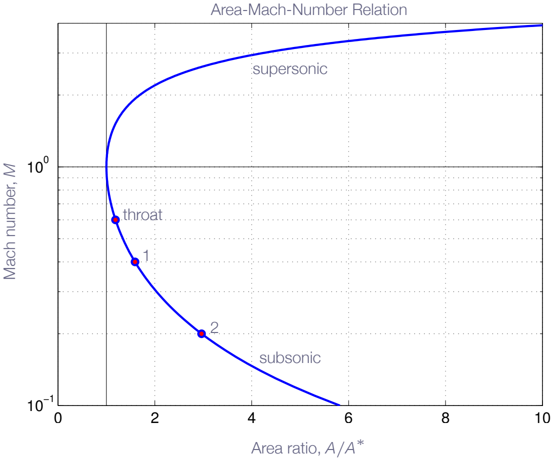
- inviscid
- steady-state
- quasi-one-dimensional
- calorically perfect gas



Nozzle Flow

Alt. 1: sub-critical (non-choked) nozzle flow

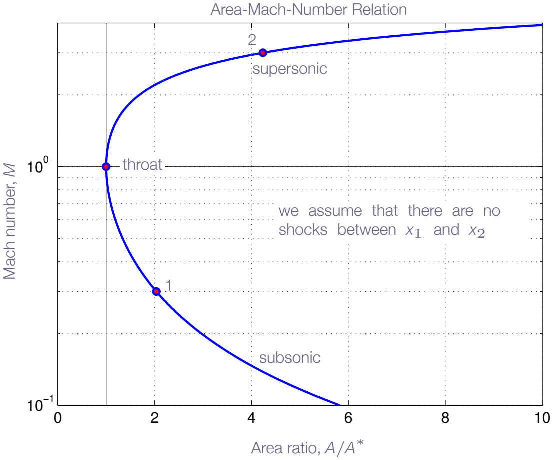
- $M < 1$ at nozzle throat
- $A_t > A^*$
- $M_1 < 1$
- $M_2 < 1$



Nozzle Flow

Alt. 2: critical (choked) nozzle flow

- $M = 1$ at nozzle throat
- $A_t = A^*$
- $M_1 < 1$
- $M_2 > 1$



Nozzle Flow

Choked nozzle flow (no shocks):

- A^* is constant throughout the nozzle
- $A_t = A^*$

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \left(\frac{A_1}{A_t}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}\left(1+\frac{1}{2}(\gamma-1)M_1^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M_2 given by the supersonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \left(\frac{A_2}{A_t}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}\left(1+\frac{1}{2}(\gamma-1)M_2^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M is uniquely determined everywhere in the nozzle, with subsonic flow upstream of throat and supersonic flow downstream of throat

Nozzle Mass Flow

For a choked nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho^* u^* A^* = \rho_2 u_2 A_2$$

$$\left.\begin{aligned} \rho^* &= \frac{\rho^*}{\rho_o}\rho_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \frac{p_o}{RT_o} \\ a^* &= \frac{a^*}{a_o}a_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \sqrt{\gamma R T_o} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}}}$$

- The maximum mass flow that can be sustained through the nozzle
- Valid for quasi-one-dimensional, inviscid, steady-state flow and calorically perfect gas

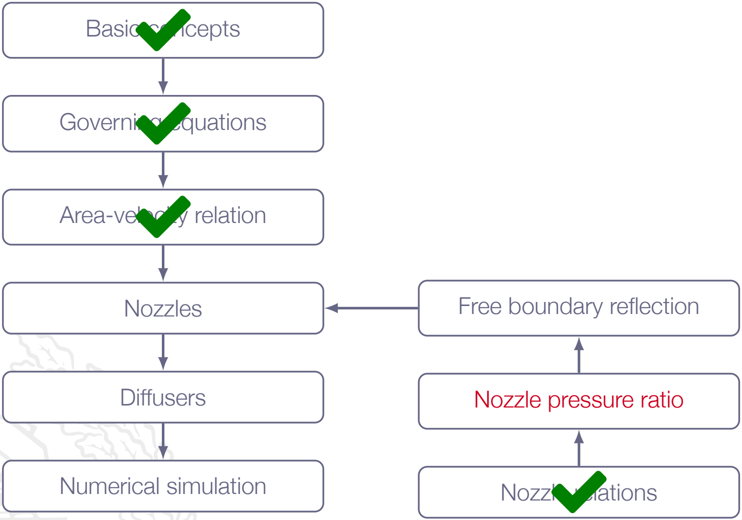
  Note: The massflow formula is valid even if there are shocks present downstream of throat!

Nozzle Mass Flow

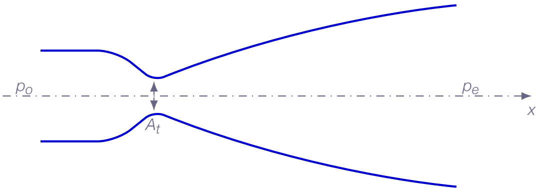
How can we increase mass flow through nozzle?

- increase p_o
- decrease T_o
- increase A_t
- decrease R
  (increase molecular weight, without changing γ)

Roadmap - Quasi-One-Dimensional Flow

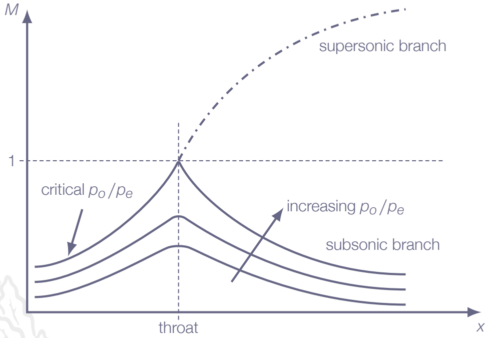


Nozzle Flow with Varying Pressure Ratio



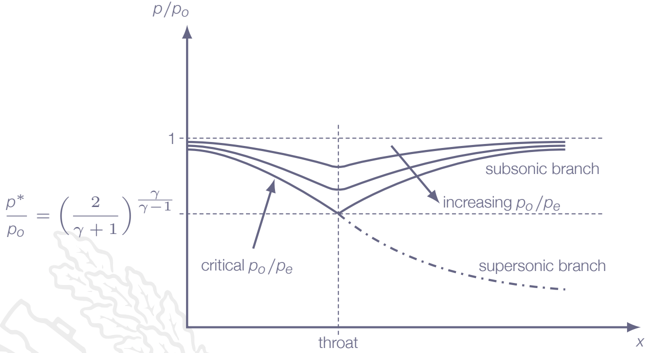
$A(x)$	area function
A_t	$\min\{A(x)\}$
p_o	inlet total pressure
p_e	outlet static pressure (ambient pressure)
p_o/p_e	pressure ratio

Nozzle Flow with Varying Pressure Ratio



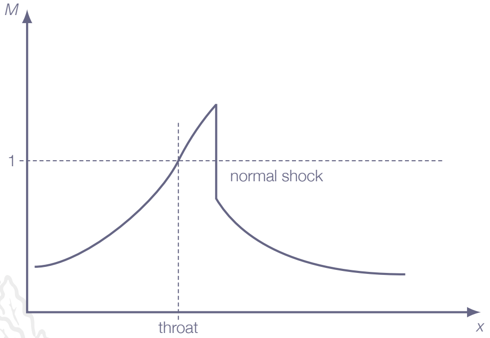
For critical p_o/p_e, a jump to supersonic solution will occur

Nozzle Flow with Varying Pressure Ratio

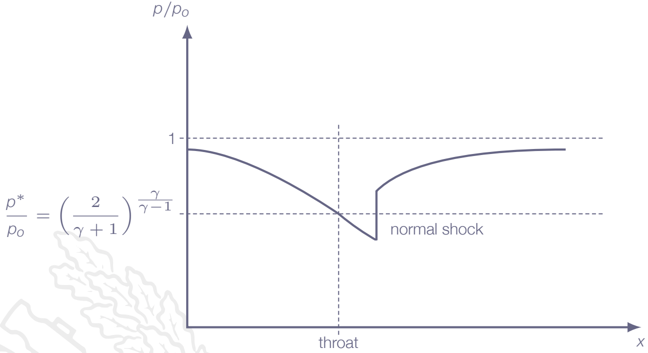


As the flow jumps to the supersonic branch downstream of the throat, a normal shock will appear in order to match the ambient pressure at the nozzle exit

Nozzle Flow with Varying Pressure Ratio



Nozzle Flow with Varying Pressure Ratio



Nozzle Flow with Varying Pressure Ratio (Summary)

$(p_o/p_e) < (p_o/p_e)_{cr}$

- the flow remains entirely subsonic
- the mass flow depends on p_e, *i.e.* the flow is not choked
- no shock is formed, therefore the flow is isentropic throughout the nozzle

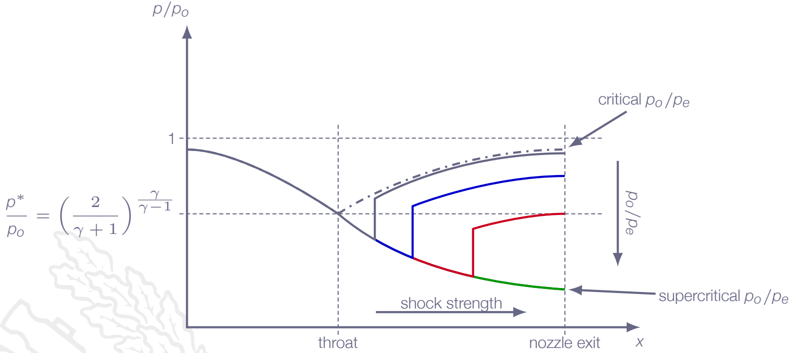
$(p_o/p_e) = (p_o/p_e)_{cr}$

- the flow just achieves $M = 1$ at the throat
- the flow will then suddenly flip to the supersonic solution downstream of the throat, for an infinitesimally small increase in (p_o/p_e)

$(p_o/p_e) > (p_o/p_e)_{cr}$

- the flow is choked (fixed mass flow), *i.e.* it does not depend on p_e
- a normal shock will appear downstream of the throat, with strength and position depending on (p_o/p_e)

Nozzle Flow with Varying Pressure Ratio



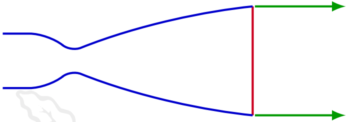
Nozzle Flow with Varying Pressure Ratio

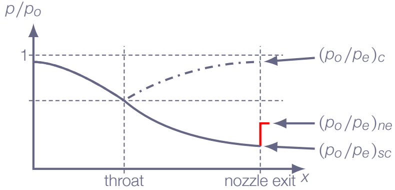
Effects of changing the pressure ratio (p_o/p_e) (*where p_e is the back pressure and p_o is the total pressure at the nozzle inlet*)

- critical value: $p_o/p_e = (p_o/p_e)_c$
  - nozzle flow reaches $M = 1$ at throat, flow becomes choked
- supercritical value: $p_o/p_e = (p_o/p_e)_{sc}$
  - nozzle flow is supersonic from throat to exit, without any interior normal shock - isentropic flow
- normal shock at exit: $(p_o/p_e) = (p_o/p_e)_{ne} < (p_o/p_e)_{sc}$
  - normal shock is still present but is located just at exit - isentropic flow inside nozzle

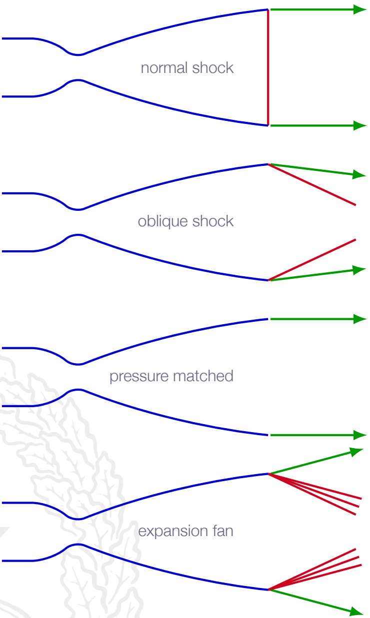
Nozzle Flow with Varying Pressure Ratio

Normal shock at exit





Nozzle Flow with Varying Pressure Ratio



$p_o/p_e = (p_o/p_e)_{ne}$
normal shock at nozzle exit

$(p_o/p_e)_{ne} < p_o/p_e < (p_o/p_e)_{sc}$
overexpanded nozzle flow

$p_o/p_e = (p_o/p_e)_{sc}$
pressure matched nozzle flow

$p_o/p_e > (p_o/p_e)_{sc}$
underexpanded nozzle flow

Nozzle Flow with Varying Pressure Ratio

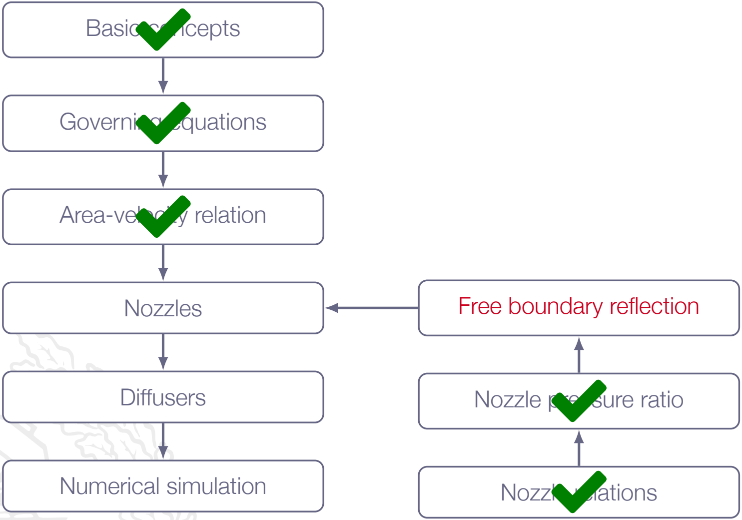
Quasi-one-dimensional theory

- When the interior normal shock is "pushed out" through the exit (by increasing (p_o/p_e), *i.e.* lowering the back pressure), it disappears completely.
- The flow through the nozzle is then shock free (and thus also isentropic since we neglect viscosity).

Three-dimensional nozzle flow

- When the interior normal shock is "pushed out" through the exit (by increasing (p_o/p_e)), an oblique shock is formed outside of the nozzle exit.
- For the exact supercritical value of (p_o/p_e) this oblique shock disappears.
- For (p_o/p_e) above the supercritical value an expansion fan is formed at the nozzle exit.

Roadmap - Quasi-One-Dimensional Flow

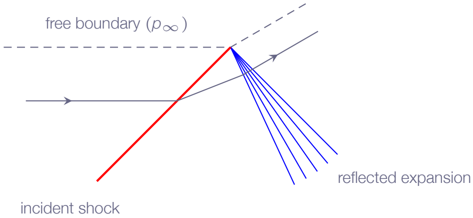


Chapter 5.6
Wave Reflection From a Free Boundary

Free-Boundary Reflection

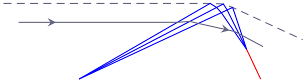
Free boundary - shear layer, interface between different fluids, etc

Free-Boundary Reflection - Shock Reflection



- No jump in pressure at the free boundary possible
- Incident shock reflects as expansion waves at the free boundary
- Reflection results in net turning of the flow

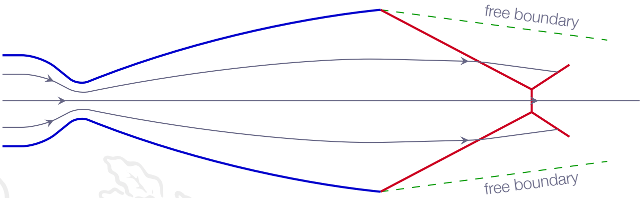
Free-Boundary Reflection - Expansion Wave Reflection



- No jump in pressure at the free boundary possible
- Incident expansion waves reflects as compression waves at the free boundary
- Finite compression waves coalesces into a shock
- Reflection results in net turning of the flow

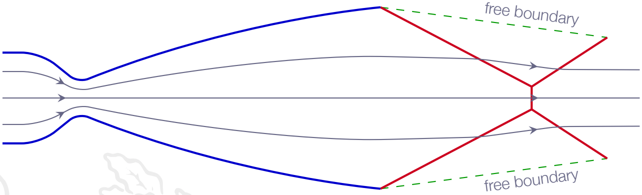
Free-Boundary Reflection - System of Reflections

overexpanded nozzle flow



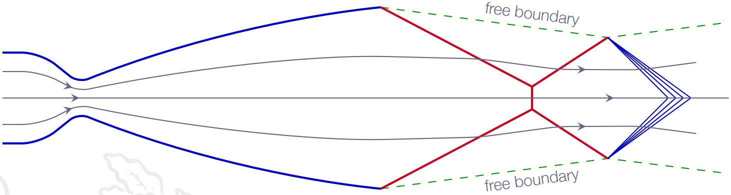
Free-Boundary Reflection - System of Reflections

shock reflection at jet centerline



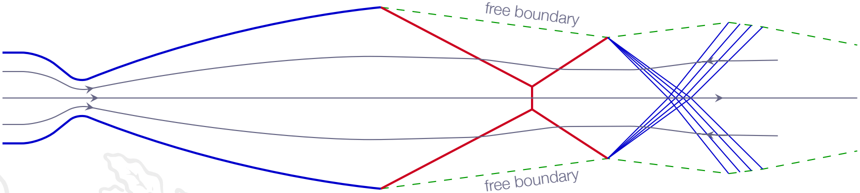
Free-Boundary Reflection - System of Reflections

shock reflection at free boundary



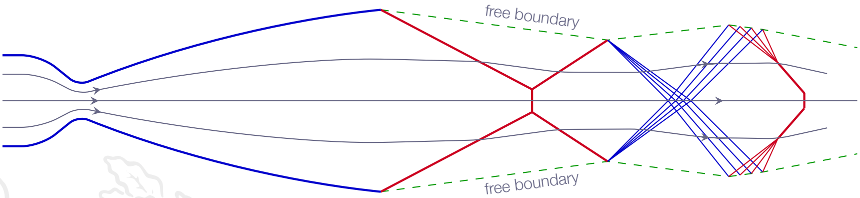
Free-Boundary Reflection - System of Reflections

expansion wave reflection at jet centerline



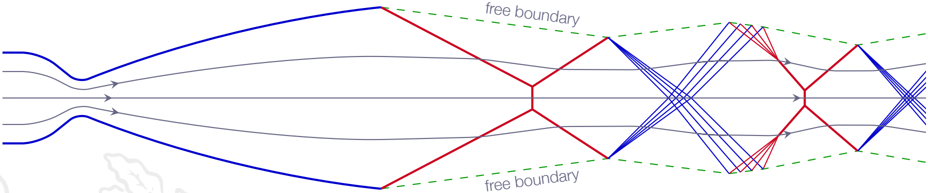
Free-Boundary Reflection - System of Reflections

expansion wave reflection at free boundary



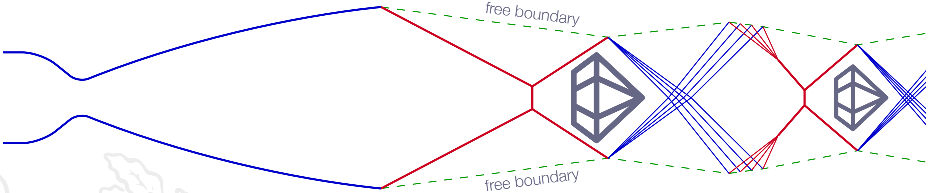
Free-Boundary Reflection - System of Reflections

repeated shock/expansion system



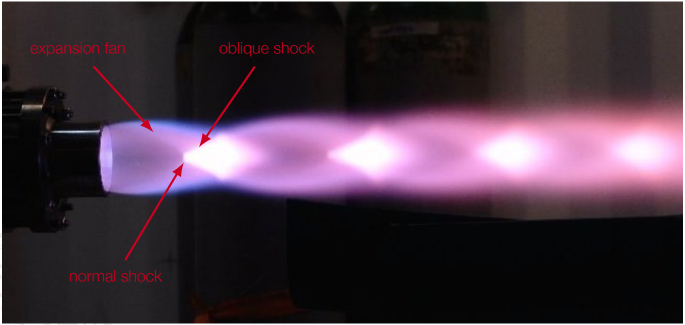
Free-Boundary Reflection - System of Reflections

shock diamonds



Free-Boundary Reflection - System of Reflections

underexpanded jet



Free-Boundary Reflection - Summary

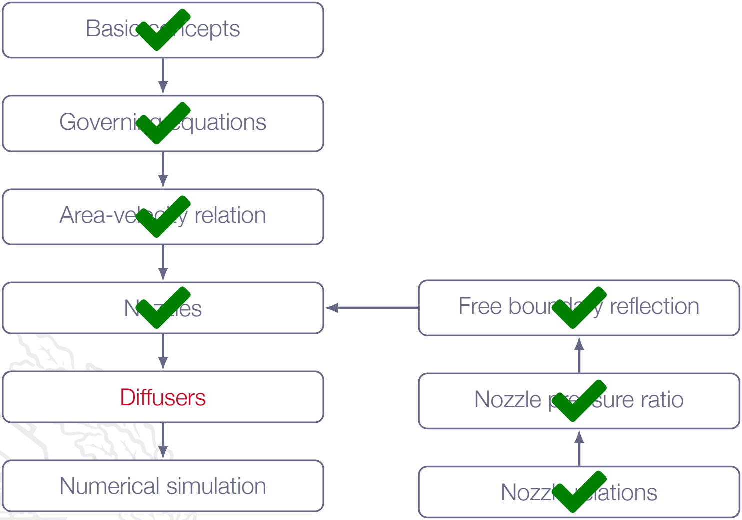
Solid-wall reflection

Compression waves reflects as compression waves
Expansion waves reflects as expansion waves

Free-boundary reflection

Compression waves reflects as expansion waves
Expansion waves reflects as compression waves

Roadmap - Quasi-One-Dimensional Flow

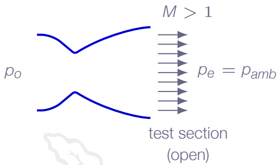


Chapter 5.5
Diffusers

Supersonic Wind Tunnel

wind tunnel with supersonic test section

open test section



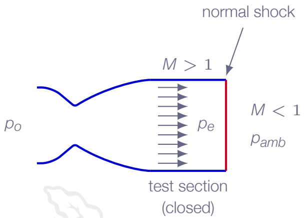
$p_o/p_e = (p_o/p_e)_{sc}$

$M = 3.0$ in test section $\Rightarrow p_o/p_e = 36.7$!!!

Supersonic Wind Tunnel

wind tunnel with supersonic test section

enclosed test section, normal shock at exit



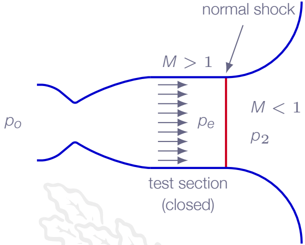
$p_o/p_{amb} = (p_o/p_e)(p_e/p_{amb}) < (p_o/p_e)_{sc}$

$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33 = 3.55$

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add subsonic diffuser after normal shock



$$\left(p_{o_2} = p_{amb}\right)$$

$p_o/p_{amb} = (p_o/p_e)(p_e/p_2)(p_2/p_{o_2})$

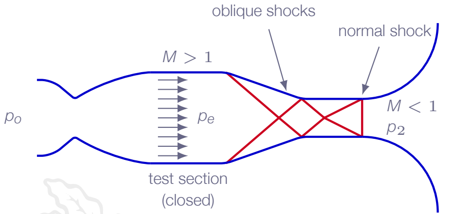
$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33/1.17 = 3.04$

Note: this corresponds exactly to total pressure loss across normal shock

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add supersonic diffuser before normal shock



$\left(p_{o_2} = p_{amb}\right)$

well-designed supersonic + subsonic diffuser ⇒

1. decreased total pressure loss
2. decreased p_o and power to drive wind tunnel

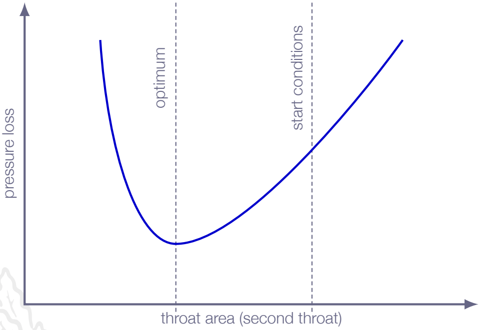
Supersonic Wind Tunnel

Main problems:

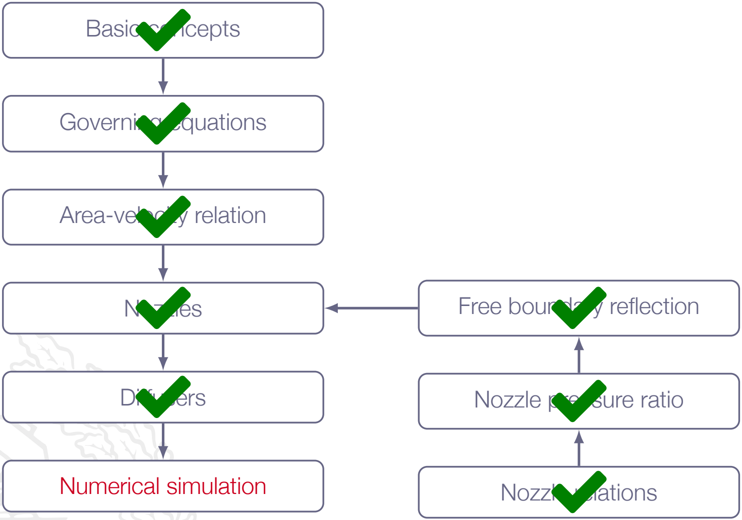
1. Design is extremely difficult due to complex 3D flow in diffuser
   - viscous effects
   - oblique shocks
   - separations
2. Starting requirements: second throat must be significantly larger than first throat

   solution:
   - variable geometry diffuser
   - second throat larger during startup procedure
   - decreased second throat to optimum value after flow is established

Supersonic Wind Tunnel



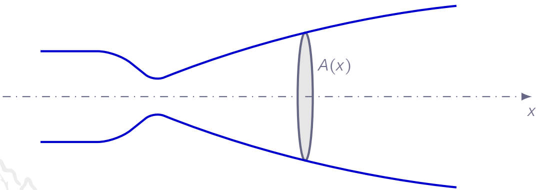
Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Euler Equations

Quasi-One-Dimensional Euler Equations

Example: choked flow through a convergent-divergent nozzle



Assumptions: inviscid, $Q = Q(x, t)$

Quasi-One-Dimensional Euler Equations

$$A(x)\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}\left[A(x)E\right] = A'(x)H$$

where $A(x)$ is the cross section area and

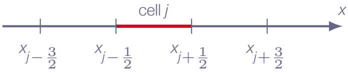
$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho e_o \end{bmatrix}, \; E(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho h_o u \end{bmatrix}, \; H(Q) = \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}$$

Numerical Approach

- Finite-Volume Method
- Method of lines, three-stage Runge-Kutta time stepping
- 3^{rd}-order characteristic upwinding scheme
- Subsonic inflow boundary condition at min(x)
  - T_o, p_o given
- Subsonic outflow boundary condition at max(x)
  - p given

Finite-Volume Spatial Discretization

$$\left(\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}\right)$$



Integration over cell j gives:

$$\frac{1}{2}\left[A(x_{j-\frac{1}{2}}) + A(x_{j+\frac{1}{2}})\right]\Delta x_j \frac{d}{dt}\bar{Q}_j + \left[A(x_{j+\frac{1}{2}})\hat{E}_{j+\frac{1}{2}} - A(x_{j-\frac{1}{2}})\hat{E}_{j-\frac{1}{2}}\right] = \left[A(x_{j+\frac{1}{2}}) - A(x_{j-\frac{1}{2}})\right]\hat{H}_j$$

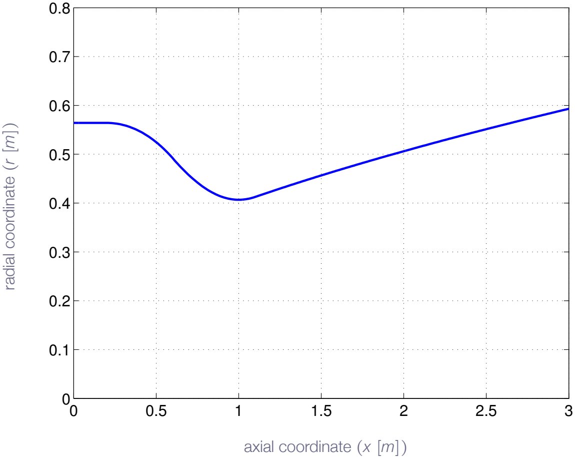
Finite-Volume Spatial Discretization

$$\bar{Q}_j = \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} QA(x)dx\right) \Big/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A(x)dx\right)$$

$$\hat{E}_{j+\frac{1}{2}} \approx E\left(Q\left(x_{j+\frac{1}{2}}\right)\right)$$

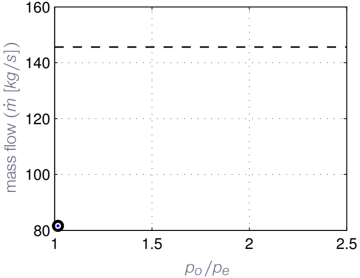
$$\hat{H}_j \approx \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} HA'(x)dx\right) \Big/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A'(x)dx\right)$$

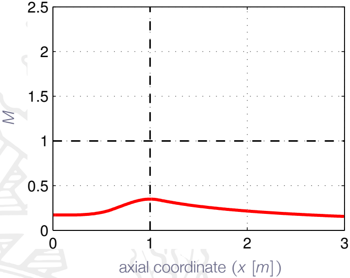
Nozzle Simulation - Back Pressure Sweep

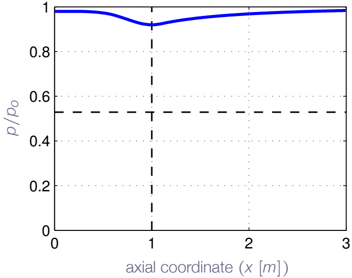


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_e	1.18	[bar]
p_o/p_e	1.02	
$\dot{m}$	81.61	[kg/s]
M_{max}	0.35	

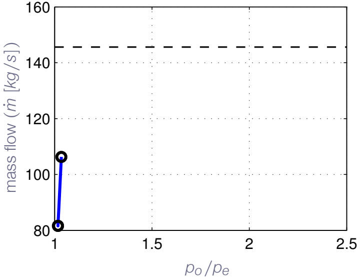


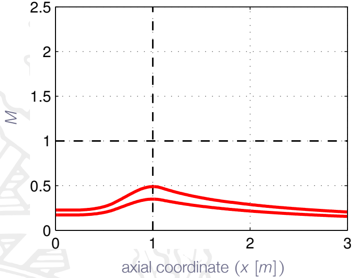


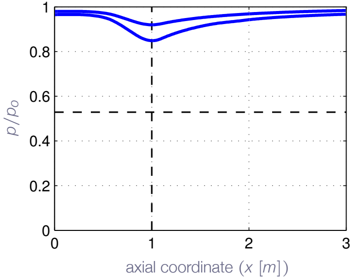


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.16 [bar]
p_o/p_e	1.03
$\dot{m}$	106.27 [kg/s]
M_{max}	0.49

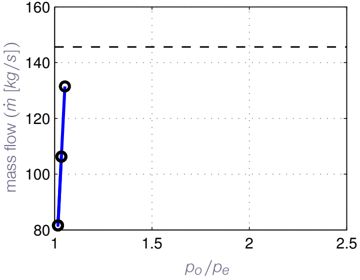


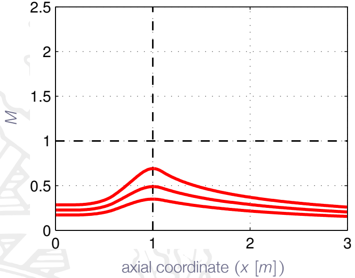


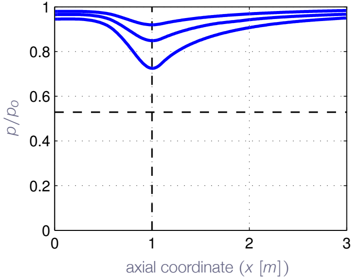


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_e	1.14	[bar]
p_o/p_e	1.05	
$\dot{m}$	131.45	[kg/s]
M_{max}	0.69	

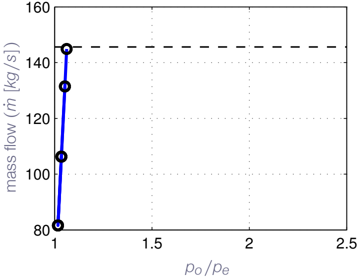


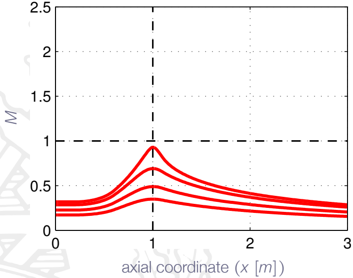


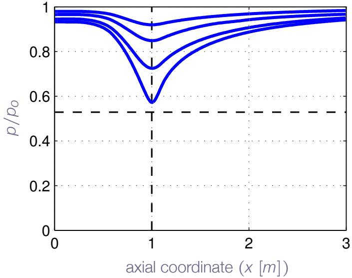


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.13 [bar]
p_o/p_e	1.06
$\dot{m}$	144.88 [kg/s]
M_{max}	0.93

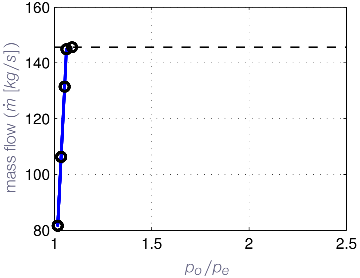


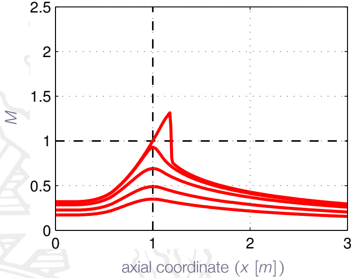


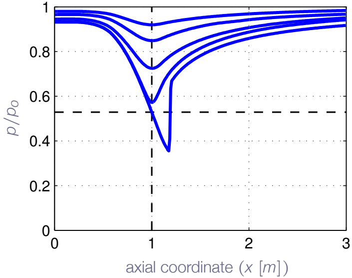


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_e	1.10	[bar]
p_o/p_e	1.09	
$\dot{m}$	145.62	[kg/s]
M_{max}	1.31	

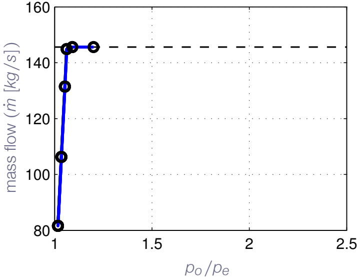


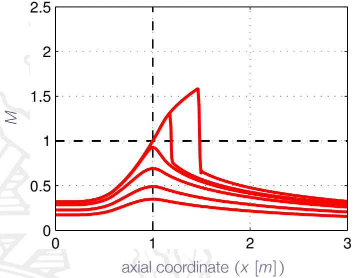


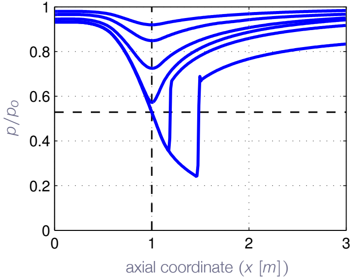


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.00 [bar]
p_o/p_e	1.20
$\dot{m}$	145.6 [kg/s]
M_{max}	1.58

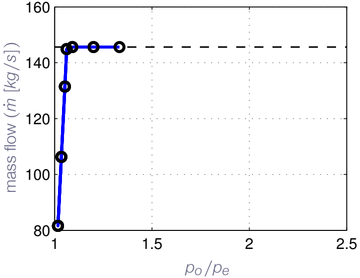


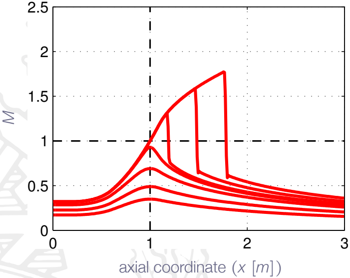


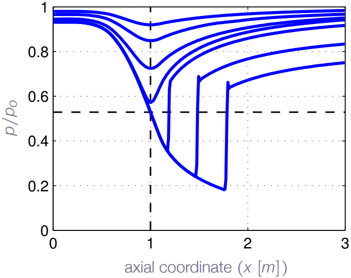


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.90 [bar]
p_o/p_e	1.33
$\dot{m}$	145.6 [kg/s]
M_{max}	1.77

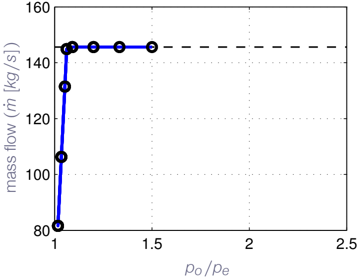


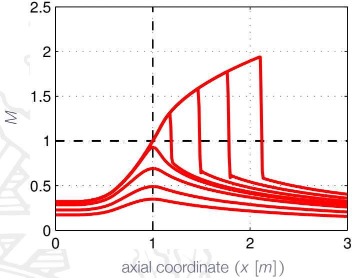


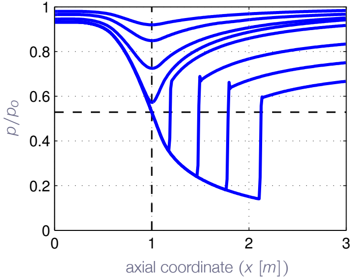


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_e	0.80	[bar]
p_o/p_e	1.50	
$\dot{m}$	145.6	[kg/s]
M_{max}	1.94	

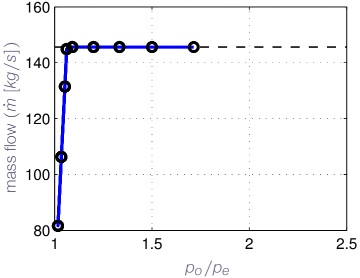


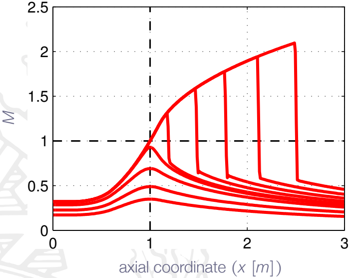


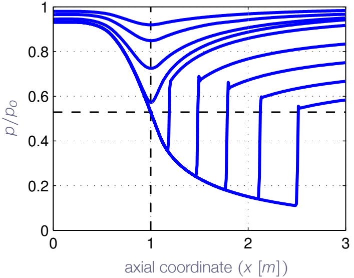


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.70 [bar]
p_o/p_e	1.71
$\dot{m}$	145.6 [kg/s]
M_{max}	2.10

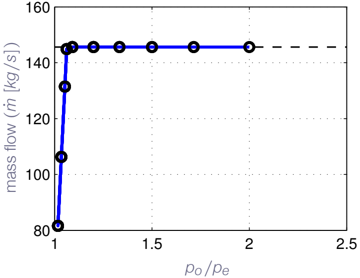


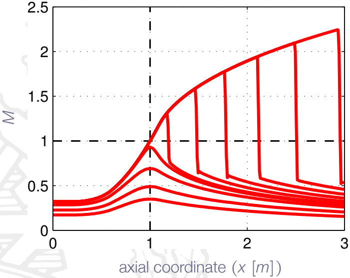


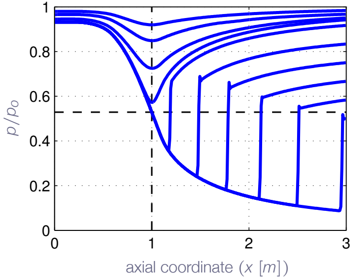


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.60 [bar]
p_o/p_e	2.00
$\dot{m}$	145.6 [kg/s]
M_{max}	2.24

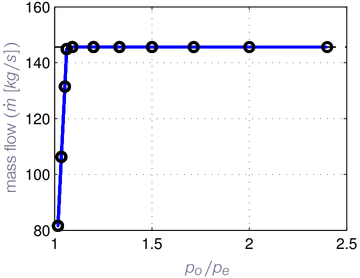


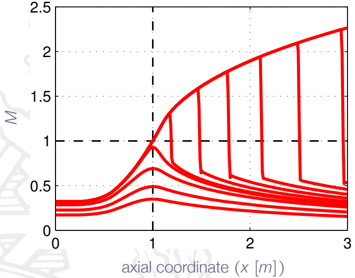


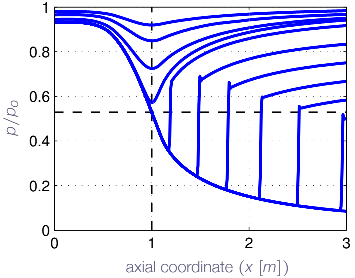


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.50 [bar]
p_o/p_e	2.40
$\dot{m}$	145.6 [kg/s]
M_{max}	2.26

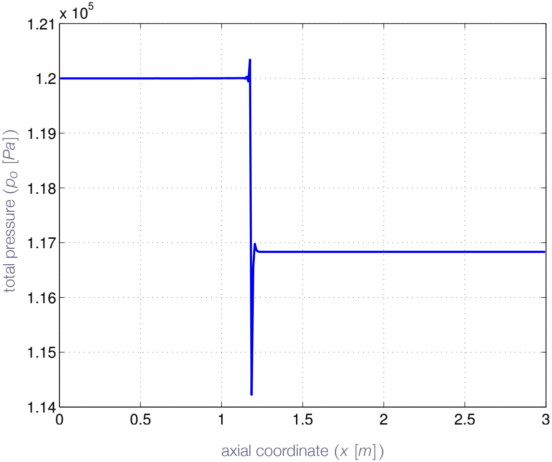






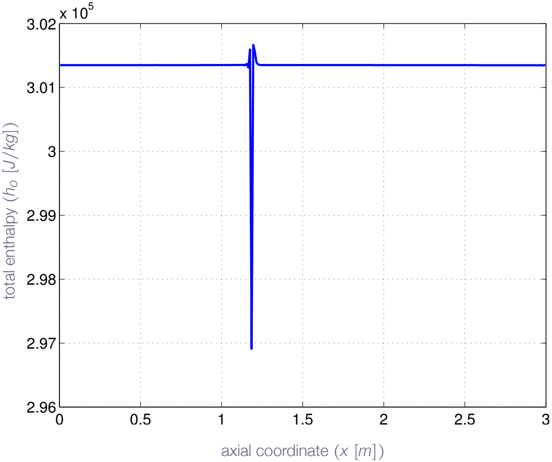
Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.10 [bar]
p_o/p_e	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

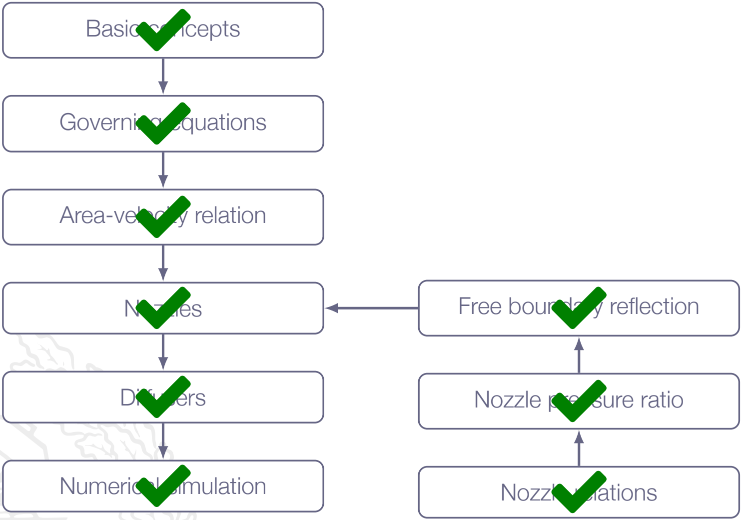


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.10 [bar]
p_o/p_e	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31



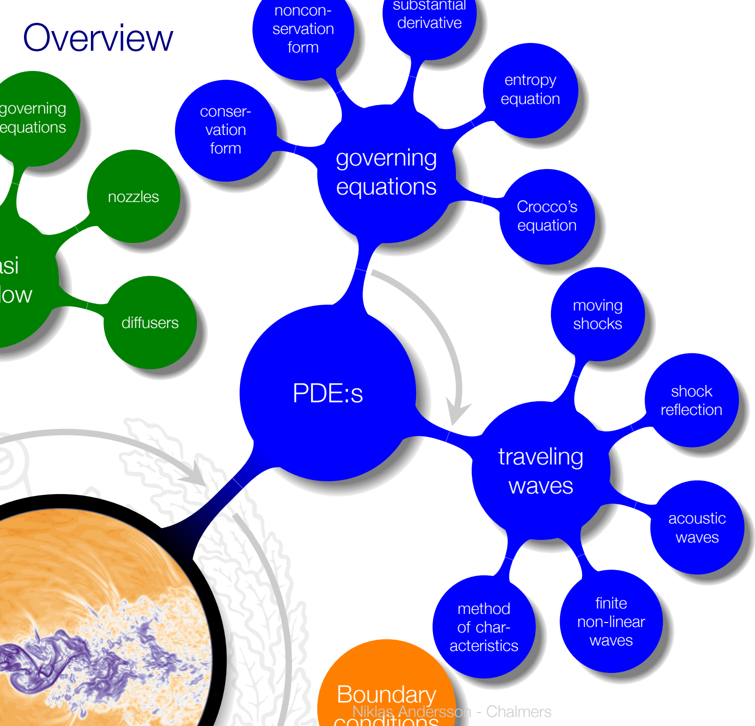
Roadmap - Quasi-One-Dimensional Flow



LECTURE 9

Chapter 7
Unsteady Wave Motion

Overview



noncon-
servation
form
substantial
derivative
entropy
equation
conser-
vation
form
governing
equations
Crocco's
equation
governing
equations
nozzles
diffusers
PDE:s
moving
shocks
shock
reflection
traveling
waves
acoustic
waves
method
of char-
acteristics
finite
non-linear
waves
Boundary
conditions

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

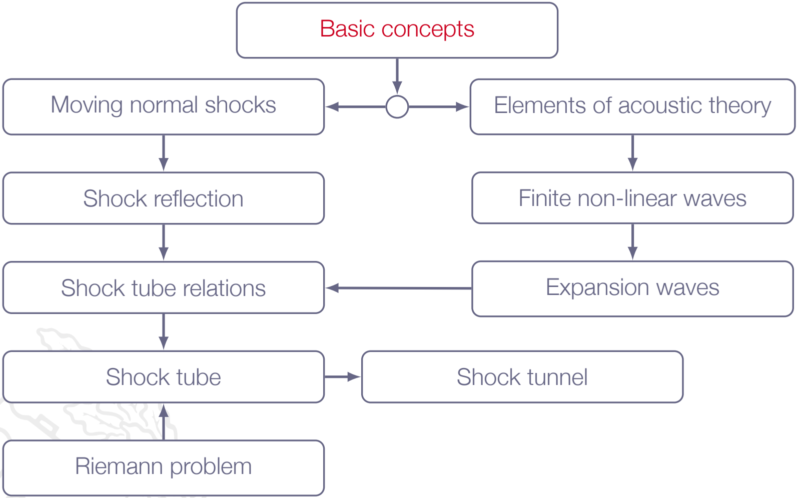
  a 1D isentropic flow*

  b normal shocks*

  j unsteady waves and discontinuities in 1D

moving normal shocks - frame of reference seems to be the key here?!

Roadmap - Unsteady Wave Motion



Unsteady Wave Motion - Example #1

Object moving with supersonic speed through the air

observer moving with the bullet

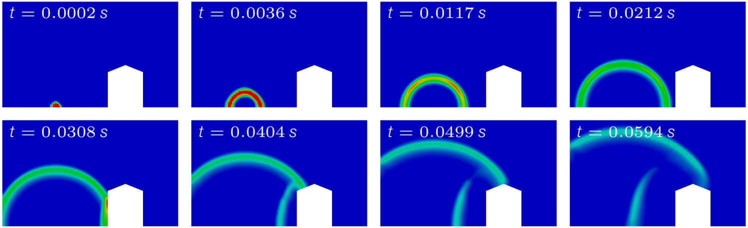
- steady-state flow
- the detached shock wave is stationary

observer at rest

- unsteady flow
- detached shock wave moves through the air (to the left)

Unsteady Wave Motion - Example #2

Shock wave from explosion



- normal shock moving spherically outwards
- Shock strength decreases with radius
- Shock speed decreases with radius

Unsteady Wave Motion

inertial frames!

Physical laws are the same for both frame of references

Shock characteristics are the same for both observers (shape, strength, etc)

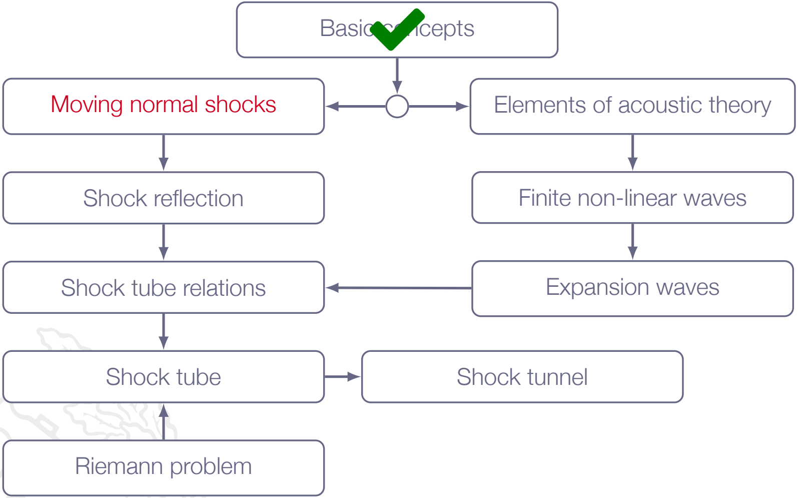
Unsteady Wave Motion

Is there a connection with stationary shock waves?

Answer: Yes!

Locally, in a moving frame of reference, the shock may be viewed as a stationary normal shock

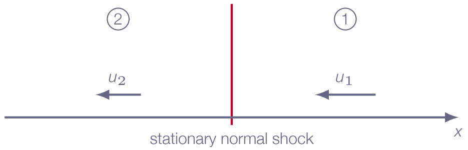
Roadmap - Unsteady Wave Motion



Chapter 7.2 Moving Normal Shock Waves

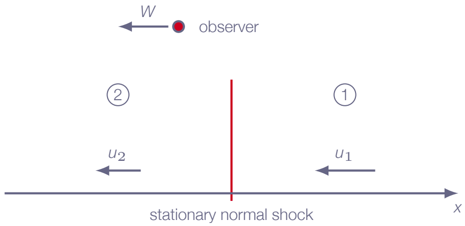
Moving Normal Shock Waves

Chapter 3: stationary normal shock



$u_1 > a_1$ (supersonic flow)
$u_2 < a_2$ (subsonic flow)
$p_2 > p_1$ (sudden compression)
$s_2 > s_1$ (shock loss)

Moving Normal Shock Waves



- Introduce observer moving to the left with speed W
  - if W is constant the observer is still in an inertial system
  - all physical laws are unchanged

- The observer sees a normal shock moving to the right with speed W
  - gas velocity ahead of shock: $u_1' = W - u_1$
  - gas velocity behind shock: $u_2' = W - u_2$

Moving Normal Shock Waves

Now, let $W = u_1 \Rightarrow$

$$u_1' = 0$$

$$u_2' = u_1 - u_2 > 0$$

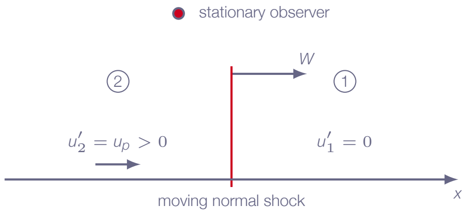
The observer now sees the shock traveling to the right with speed $W = u_1$ into a stagnant gas, leaving a compressed gas ($p_2 > p_1$) with velocity $u_2' > 0$ behind it

Introducing u_p:

$$u_p = u_2' = u_1 - u_2$$

Moving Normal Shock Waves

Analogy:



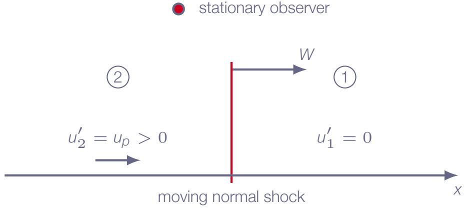
Case 1

- stationary normal shock
- observer moving with velocity W

Case 2

- normal shock moving with velocity W
- stationary observer

Moving Normal Shock Waves - Governing Equations



For stationary normal shocks we have:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

With $(u_1 = W)$ and $(u_2 = W - u_p)$ we get:

$$\rho_1 W = \rho_2 (W - u_p)$$
$$\rho_1 W^2 + p_1 = \rho_2 (W - u_p)^2 + p_2$$
$$h_1 + \frac{1}{2}W^2 = h_2 + \frac{1}{2}(W - u_p)^2$$

Moving Normal Shock Waves - Relations

Starting from the governing equations

$$\rho_1 W = \rho_2 (W - u_p)$$
$$\rho_1 W^2 + p_1 = \rho_2 (W - u_p)^2 + p_2$$
$$h_1 + \frac{1}{2} W^2 = h_2 + \frac{1}{2} (W - u_p)^2$$

and using $h = e + \dfrac{p}{\rho}$

it is possible to show that

$$e_2 - e_1 = \frac{p_1 + p_2}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

Moving Normal Shock Waves - Relations

$$e_2 - e_1 = \frac{p_1 + p_2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

same Hugoniot equation as for stationary normal shock

This means that we will have same shock strength, *i.e.* same jumps in density, velocity, pressure, etc

Moving Normal Shock Waves - Relations

Starting from the Hugoniot equation one can show that

$$\frac{\rho_2}{\rho_1} = \frac{1 + \dfrac{\gamma+1}{\gamma-1}\left(\dfrac{p_2}{p_1}\right)}{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}$$

and

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\left[\frac{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}{1 + \dfrac{\gamma+1}{\gamma-1}\left(\dfrac{p_2}{p_1}\right)}\right]$$

Moving Normal Shock Waves - Relations

For calorically perfect gas and stationary normal shock:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_s^2 - 1)$$

same as eq. (3.57) in Anderson with $M_1 = M_s$

where

$$M_s = \frac{W}{a_1}$$

- M_s is simply the speed of the shock (W), traveling into the stagnant gas, normalized by the speed of sound in this stagnant gas (a_1)
  - $M_s > 1$, otherwise there is no shock!
  - shocks always moves faster than sound - no warning before it hits you ☺

Moving Normal Shock Waves - Relations

Re-arrange:

$$M_s = \sqrt{\frac{\gamma + 1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

(speed of shock directly linked to pressure ratio)

$$M_s = \frac{W}{a_1} \Rightarrow$$

$$W = a_1 M_s = a_1\sqrt{\frac{\gamma + 1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

Moving Normal Shock Waves - Relations

From the continuity equation we get:

$$u_p = W\left(1 - \frac{\rho_1}{\rho_2}\right) > 0$$

After some derivation we obtain:

$$u_p = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma}{\gamma+1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma-1}{\gamma+1}}\right]^{1/2}$$

Moving Normal Shock Waves - Relations

Induced Mach number:

$$M_p = \frac{u_p}{a_2} = \frac{u_p}{a_1}\frac{a_1}{a_2} = \frac{u_p}{a_1}\sqrt{\frac{T_1}{T_2}}$$

inserting u_p/a_1 and T_1/T_2 from relations on previous slides we get:

$$M_p = \frac{1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} + \frac{p_2}{p_1}}\right]^{1/2}\left[\frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right) + \left(\frac{p_2}{p_1}\right)^2}\right]^{1/2}$$

Moving Normal Shock Waves - Relations

Note that

$$\lim_{\frac{p_2}{p_1}\to\infty} M_p \to \sqrt{\frac{2}{\gamma(\gamma-1)}}$$

for air ($\gamma = 1.4$)

$$\lim_{\frac{p_2}{p_1}\to\infty} M_p \to 1.89$$

Moving Normal Shock Waves - Relations

Moving normal shock with $p_2/p_1 = 10$

($p_1 = 10\ bar$, $T_1 = 300\ K$, $\gamma = 1.4$)

$\Rightarrow M_s = 2.95$ and $W = 1024.2\ m/s$

The shock is advancing with almost three times the speed of sound!

Behind the shock the induced velocity is $u_p = 756.2\ m/s \Rightarrow$ supersonic flow ($a_2 = 562.1\ m/s$)

May be calculated by formulas 7.13, 7.16, 7.10, 7.11 or by using Table A.2 for stationary normal shock ($u_1 = W$, $u_2 = W - u_p$)

Moving Normal Shock Waves - Relations

Note that $h_{o_1} \neq h_{o_2}$

constant total enthalpy is only valid for stationary shocks!

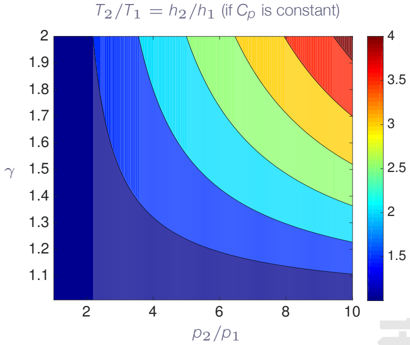
shock is uniquely defined by pressure ratio p_2/p_1

$$u_1 = 0$$

$$h_{o_1} = h_1 + \frac{1}{2}u_1^2 = h_1$$

$$h_{o_2} = h_2 + \frac{1}{2}u_2^2$$

$$h_2 > h_1 \Rightarrow h_{o_2} > h_{o_1}$$



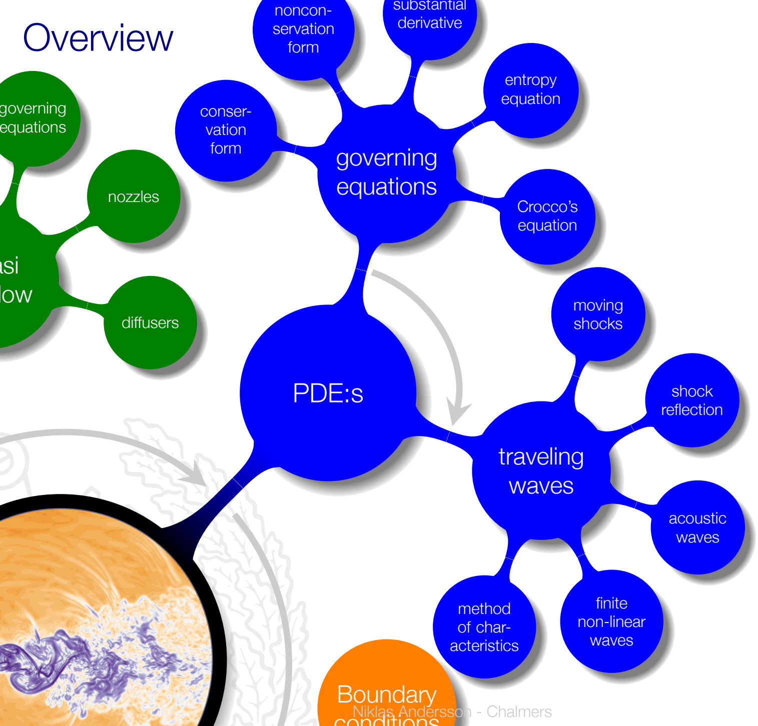
Moving Normal Shock Waves - Relations

Gas/Vapor	**Ratio of specific heats** (γ)	**Gas constant** R
Acetylene	1.23	319
Air (standard)	1.40	287
Ammonia	1.31	530
Argon	1.67	208
Benzene	1.12	100
Butane	1.09	143
Carbon Dioxide	1.29	189
Carbon Disulphide	1.21	120
Carbon Monoxide	1.40	297
Chlorine	1.34	120
Ethane	1.19	276
Ethylene	1.24	296
Helium	1.67	2080
Hydrogen	1.41	4120
Hydrogen chloride	1.41	230
Methane	1.30	518
Natural Gas (Methane)	1.27	500
Nitric oxide	1.39	277
Nitrogen	1.40	297
Nitrous oxide	1.27	180
Oxygen	1.40	260
Propane	1.13	189
Steam (water)	1.32	462
Sulphur dioxide	1.29	130

LECTURE 10

Chapter 7
Unsteady Wave Motion

Overview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PDE:s
moving
shocks
shock
reflection
traveling
waves
acoustic
waves
method
of char-
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finite
non-linear
waves
Boundary
conditions

Addressed Learning Outcomes

3 Describe typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)

8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

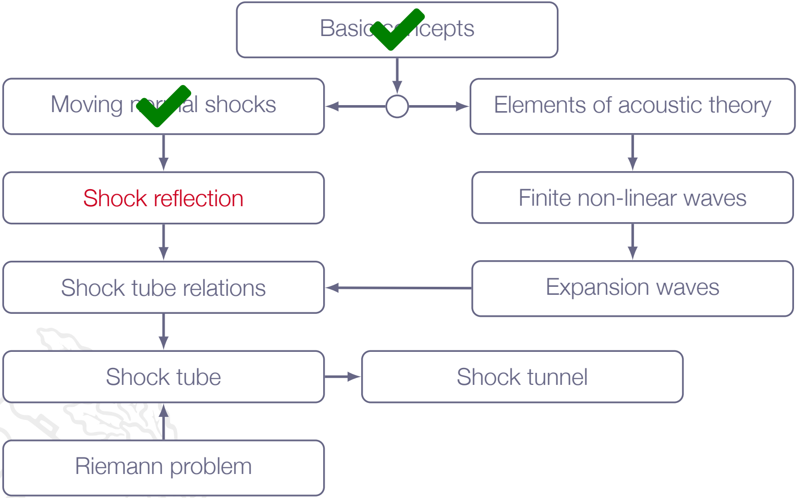
 j unsteady waves and discontinuities in 1D

9 Solve engineering problems involving the above-mentioned phenomena (8a-8k)

12 Explain the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions

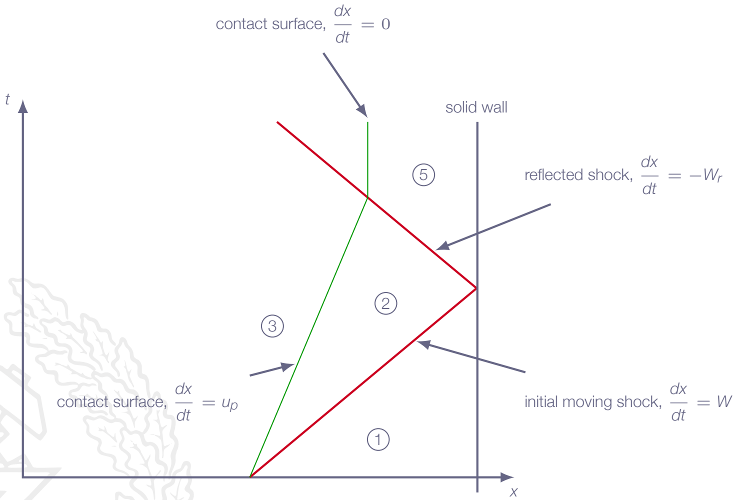
what happens when a moving shock approaches a wall?

Roadmap - Unsteady Wave Motion



Chapter 7.3
Reflected Shock Wave

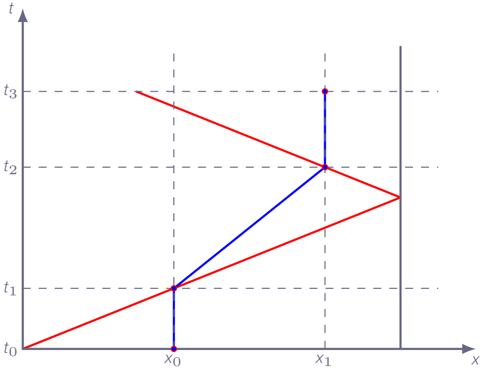
Shock Reflection



Shock Reflection - Particle Path

A fluid particle located at x_0 at time t_0 (a location ahead of the shock) will be affected by the moving shock and follow the blue path

time	location	velocity
t_0	x_0	0
t_1	x_0	u_p
t_2	x_1	u_p
t_3	x_1	0



Shock Reflection Relations

- velocity ahead of reflected shock: $W_r + u_p$
- velocity behind reflected shock: W_r

Continuity:

$$\rho_2(W_r + u_p) = \rho_5 W_r$$

Momentum:

$$p_2 + \rho_2(W_r + u_p)^2 = p_5 + \rho_5 W_r^2$$

Energy:

$$h_2 + \frac{1}{2}(W_r + u_p)^2 = h_5 + \frac{1}{2}W_r^2$$

Shock Reflection Relations

Reflected shock is determined such that $u_5 = 0$

$$\frac{M_r}{M_r^2 - 1} = \frac{M_s}{M_s^2 - 1}\sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}(M_s^2 - 1)\left(\gamma + \frac{1}{M_s^2}\right)}$$

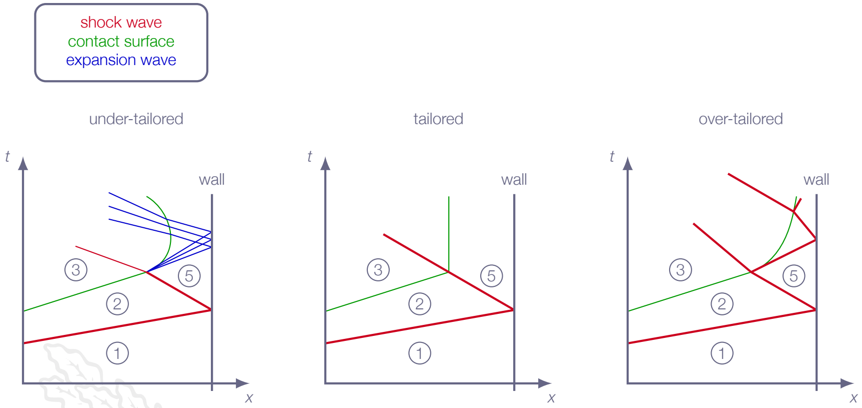
where

$$M_r = \frac{W_r + u_p}{a_2}$$

Tailored v.s. Non-Tailored Shock Reflection

- The time duration of condition 5 is determined by what happens after interaction between reflected shock and contact discontinuity
- For special choice of initial conditions (tailored case), this interaction is negligible, thus prolonging the duration of condition 5

Tailored v.s. Non-Tailored Shock Reflection



Under-tailored conditions:
Mach number of incident wave lower than in tailored conditions

Over-tailored conditions:
Mach number of incident wave higher than in tailored conditions

Shock Reflection - Example

Shock reflection in shock tube ($\gamma = 1.4$)
(Example 7.1 in Anderson)

Incident shock (given data)

p_2/p_1	10.0
M_s	2.95
T_2/T_1	2.623
p_1	1.0 [bar]
T_1	300.0 [K]

Calculated data

M_r 2.09

Table A.2

p_5/p_2	4.978
T_5/T_2	1.77

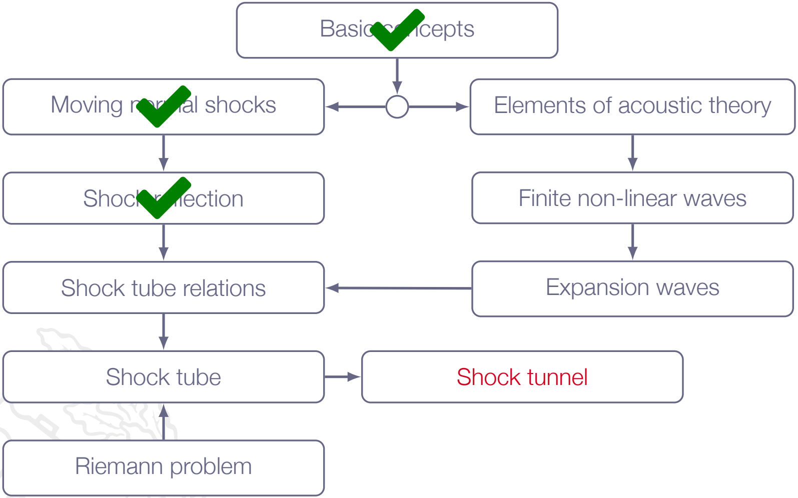
$$p_5 = \left(\frac{p_5}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 = 49.78$$

$$T_5 = \left(\frac{T_5}{T_2}\right)\left(\frac{T_2}{T_1}\right)T_1 = 1393$$

Shock Reflection - Shock Tube

- Very high pressure and temperature conditions in a specified location with very high precision (p_5, T_5)
  - measurements of thermodynamic properties of various gases at extreme conditions, *e.g.* dissociation energies, molecular relaxation times, etc.
  - measurements of chemical reaction properties of various gas mixtures at extreme conditions

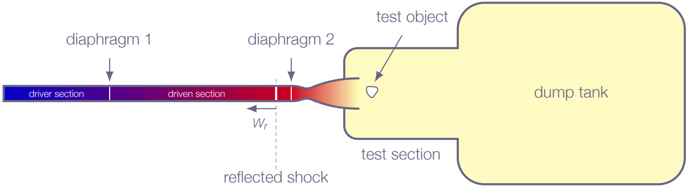
Roadmap - Unsteady Wave Motion



Shock Tunnel

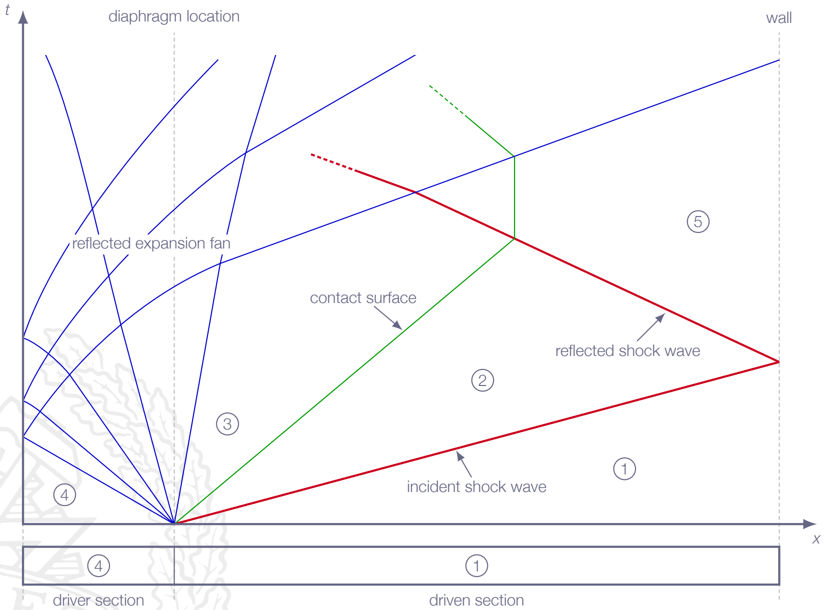
- Addition of a convergent-divergent nozzle to a shock tube configuration
- Capable of producing flow conditions which are close to those during the reentry of a space vehicles into the earth's atmosphere
  - high-enthalpy, hypersonic flows (short time)
  - real gas effects
- Example - Aachen TH2:
  - velocities up to 4 km/s
  - stagnation temperatures of several thousand degrees

Shock Tunnel



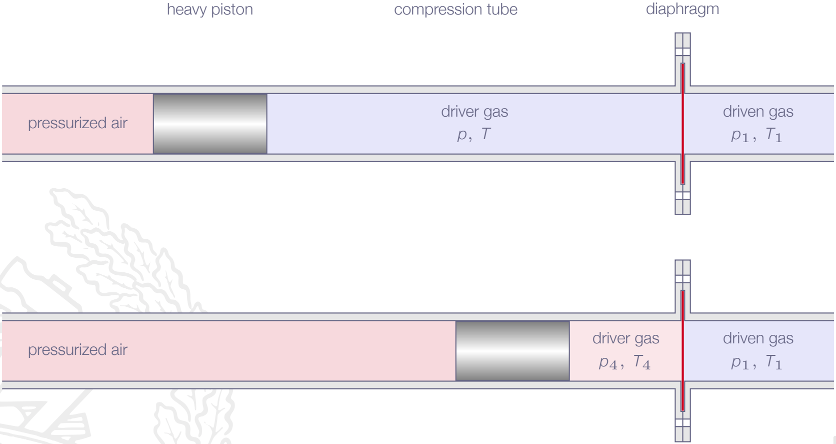
1. High pressure in region 4 (driver section)
   - diaphragm 1 burst
   - primary shock generated
2. Primary shock reaches end of shock tube
   - shock reflection
3. High pressure in region 5
   - diaphragm 2 burst
   - nozzle flow initiated
   - hypersonic flow in test section

Shock Tunnel

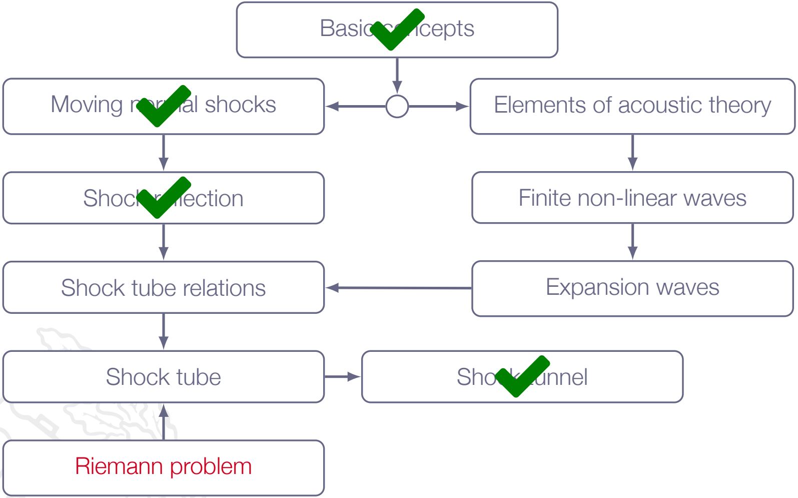


Shock Tunnel

By adding a compression tube to the shock tube a very high p_4 and T_4 may be achieved for any gas in a fairly simple manner



Roadmap - Unsteady Wave Motion



Riemann Problem

The shock tube problem is a special case of the general Riemann Problem

> *"... A Riemann problem, named after Bernhard Riemann, consists of an initial value problem composed by a conservation equation together with piecewise constant data having a single discontinuity ..."*

Wikipedia

Riemann Problem

May show that solutions to the shock tube problem have the general form:

$$
\begin{aligned}
p &= p(x/t) \\
\rho &= \rho(x/t) \\
u &= u(x/t) \\
T &= T(x/t) \\
a &= a(x/t)
\end{aligned}
$$

where $x = 0$ denotes the position of the initial jump between states 1 and 4

Riemann Problem - Shock Tube

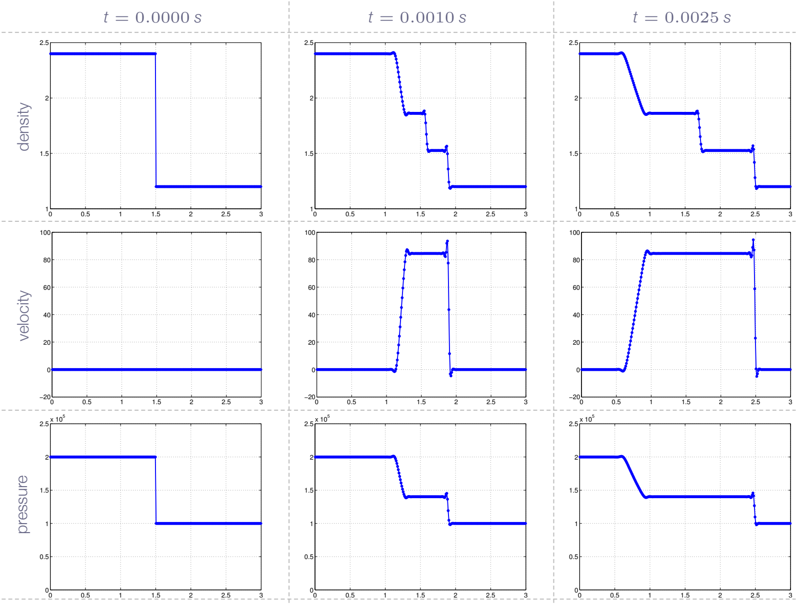
Shock tube simulation:

- left side conditions (state 4):
  - $\rho = 2.4\ kg/m^3$
  - $u = 0.0\ m/s$
  - $p = 2.0\ bar$

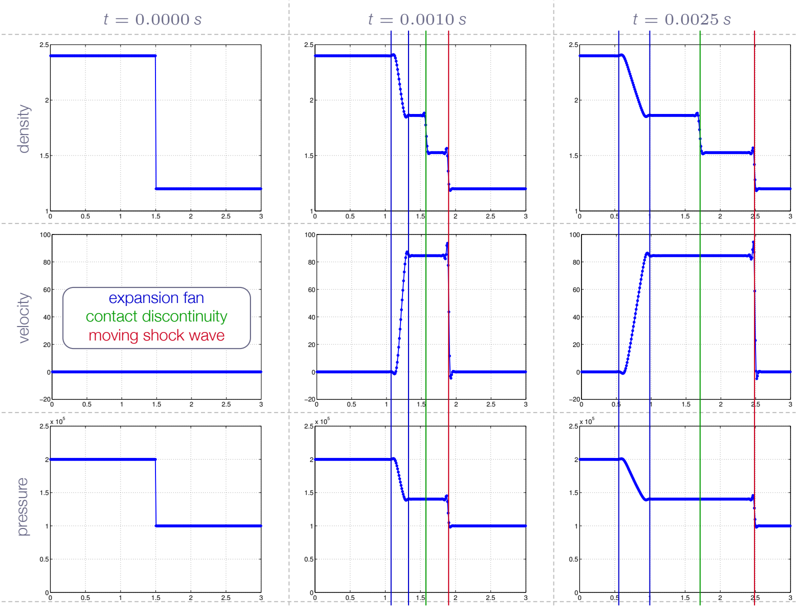
- right side conditions (state 1):
  - $\rho = 1.2\ kg/m^3$
  - $u = 0.0\ m/s$
  - $p = 1.0\ bar$

- Numerical method
  - Finite-Volume Method (FVM) solver
  - three-stage Runge-Kutta time stepping
  - third-order characteristic upwinding scheme
  - local artificial damping

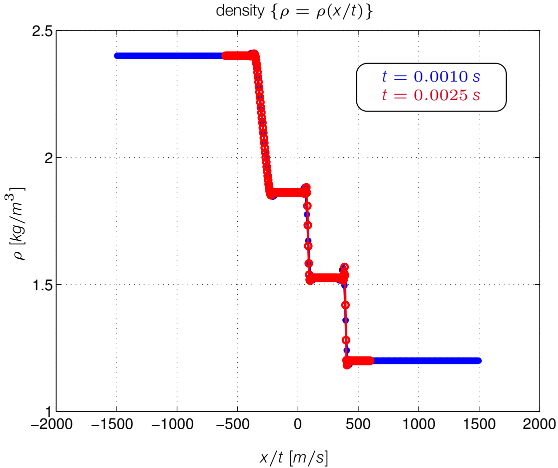
Riemann Problem - Shock Tube



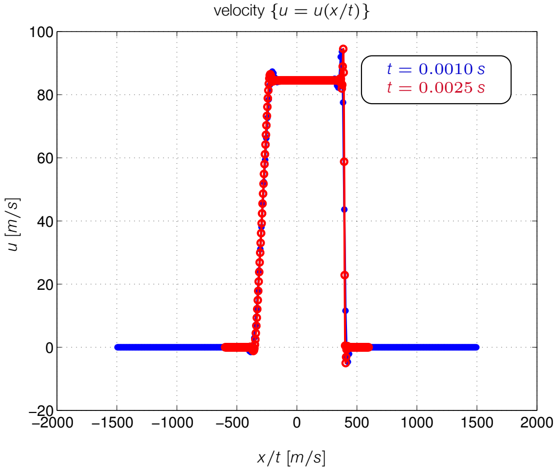
Riemann Problem - Shock Tube



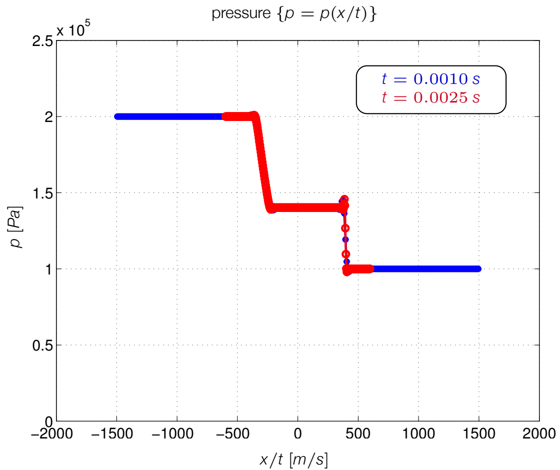
Riemann Problem - Shock Tube



Riemann Problem - Shock Tube



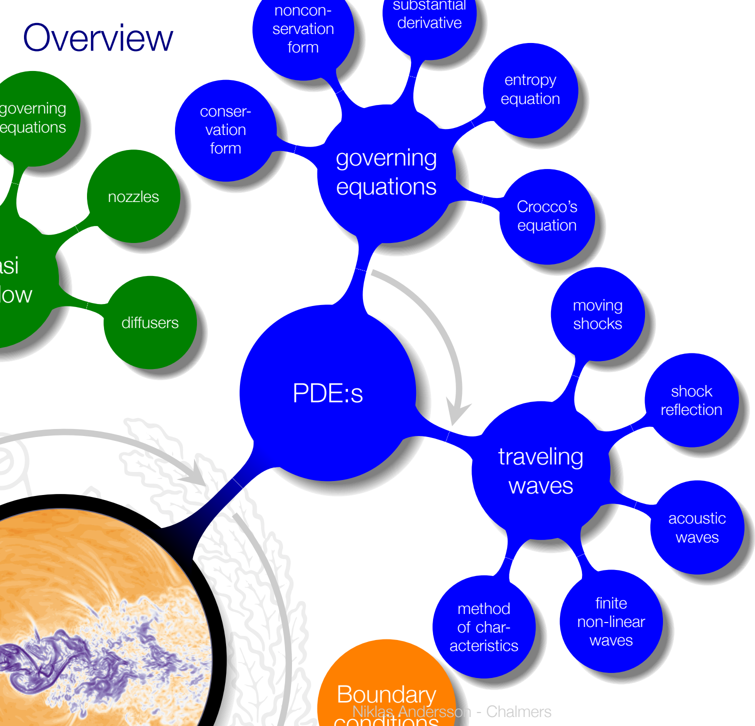
Riemann Problem - Shock Tube



LECTURE 11

Chapter 7
Unsteady Wave Motion

Overview


PDE:s
governing equations
noncon-servation form
substantial derivative
entropy equation
conser-vation form
Crocco's equation
traveling waves
moving shocks
shock reflection
acoustic waves
finite non-linear waves
method of char-acteristics
governing equations
nozzles
diffusers
Boundary

Addressed Learning Outcomes

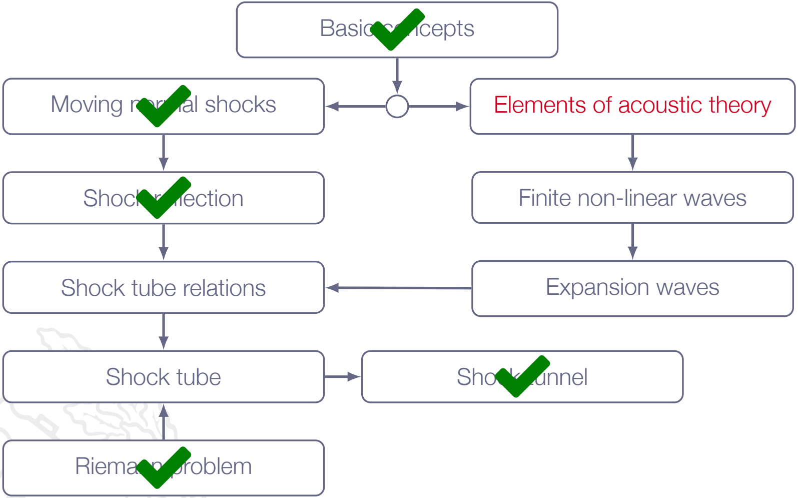
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   j unsteady waves and discontinuities in 1D
   k basic acoustics

11 Explain how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations

method of characteristics - a central element in classic compressible flow theory

Roadmap - Unsteady Wave Motion



Chapter 7.5
Elements of Acoustic Theory

Sound Waves

- Weakest audible sound wave (0 dB): Δp ~0.00002 Pa
- Loud sound wave (94 dB): Δp ~1 Pa
- Threshold of pain (120 dB): Δp ~20 Pa
- Harmful sound wave (130 dB): Δp ~60 Pa

Example:

$\Delta p \sim$ 1 Pa gives $\Delta \rho$ ~0.000009 kg/m^3 and Δu ~0.0025 m/s

Elements of Acoustic Theory

PDE:s for conservation of mass and momentum are derived in Chapter 6:

	conservation form	non-conservation form
mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$
momentum	$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = 0$	$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0$

Elements of Acoustic Theory

For adiabatic inviscid flow we also have the entropy equation as

$$\frac{Ds}{Dt} = 0$$

Assume one-dimensional flow

$$\left.\begin{aligned} \rho &= \rho(x,t) \\ \mathbf{v} &= u(x,t)\mathbf{e}_x \\ p &= p(x,t) \\ \dots \end{aligned}\right\} \Rightarrow$$

continuity $\displaystyle \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$

momentum $\displaystyle \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$

s=constant

can $\displaystyle \frac{\partial p}{\partial x}$ be expressed in terms of density?

Elements of Acoustic Theory

From Chapter 1: any thermodynamic state variable is uniquely defined by any tow other state variables

$$p = p(\rho, s) \Rightarrow dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

s=constant gives

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho = a^2 d\rho$$

$$\Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \\[2ex] \rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + a^2\dfrac{\partial \rho}{\partial x} = 0 \end{cases}$$

Elements of Acoustic Theory

Assume small perturbations around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Assume small perturbations around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Speed of sound is a thermodynamic state variable
$\Rightarrow a^2 = a^2(\rho, s)$. With entropy constant $\Rightarrow a^2 = a^2(\rho)$

Taylor expansion around a_∞ with $(\Delta\rho = \rho - \rho_\infty)$ gives

$$a^2 = a_\infty^2 + \left(\frac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}(a^2)\right)_\infty (\Delta\rho)^2 + ...$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}(\Delta\rho) + \Delta u \frac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\frac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\frac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \frac{\partial}{\partial x}(\Delta u) + \left[a_\infty^2 + \left(\frac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + ...\right]\frac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory - Acoustic Equations

Since $\Delta\rho$ and Δu are assumed to be small ($\Delta\rho \ll \rho_\infty$, $\Delta u \ll a$)

- products of perturbations can be neglected
- higher-order terms in the Taylor expansion can be neglected

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \rho_\infty \dfrac{\partial}{\partial x}(\Delta u) = 0 \\ \\ \rho_\infty \dfrac{\partial}{\partial t}(\Delta u) + a_\infty^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Note: Only valid for small perturbations (sound waves)

This type of derivation is based on linearization, *i.e.* the acoustic equations are linear

Elements of Acoustic Theory - Acoustic Equations

Acoustic equations:

> *"... describe the motion of gas induced by the passage of a sound wave ..."*

Elements of Acoustic Theory - Wave Equation

Combining linearized continuity and the momentum equations we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

(combine the time derivative of the continuity eqn. and the divergence of the momentum eqn.)

General solution:

$$\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$$

wave traveling in positive x-direction with speed a_∞

wave traveling in negative x-direction with speed a_∞

F and G may be arbitrary functions

Wave shape is determined by functions F and G

Elements of Acoustic Theory - Wave Equation

Spatial and temporal derivatives of F are obtained according to

$$\left\{\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial F}{\partial(x - a_\infty t)}\frac{\partial(x - a_\infty t)}{\partial t} = -a_\infty F' \\ \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial(x - a_\infty t)}\frac{\partial(x - a_\infty t)}{\partial x} = F' \end{aligned}\right.$$

spatial and temporal derivatives of G can of course be obtained in the same way...

Elements of Acoustic Theory - Wave Equation

with $\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$ and the derivatives of F and G we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 F'' + a_\infty^2 G''$$

and

$$\frac{\partial^2}{\partial x^2}(\Delta\rho) = F'' + G''$$

which gives

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho) = 0$$

i.e., the proposed solution fulfils the wave equation

Elements of Acoustic Theory - Wave Equation

F and G may be arbitrary functions, assume $G = 0$

$$\Delta\rho(x,t) = F(x - a_\infty t)$$

If $\Delta\rho$ is constant (constant wave amplitude), $(x - a_\infty t)$ must be a constant which implies

$$x = a_\infty t + c$$

where c is a constant

$$\frac{dx}{dt} = a_\infty$$

Elements of Acoustic Theory - Wave Equation

We want a relation between $\Delta\rho$ and Δu

$\Delta\rho(x,t) = F(x - a_\infty t)$ (wave in positive x direction) gives:

$$\frac{\partial}{\partial t}(\Delta\rho) = -a_\infty F' \qquad \text{and} \qquad \frac{\partial}{\partial x}(\Delta\rho) = F'$$

$$\underbrace{\frac{\partial}{\partial t}(\Delta\rho)}_{-a_\infty F'} + a_\infty \underbrace{\frac{\partial}{\partial x}(\Delta\rho)}_{F'} = 0$$

or

$$\frac{\partial}{\partial x}(\Delta\rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta\rho)$$

Elements of Acoustic Theory - Wave Equation

Linearized momentum equation:

$$\rho_\infty \frac{\partial}{\partial t}(\Delta u) = -a_\infty^2 \frac{\partial}{\partial x}(\Delta \rho) \Rightarrow$$

$$\frac{\partial}{\partial t}(\Delta u) = -\frac{a_\infty^2}{\rho_\infty}\frac{\partial}{\partial x}(\Delta \rho) = \left\{ \frac{\partial}{\partial x}(\Delta \rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta \rho) \right\} = \frac{a_\infty}{\rho_\infty}\frac{\partial}{\partial t}(\Delta \rho)$$

$$\frac{\partial}{\partial t}\left(\Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho\right) = 0 \Rightarrow \Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho = const$$

In undisturbed gas $\Delta u = \Delta \rho = 0$ which implies that the constant must be zero and thus

$$\boxed{\Delta u = \frac{a_\infty}{\rho_\infty}\Delta \rho}$$

Elements of Acoustic Theory - Wave Equation

Similarly, for $\Delta\rho(x,t) = G(x + a_\infty t)$ (wave in negative x direction) we obtain:

$$\boxed{\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho}$$

Also, since $\Delta p = a_\infty^2\Delta\rho$ we get:

Right going wave (+x direction) $\Delta u = \frac{a_\infty}{\rho_\infty}\Delta\rho = \frac{1}{a_\infty\rho_\infty}\Delta p$

Left going wave (-x direction) $\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho = -\frac{1}{a_\infty\rho_\infty}\Delta p$

Elements of Acoustic Theory - Wave Equation

- Δu denotes induced mass motion and is positive in the positive x-direction

$$\Delta u = \pm \frac{a_\infty \Delta \rho}{\rho_\infty} = \pm \frac{\Delta p}{a_\infty \rho_\infty}$$

- condensation (the part of the sound wave where $\Delta \rho > 0$): Δu is always in the same direction as the wave motion
- rarefaction (the part of the sound wave where $\Delta \rho < 0$): Δu is always in the opposite direction as the wave motion

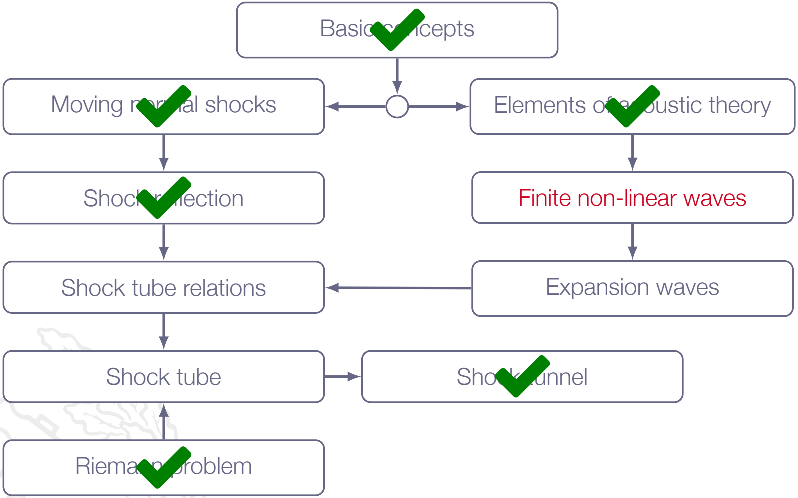
Elements of Acoustic Theory - Wave Equation *Summary*

Combining linearized continuity and the momentum equations we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

- Due to the assumptions made, the equation is not exact
- More and more accurate as the perturbations becomes smaller and smaller
- How should we describe waves with larger amplitudes?

Roadmap - Unsteady Wave Motion



Chapter 7.6
Finite (Non-Linear) Waves

Finite (Non-Linear) Waves

When $\Delta\rho,\ \Delta u,\ \Delta p,\ \ldots$ Become large, the linearized acoustic equations become poor approximations

Non-linear equations must be used

One-dimensional non-linear continuity and momentum equations

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

We still assume isentropic flow, $ds = 0$

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t} \qquad \frac{\partial \rho}{\partial x} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x}$$

Inserted in the continuity equation this gives:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

Add $1/(\rho a)$ times the continuity equation to the momentum equation:

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

If we instead subtraction $1/(\rho a)$ times the continuity equation from the momentum equation, we get:

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] - \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] = 0$$

Finite (Non-Linear) Waves

Since $u = u(x,t)$, we have:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt$$

Let $\frac{dx}{dt} = u + a$ gives

$$du = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right]dt$$

Interpretation: change of u in the direction of line $\frac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

In the same way we get:

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}\frac{dx}{dt}dt$$

and thus

$$dp = \left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right]dt$$

Finite (Non-Linear) Waves

Now, if we combine

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

$$du = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] dt$$

$$dp = \left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] dt$$

we get

$$\boxed{\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0}$$

Characteristic Lines

Thus, along a line $dx = (u + a)dt$ we have

$$du + \frac{dp}{\rho a} = 0$$

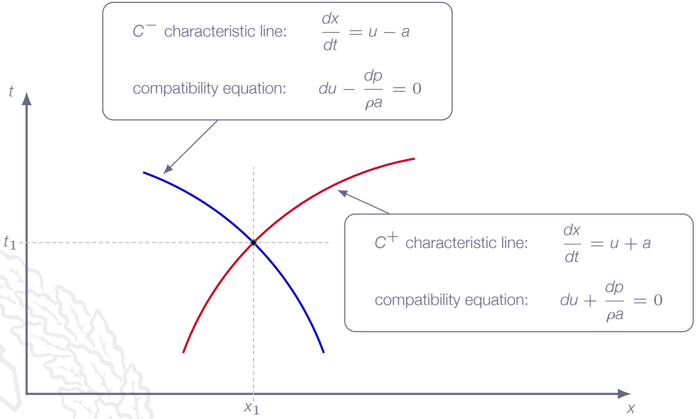
In the same way we get along a line where $dx = (u - a)dt$

$$du - \frac{dp}{\rho a} = 0$$

Characteristic Lines

- We have found a path through a point (x_1, t_1) along which the governing partial differential equations reduces to ordinary differential equations
- These paths or lines are called characteristic lines
- The C^+ and C^- characteristic lines are physically the paths of right- and left-running sound waves in the xt-plane

Characteristic Lines



Characteristic Lines

summary:

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^- \text{ characteristic}$$

or

$$du + \frac{dp}{\rho a} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$du - \frac{dp}{\rho a} = 0 \quad \text{along } C^- \text{ characteristic}$$

Riemann Invariants

Integration gives:

$$J^{+} = u + \int \frac{dp}{\rho a} = \text{constant along } C^{+} \text{ characteristic}$$

$$J^{-} = u - \int \frac{dp}{\rho a} = \text{constant along } C^{-} \text{ characteristic}$$

We need to rewrite $\dfrac{dp}{\rho a}$ to be able to perform the integrations

Riemann Invariants

Isentropic processes:

$$p = c_1 T^{\gamma/(\gamma-1)} = c_2 a^{2\gamma/(\gamma-1)}$$

where c_1 and c_2 are constants

$$\Rightarrow dp = c_2 \left(\frac{2\gamma}{\gamma - 1}\right) a^{[2\gamma/(\gamma-1)-1]} da$$

Assume calorically perfect gas:

$$a^2 = \frac{\gamma p}{\rho} \Rightarrow \rho = \frac{\gamma p}{a^2}$$

with $p = c_2 a^{2\gamma/(\gamma-1)}$ we get

$$\rho = c_2 \gamma a^{[2\gamma/(\gamma-1)-2]}$$

Riemann Invariants

$$J^+ = u + \int \frac{dp}{\rho a} = u + \int \frac{c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]}}{c_2\gamma a^{[2\gamma/(\gamma-1)-1]}} da = u + \int \frac{2da}{\gamma - 1}$$

$$J^+ = u + \frac{2a}{\gamma - 1}$$

$$J^- = u - \frac{2a}{\gamma - 1}$$

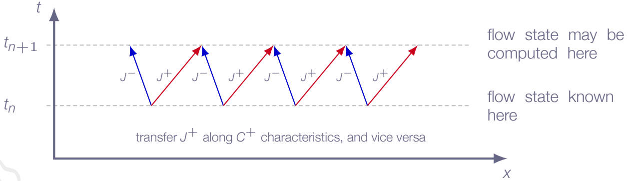
Riemann Invariants

If J^+ and J^- are known at some point (x, t), then

$$\begin{cases} J^+ + J^- = 2u \\ \\ J^+ - J^- = \dfrac{4a}{\gamma - 1} \end{cases} \Rightarrow \begin{cases} u = \dfrac{1}{2}(J^+ + J^-) \\ \\ a = \dfrac{\gamma - 1}{4}(J^+ - J^-) \end{cases}$$

Flow state is uniquely defined!

Method of Characteristics



Summary

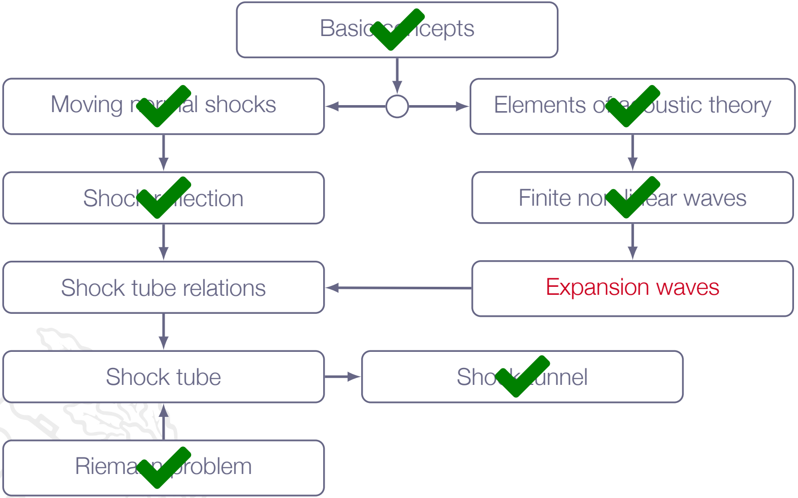
Acoustic waves

- $\Delta\rho$, Δu, etc - very small
- All parts of the wave propagate with the same velocity a_∞
- The wave shape stays the same
- The flow is governed by linear relations

Finite (non-linear) waves

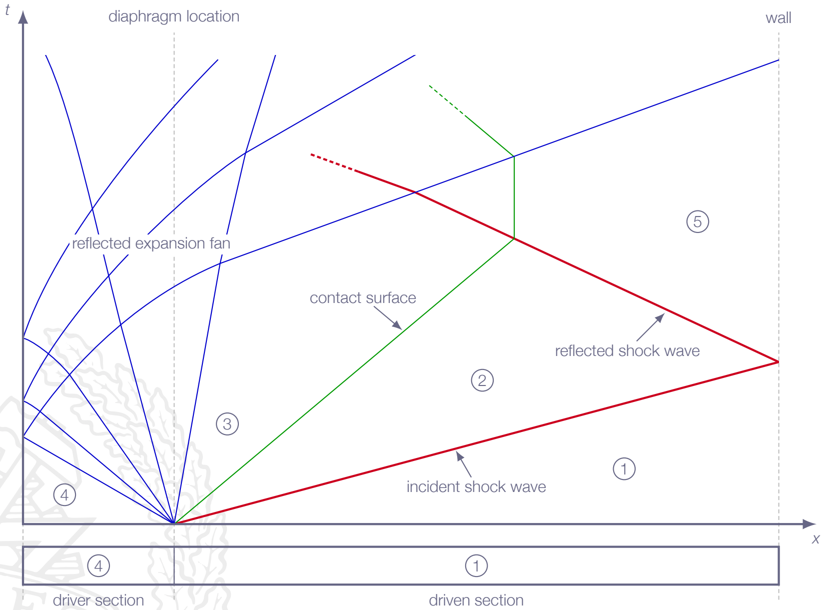
- $\Delta\rho$, Δu, etc - can be large
- Each local part of the wave propagates at the local velocity $(u + a)$
- The wave shape changes with time
- The flow is governed by non-linear relations

Roadmap - Unsteady Wave Motion



Chapter 7.7
Incident and Reflected Expansion Waves

Expansion Waves



Expansion Waves

Properties of a left-running expansion wave

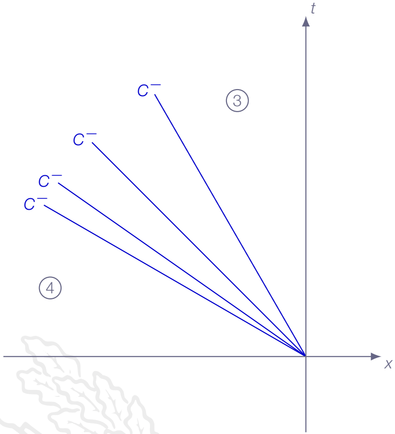
1. All flow properties are constant along C^- characteristics
2. The wave head is propagating into region 4 (high pressure)
3. The wave tail defines the limit of region 3 (lower pressure)
4. Regions 3 and 4 are assumed to be constant states

For calorically perfect gas:

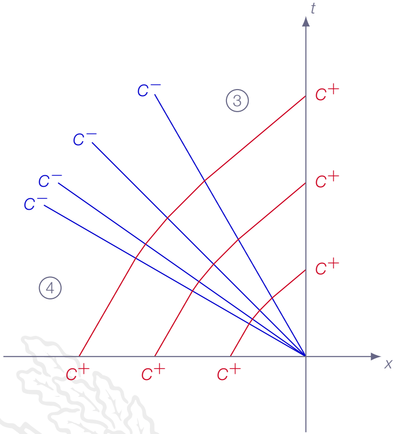
$$J^+ = u + \frac{2a}{\gamma - 1} \quad \text{is constant along } C^+ \text{ lines}$$

$$J^- = u - \frac{2a}{\gamma - 1} \quad \text{is constant along } C^- \text{ lines}$$

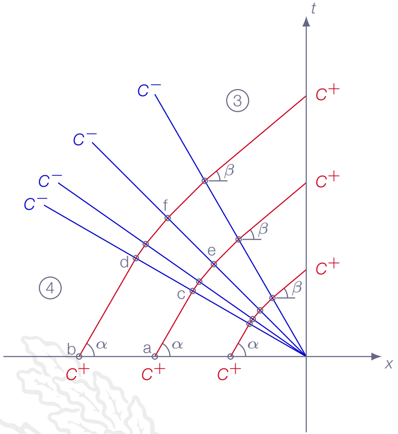
Expansion Waves



Expansion Waves



Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

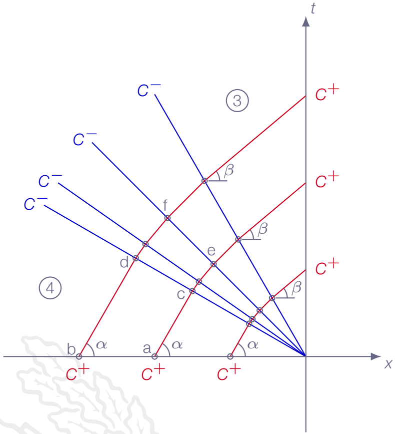
since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

$$u_e = \frac{1}{2}(J_e^+ + J_e^-),\ u_f = \frac{1}{2}(J_f^+ + J_f^-),\ \Rightarrow u_e = u_f$$

$$a_e = \frac{\gamma - 1}{4}(J_e^+ - J_e^-),\ a_f = \frac{\gamma - 1}{4}(J_f^+ - J_f^-),\ \Rightarrow a_e = a_f$$

Expansion Waves

Along each C^- line u and a are constants which means that

$$\frac{dx}{dt} = u - a = const$$

C^- characteristics are straight lines in xt-space

Expansion Waves

The start and end conditions are the same for all C^+ lines

J^+ invariants have the same value for all C^+ characteristics

C^- characteristics are straight lines in xt-space

Simple expansion waves centered at $(x, t) = (0, 0)$

Expansion Waves

In a left-running expansion fan:

- J^+ is constant throughout expansion fan, which implies:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = u_3 + \frac{2a_3}{\gamma - 1}$$

- J^- is constant along C^- lines, but varies from one line to the next, which means that

$$u - \frac{2a}{\gamma - 1}$$

is constant along each C^- line

Expansion Waves

Since $u_4 = 0$ we obtain:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = \frac{2a_4}{\gamma - 1} \Rightarrow$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}$$

with $a = \sqrt{\gamma RT}$ we get

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

Expansion Wave Relations

Isentropic flow $\Rightarrow$ we can use the isentropic relations

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

$$\frac{p}{p_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2}{\gamma-1}}$$

complete description in terms of u/a_4

Expansion Wave Relations

Since C^- characteristics are straight lines, we have:

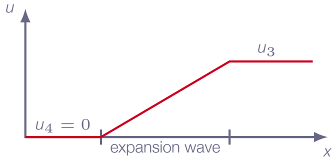
$$\frac{dx}{dt} = u - a \Rightarrow x = (u - a)t$$

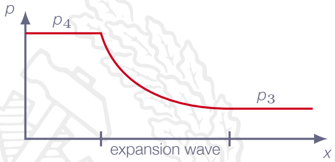
$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4} \Rightarrow a = a_4 - \frac{1}{2}(\gamma - 1)u \Rightarrow$$

$$x = \left[u - a_4 + \frac{1}{2}(\gamma - 1)u\right]t = \left[\frac{1}{2}(\gamma - 1)u - a_4\right]t \Rightarrow$$

$$u = \frac{2}{\gamma + 1}\left[a_4 + \frac{x}{t}\right]$$

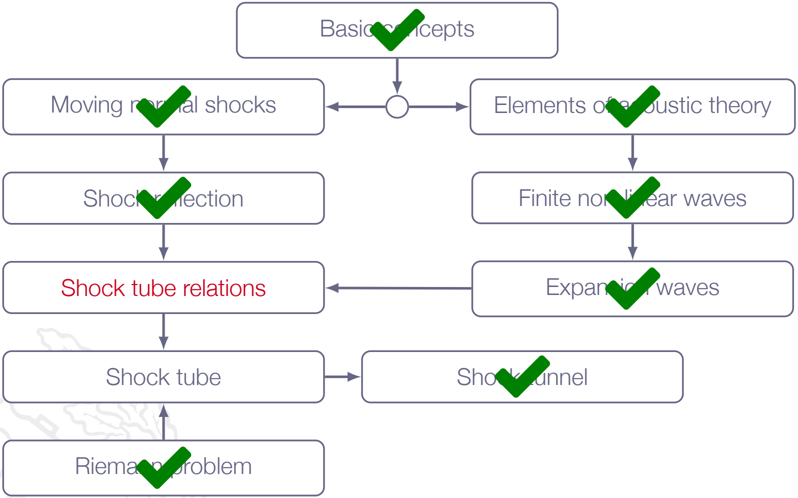
Expansion Wave Relations





- Expansion wave head is advancing to the left with speed a_4 into the stagnant gas
- Expansion wave tail is advancing with speed $u_3 - a_3$, which may be positive or negative, depending on the initial states

Roadmap - Unsteady Wave Motion



Chapter 7.8
Shock Tube Relations

Shock Tube Relations

$$u_p = u_2 = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma_1}{\gamma_1 + 1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma_1 - 1}{\gamma_1 + 1}}\right]^{1/2}$$

$$\frac{p_3}{p_4} = \left[1 - \frac{\gamma_4 - 1}{2}\left(\frac{u_3}{a_4}\right)\right]^{2\gamma_4/(\gamma_4 - 1)}$$

solving for u_3 gives

$$u_3 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4 - 1)/(2\gamma_4)}\right]$$

Shock Tube Relations

But, $p_3 = p_2$ and $u_3 = u_2$ (no change in velocity and pressure over contact discontinuity)

$$\Rightarrow u_2 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

We have now two expressions for u_2 which gives us

$$\frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma_1}{\gamma_1+1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2} = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

Rearranging gives:

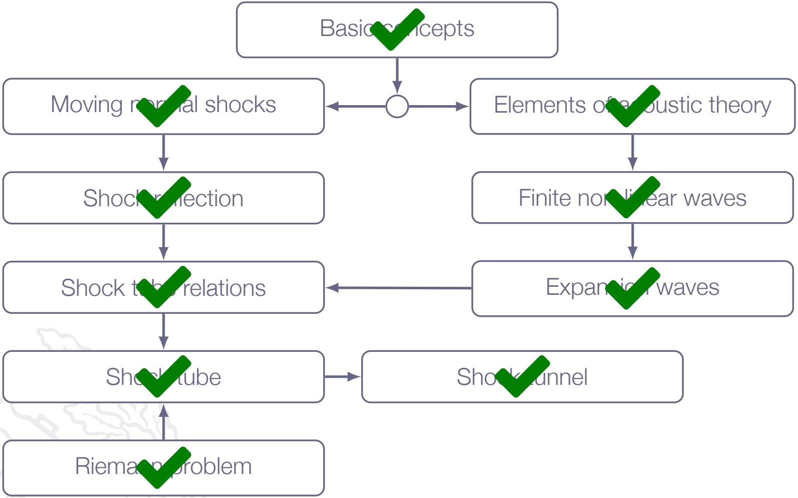
$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left\{1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1\left[2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1)\right]}}\right\}^{-2\gamma_4/(\gamma_4-1)}$$

- p_2/p_1 as implicit function of p_4/p_1
- for a given p_4/p_1, p_2/p_1 will increase with decreased a_1/a_4

$$a = \sqrt{\gamma RT} = \sqrt{\gamma(R_u/M)T}$$

- the speed of sound in a light gas is higher than in a heavy gas
  - driver gas: low molecular weight, high temperature
  - driven gas: high molecular weight, low temperature

Roadmap - Unsteady Wave Motion



LECTURE 12

Chapter 12
The Time-Marching Technique


method of char- acteristics
finite non-linear waves
Boundary conditions
Shock handling
CFD
Spatial dis- cretization
equilibrium gas
Time integration
Numerical schemes
Boltzmann distribution

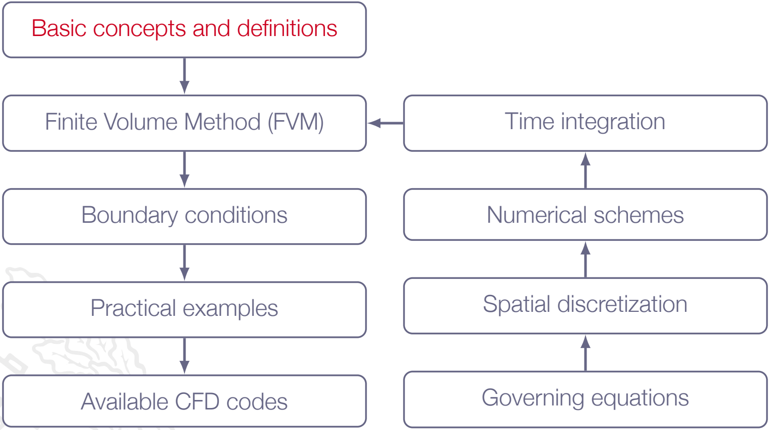
Addressed Learning Outcomes

12 Explain the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions

15 Explain the limitations in fluid flow simulation software

time for CFD!

Roadmap - The Time-Marching Technique



The Time-Marching Technique

Note:

Anderson's text is here rather out-of-date, it was written during the 70's and has not really been updated since then.
The additional material covered in this lecture is an attempt to amend this.

The Time-Marching Technique

The problems that we like to investigate numerically within the field of compressible flows can be categorized as

steady-state compressible flows

unsteady compressible flows

The Time-marching method is a solver framework that addresses both problem categories

The Time-Marching Technique

The time-marching approach is a good alternative for simulating flows where there are both supersonic and subsonic regions

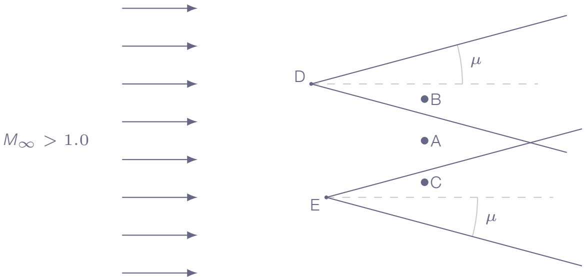
supersonic/hyperbolic:

- perturbations propagate in preferred directions
- zone of influence/zone of dependence
- PDEs can be transformed into ODEs

subsonic/elliptic:

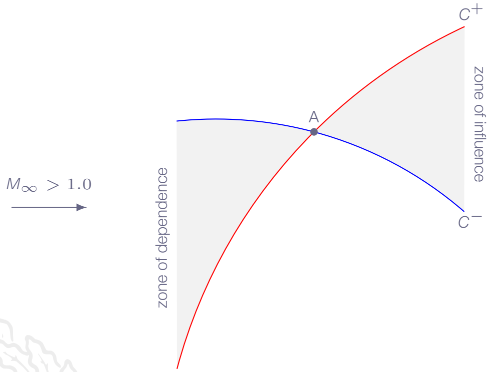
- perturbations propagate in all directions

Zone of Influence and Zone of Dependence



- A, B and C at the same axial position in the flow
- D and E are located upstream of A, B and C
- Mach waves generated at D will affect the flow in B but not in A and C
- Mach waves generated at E will affect the flow in C but not in A and B
- The flow in A is unaffected by the both D and E

Zone of Influence and Zone of Dependence



The zone of dependence for point A and the zone of influence of point A are defined by C^+ and C^- characteristic lines

The Time-Marching Technique

Steady-state problems:

1. define simple initial solution
2. apply specified boundary conditions
3. march in time until steady-state solution is reached

Unsteady problems:

1. apply specified initial solution
2. apply specified boundary conditions
3. march in time for specified total time to reach a desired unsteady solution

establish fully developed flow before initiating data sampling

Characterization of CFD Methods - Discretization

Discretization in space and time:

- most common approach: Method of Lines
  1. discretize in space ⇒
     system of ordinary differential equations (ODEs)
  2. discretize in time ⇒
     time-stepping scheme for system of ODEs

Spatial discretization techniques:

- Finite-Difference Method (FDM)
- Finite-Volume Method (FVM)
- Finite-Element Method (FEM)

Characterization of CFD Methods - Time Stepping

Temporal discretization techniques:

1. Explicit
   - mostly for transonic/supersonic steady-state and unsteady flows
   - short time steps
   - usually very stable
2. Implicit
   - mostly for subsonic/transonic steady-state flows
   - longer time steps possible

for high-supersonic flows, explicit solvers may very well outperform implicit solvers

Characterization of CFD Methods - Equations

Equations solved:

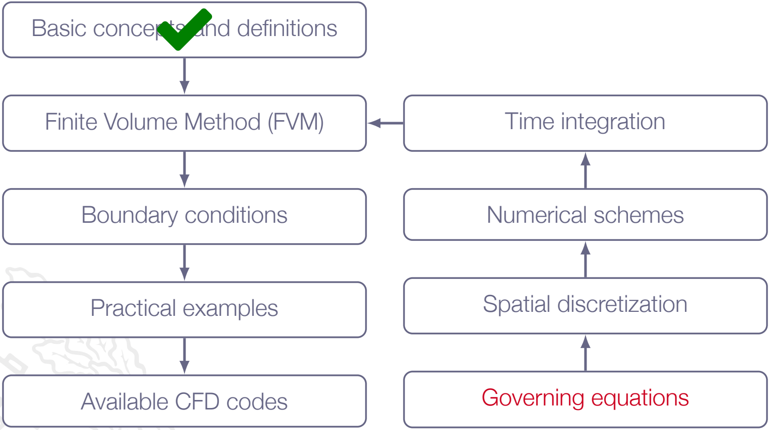
1. Density-based
   - solve for density in the continuity equation
   - mostly for transonic/supersonic steady-state and unsteady flows
2. Pressure-based
   - the continuity and momentum equations are combined to form a pressure correction equation
   - mostly for subsonic/transonic steady-state flows

Characterization of CFD Methods - Solver Approach

Solution procedure:

1. Fully coupled
   - all equations (continuity, moentum, energy, ...) are solved simultaneously
   - mostly for transonic/supersonic steady-state and unsteady flows
2. Segregated
   - solve the equations in sequence
   - mostly for subsonic steady-state flows

Roadmap - The Time-Marching Technique



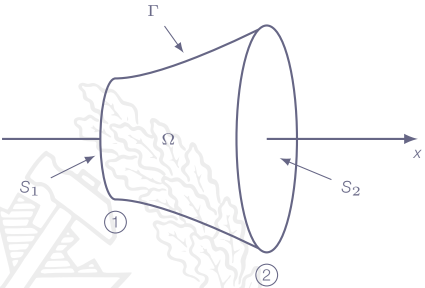
Explicit Finite-Volume Method

Governing Equations

Quasi-One-Dimensional Flow - Conceptual Idea

Introduce cross-section-averaged flow quantities $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \ldots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Quasi-One-Dimensional Flow - Governing Equations

Governing equations (general form):

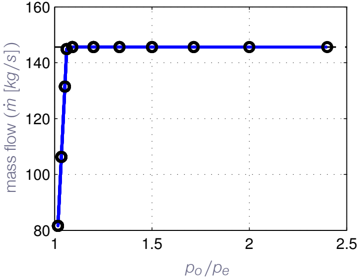
$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} + \oiint_{\partial\Omega} \rho \mathbf{v}\cdot\mathbf{n} dS = 0$$

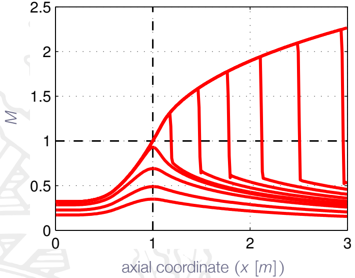
$$\frac{d}{dt}\iiint_{\Omega} \rho u d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS = 0$$

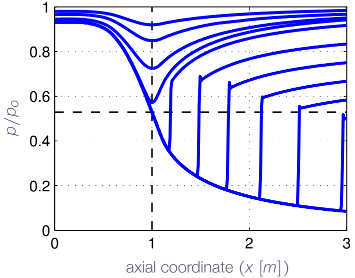
$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \rho h_o(\mathbf{v}\cdot\mathbf{n}) dS = 0$$

Example: Nozzle Simulation (Back Pressure Sweep)

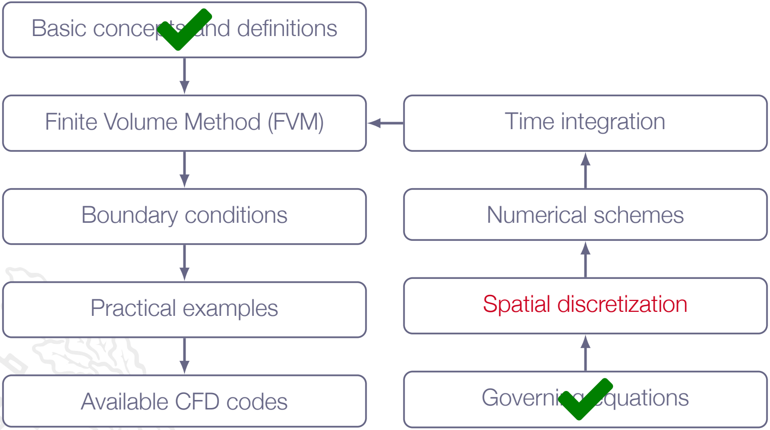
p_o	1.20 [bar]
p_e	0.50 [bar]
p_o/p_e	2.40
$\dot{m}$	145.6 [kg/s]
M_{max}	2.26







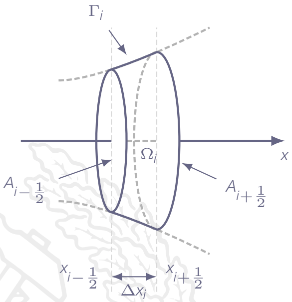
Roadmap - The Time-Marching Technique



Spatial Discretization

Quasi-One-Dimensional Flow - Spatial Discretization

Let's look at a small tube segment with length Δx



Streamtube with area $A(x)$

$$A_{i-\frac{1}{2}} = A(x_{i-\frac{1}{2}})$$

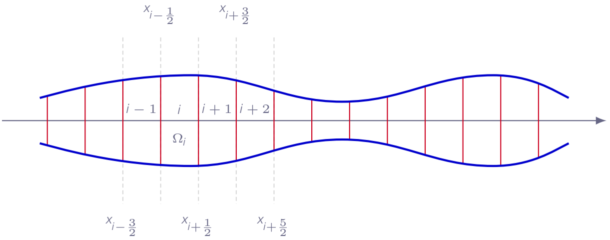
$$A_{i+\frac{1}{2}} = A(x_{i+\frac{1}{2}})$$

$$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

Ω_i - control volume enclosed by $A_{i-\frac{1}{2}}$, $A_{i+\frac{1}{2}}$, and Γ_i

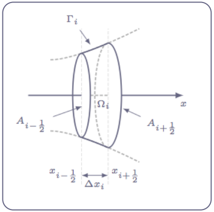
$\Rightarrow$ spatial discretization

Quasi-One-Dimensional Flow - Spatial Discretization



- Integer indices (i, $i+1$, ...):
  control volumes or cells
- Fractional indices ($i+\dfrac{1}{2}$, $i+\dfrac{3}{2}$, ...):
  interfaces between control volumes or cell faces
- Apply control volume formulations for mass, momentum, energy to control volume Ω_i

Quasi-One-Dimensional Flow



cell-averaged quantity
face-averaged quantity

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega_i}\rho d\mathcal{V}}_{VOL_i\frac{d}{dt}\bar{\rho}_i} + \underbrace{\iint\limits_{x_{i-\frac{1}{2}}}\rho\mathbf{v}\cdot\mathbf{n}dS}_{-\overline{(\rho u)}_{i-\frac{1}{2}}A_{i-\frac{1}{2}}} + \underbrace{\iint\limits_{x_{i+\frac{1}{2}}}\rho\mathbf{v}\cdot\mathbf{n}dS}_{\overline{(\rho u)}_{i+\frac{1}{2}}A_{i+\frac{1}{2}}} + \underbrace{\iint\limits_{\Gamma_i}\rho\mathbf{v}\cdot\mathbf{n}dS}_{0} = 0$$

where

$$VOL_i = \iiint\limits_{\Omega_i} d\mathcal{V}$$

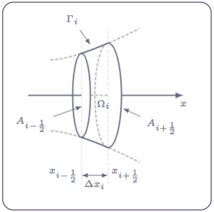
$$\overline{(\rho u)}_{i-\frac{1}{2}} = \frac{1}{A_{i-\frac{1}{2}}}\iint\limits_{x_{i-\frac{1}{2}}}\rho u dS$$

$$\bar{\rho}_i = \frac{1}{VOL_i}\iiint\limits_{\Omega_i}\rho d\mathcal{V}$$

$$\overline{(\rho u)}_{i+\frac{1}{2}} = \frac{1}{A_{i+\frac{1}{2}}}\iint\limits_{x_{i+\frac{1}{2}}}\rho u dS$$

Quasi-One-Dimensional Flow

cell-averaged quantity
face-averaged quantity
source term

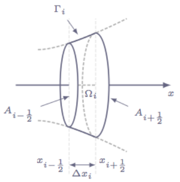


Conservation of momentum:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega_i} \rho u d\mathscr{V}}_{VOL_i \frac{d}{dt}\overline{(\rho u)}_i} + \underbrace{\iint\limits_{x_{i-\frac{1}{2}}} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS}_{-\overline{(\rho u^2+p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}}} +$$

$$+\underbrace{\iint\limits_{x_{i+\frac{1}{2}}} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS}_{\overline{(\rho u^2+p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}}} + \underbrace{\iint\limits_{\Gamma_i} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS}_{-\iint_{\Gamma_i} p dA} = 0$$

Quasi-One-Dimensional Flow



cell-averaged quantity

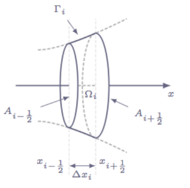
face-averaged quantity

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega_i} \rho e_o d\mathscr{V}}_{VOL_i \frac{d}{dt}\overline{(\rho e_o)}_i} + \underbrace{\iint\limits_{x_{i-\frac{1}{2}}} \rho h_o (\mathbf{v}\cdot\mathbf{n}) dS}_{-\overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}}} +$$

$$+ \underbrace{\iint\limits_{x_{i+\frac{1}{2}}} \rho h_o (\mathbf{v}\cdot\mathbf{n}) dS}_{\overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}}} + \underbrace{\iint\limits_{\Gamma_i} \rho h_o (\mathbf{v}\cdot\mathbf{n}) dS}_{0} = 0$$

Quasi-One-Dimensional Flow



Lower order term due to varying stream tube area:

$$\iint\limits_{\Gamma_i} p dA \approx \bar{p}_i \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right)$$

where $\bar{p}_i$ is calculated from cell-averaged quantities (DOFs)

$$\left\{ \bar{\rho}, \overline{(\rho u)}, \overline{(\rho e_o)} \right\}_i$$

as

$$\bar{p}_i = (\gamma - 1) \left(\overline{(\rho e_o)}_i - \frac{1}{2} \bar{\rho}_i \bar{u}_i \right), \quad \bar{u}_i = \frac{\overline{(\rho u)}_i}{\bar{\rho}_i}$$

Quasi-One-Dimensional Flow - Spatial Discretization

cell-averaged quantity
face-averaged quantity
source term

$$VOL_i \frac{d}{dt}\bar{\rho}_i - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

$$VOL_i \frac{d}{dt}\overline{(\rho u)}_i - \overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} =$$

$$= \bar{p}_i \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right)$$

$$VOL_i \frac{d}{dt}\overline{(\rho e_o)}_i - \overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

Application of these equations to all cells $i \in \{1, 2, ..., N\}$ of the computational domain results in a system of ODEs

Spatial Discretization - Summary

Steps to achieve spatial discretization:

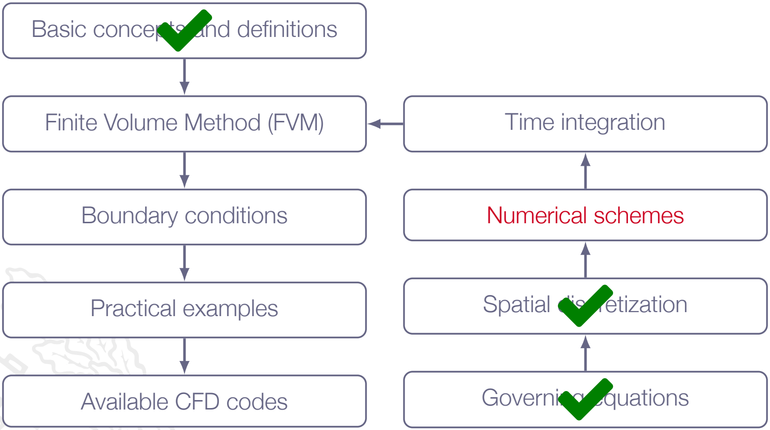
1. Choose primary variables (Degrees of Freedom or DOFs)
2. Approximate all other quantities in terms of DOFs

$\Rightarrow$ System of ordinary differential equations (ODEs)

Degrees of freedom:

- Choose $\left\{\bar{\rho}, \overline{(\rho u)}, \overline{(\rho e_o)}\right\}_i$ in all control volumes Ω_i, $i \in \{1, 2, ..., N\}$ as degrees of freedom, or primary variables
- Note that these are cell-averaged quantities

Roadmap - The Time-Marching Technique



Numerical Schemes

Flux Term Approximation

$$\begin{Bmatrix} \overline{(\rho u)} \\ \overline{(\rho u^2 + p)} \\ \overline{(\rho u h_o)} \end{Bmatrix}_{i+\frac{1}{2}} = f\left(\begin{Bmatrix} \overline{\rho} \\ \overline{(\rho u)} \\ \overline{(\rho e_o)} \end{Bmatrix}_i, \begin{Bmatrix} \overline{\rho} \\ \overline{(\rho u)} \\ \overline{(\rho e_o)} \end{Bmatrix}_{i+1}, \dots \right)$$

cell face values

cell-averaged values

Simple example:

$$\overline{(\rho u)}_{i+\frac{1}{2}} \approx \frac{1}{2}\left[\overline{(\rho u)}_i + \overline{(\rho u)}_{i+1}\right]$$

Flux Term Approximation

More complex approximations usually needed

High-order schemes:

- increased accuracy
- more cell values involved (*wider flux molecule*)
- boundary conditions more difficult to implement

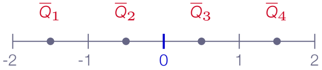
Optimized numerical dissipation:

- upwind type of flux scheme

Shock handling:

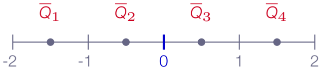
- non-linear treatment needed (*e.g.* TVD schemes)
- artificial damping

Flux Term Approximation



$$Q(x) = A + Bx + Cx^2 + Dx^3$$

Flux Term Approximation

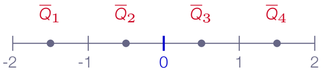


$$\overline{Q}_1 = \frac{1}{VOL_1}\int_{-2}^{-1} Q(x)dx$$

$$VOL_1 = A_1\Delta x_1 = \{A_1 = 1.0,\ \Delta x_1 = 1.0\} = 1.0$$

$$\Rightarrow \overline{Q}_1 = \int_{-2}^{-1} Q(x)dx$$

Flux Term Approximation



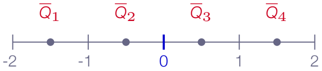
$$\overline{Q}_1 = \int_{-2}^{-1} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{-2}^{-1}$$

$$\overline{Q}_2 = \int_{-1}^{0} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{-1}^{0}$$

$$\overline{Q}_3 = \int_{0}^{1} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{0}^{1}$$

$$\overline{Q}_4 = \int_{1}^{2} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{1}^{2}$$

Flux Term Approximation



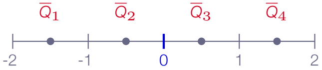
$$\overline{Q}_1 = A - \frac{3}{2}B + \frac{7}{3}C - \frac{15}{4}D$$

$$\overline{Q}_2 = A - \frac{1}{2}B + \frac{1}{3}C - \frac{1}{4}D$$

$$\overline{Q}_3 = A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D$$

$$\overline{Q}_4 = A + \frac{3}{2}B + \frac{7}{3}C + \frac{15}{4}D$$

Flux Term Approximation



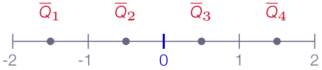
$$A = \frac{1}{12}\left[-\overline{Q}_1 + 7\overline{Q}_2 + 7\overline{Q}_3 - \overline{Q}_4\right]$$

$$B = \frac{1}{12}\left[\overline{Q}_1 - 15\overline{Q}_2 + 15\overline{Q}_3 - \overline{Q}_4\right]$$

$$C = \frac{1}{4}\left[\overline{Q}_1 - \overline{Q}_2 - \overline{Q}_3 + \overline{Q}_4\right]$$

$$D = \frac{1}{6}\left[-\overline{Q}_1 + 3\overline{Q}_2 - 3\overline{Q}_3 + \overline{Q}_4\right]$$

Flux Term Approximation



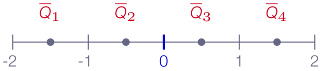
$$Q_0 = Q(0) + \delta Q'''(0) \Rightarrow Q_0 = A + 6\delta D$$

$\delta = 0 \Rightarrow$ fourth-order central scheme

$\delta = 1/12 \Rightarrow$ third-order upwind scheme

$\delta = 1/96 \Rightarrow$ third-order low-dissipation upwind scheme

Flux Term Approximation



$$Q_0 = A + 6\delta D = \{\delta = 1/12\} = -\frac{1}{6}\overline{Q}_1 + \frac{5}{6}\overline{Q}_2 + \frac{1}{3}\overline{Q}_3$$

$$Q_{0_{left}} = -\frac{1}{6}\overline{Q}_1 + \frac{5}{6}\overline{Q}_2 + \frac{1}{3}\overline{Q}_3$$

$$Q_{0_{right}} = -\frac{1}{6}\overline{Q}_4 + \frac{5}{6}\overline{Q}_3 + \frac{1}{3}\overline{Q}_2$$

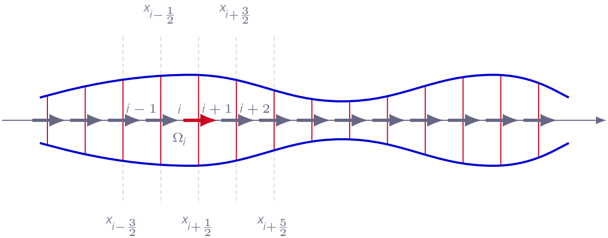
method of characteristics used in order to decide whether left- or right-upwinded flow quantities should be used

Flux Term Approximation

High-order numerical schemes:

- low numerical dissipation (smearing due to amplitudes errors)
- low dispersion errors (wiggles due to phase errors)

Conservative Scheme



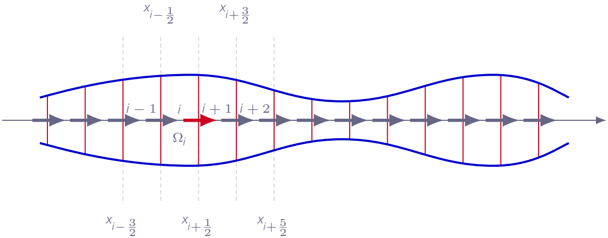
mass conservation:

cell (i):
$$VOL_i \frac{d}{dt}\bar{\rho}_i + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} = 0$$

cell ($i+1$):
$$VOL_{i+1} \frac{d}{dt}\bar{\rho}_{i+1} + \overline{(\rho u)}_{i+\frac{3}{2}} A_{i+\frac{3}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

(similarly for momentum and energy conservation)

Conservative Scheme



mass conservation:

cell (i):

$$VOL_i \frac{d}{dt}\bar{\rho}_i + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} = 0$$

cell $(i+1)$:

$$VOL_{i+1} \frac{d}{dt}\bar{\rho}_{i+1} + \overline{(\rho u)}_{i+\frac{3}{2}} A_{i+\frac{3}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

(similarly for momentum and energy conservation)

Conservative Scheme

Conservative scheme

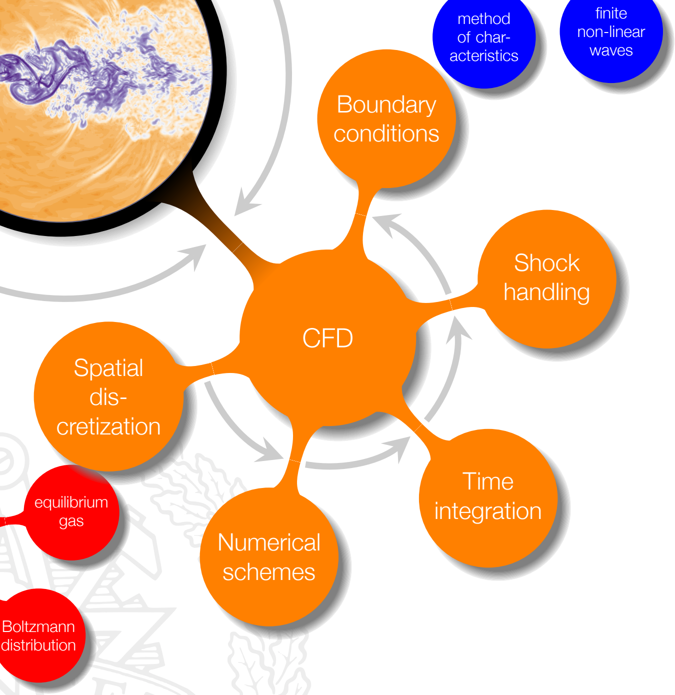
"The flux leaving one control volume equals the flux entering neighbouring control volume"

Conservation property for mass, momentum and energy is crucial for the correct prediction of shocks*

* correct prediction of shocks:
strength
position
velocity

LECTURE 13

Chapter 12
The Time-Marching Technique


method of char-acteristics
finite non-linear waves
Boundary conditions
Shock handling
CFD
Spatial dis-cretization
Time integration
Numerical schemes
equilibrium gas
Boltzmann distribution

Addressed Learning Outcomes

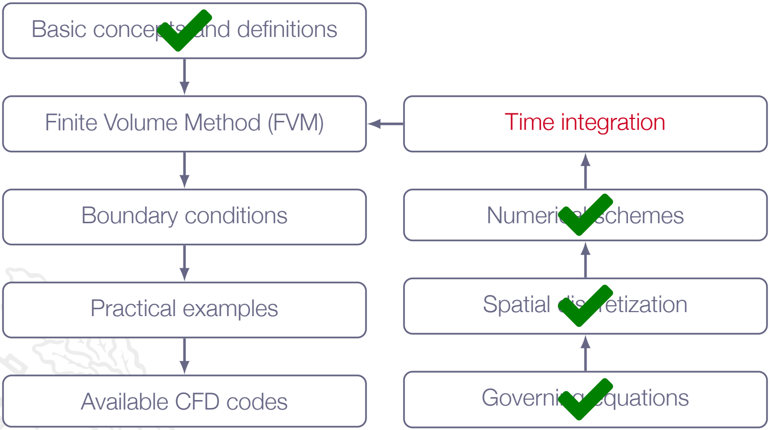
12 Explain the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions

14 Analyze and verify the quality of the numerical solution

15 Explain the limitations in fluid flow simulation software

what about boundary conditions?

Roadmap - The Time-Marching Technique



Time Stepping

Time Stepping

The system of ODEs obtained from the spatial discretization in vector notation

$$\frac{d}{dt}\mathbf{Q} = \mathbf{F}(\mathbf{Q})$$

- $\mathbf{Q}$ is a vector containing all DOFs in all cells
- $\mathbf{F}(\mathbf{Q})$ is the time derivative of $\mathbf{Q}$ resulting from above mentioned flux approximations
  non-linear vector-valued function

Time Stepping

Three-stage Runge-Kutta - *one example of many*:

- Explicit time-marching scheme
- Second-order accurate

Time Stepping - Three-stage Runge-Kutta

$$\frac{d}{dt}\mathbf{Q} = \mathbf{F}(\mathbf{Q})$$

Let $\mathbf{Q}^n = \mathbf{Q}(t_n)$ and $\mathbf{Q}^{n+1} = \mathbf{Q}(t_{n+1})$

- t_n is the current time level and t_{n+1} is the next time level
- $\Delta t = t_{n+1} - t_n$ is the solver time step

Algorithm:

$$\begin{aligned}
1.\quad \mathbf{Q}^{*} &= \mathbf{Q}^n + \Delta t \mathbf{F}(\mathbf{Q}^n) \\
2.\quad \mathbf{Q}^{**} &= \mathbf{Q}^n + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^n) + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^{*}) \\
3.\quad \mathbf{Q}^{n+1} &= \mathbf{Q}^n + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^n) + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^{**})
\end{aligned}$$

DOFs in all cells updated from time level t_n to time level t_{n+1}, repeat procedure for $t_{n+2}, t_{n+3}, \ldots$

Time Stepping - Explicit Schemes

Properties of explicit time-stepping schemes:

+ Easy to implement in computer codes
+ Efficient execution on most computers
+ Easy to adapt for parallel execution on distributed memory systems (*e.g.* Linux clusters)

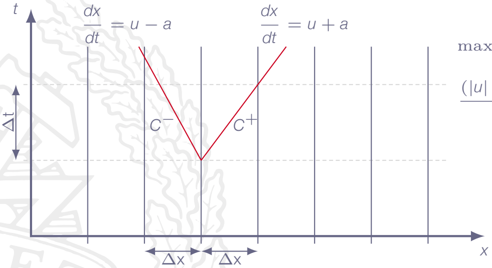
- Time step limitation (CFL number)
- Convergence to steady-state often slow (there are, however, some remedies for this)

Time Stepping - Explicit Schemes

Courant-Friedrich-Levy (CFL) number - *one-dimensional case*:

$$CFL_i = \frac{\Delta t(|u_i| + a_i)}{\Delta x_i} \leq 1$$

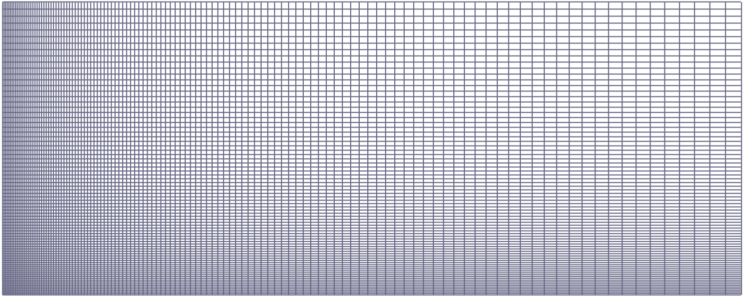
Interpretation: The fastest characteristic (C^+ or C^-) must not travel longer than Δx during one time step



$$\max(|u-a|, |u+a|)\Delta t = (|u|+a)\Delta t \leq \Delta x \Rightarrow$$

$$\frac{(|u|+a)\Delta t}{\Delta x} = CFL \leq 1$$

Time Stepping - Explicit Schemes



Steady-state problems:

- local time stepping
- each cell has an individual time step
- Δt_i maximum allowed value based on CFL criteria

Unsteady problems:

- time accurate
- all cells have the same time step
- $\Delta t_i = \min\{\Delta t_1, ..., \Delta t_N\}$

Explicit Finite-Volume Method - Summary

The described numerical scheme is an example of a density-based, fully coupled scheme

Explicit Finite-Volume Method - Summary

- density-based schemes
  - solve for density in the continuity equation
  - in general preferred for high-Mach-number flows and for unsteady compressible flows
- pressure-based schemes
  - the continuity and momentum equations are combined to form a pressure correction equation
  - were first used for incompressible flows but have been adapted for compressible flows also
  - quite popular for steady-state subsonic/transonic flows

Explicit Finite-Volume Method - Summary

- fully-copuled schemes
  - all equations (continuity, momentum, energy) are solved for simultaneously
- segregated schemes
  - alternate between the solution of the velocity field and the pressure field (pressure-based solver)

Explicit Finite-Volume Method - Summary

Spatial discretization:

- Control volume formulations of conservation equations are applied to the âcellsâ of the discretized domain
- Cell-averaged flow quantities ($\overline{\rho}, \overline{\rho u}, \overline{\rho e_o}$) are chosen as degrees of freedom (DOFs)
- Flux terms are approximated in terms of the chosen DOFs
  - high-order, upwind type of flux approximation is used for optimum results
- A fully conservative scheme is obtained
  - the flux leaving one cell is identical to the flux entering the neighboring cell
- The result of the spatial discretization is a system of ODEs

Explicit Finite-Volume Method - Summary

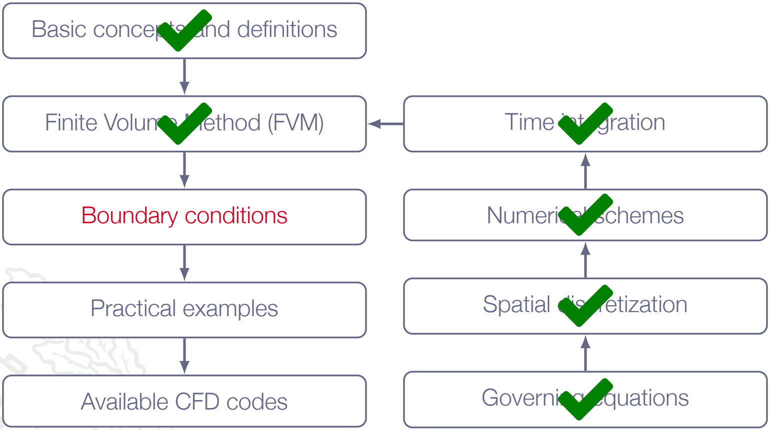
Time marching:

- Three-stage, second-order accurate Runge-Kutta scheme
  - Explicit time-stepping
  - Time step length limited by the *CFL* condition (*CFL* ≤ 1)

Classification of numerical scheme:

- density-based
  - includes the continuity equation
- fully coupled
  - all equations are solved simultaneously

Roadmap - The Time-Marching Technique



Boundary Conditions

Boundary Conditions

Boundary conditions are very important for numerical simulation of compressible flows

Main reason: both flow and acoustics involved!

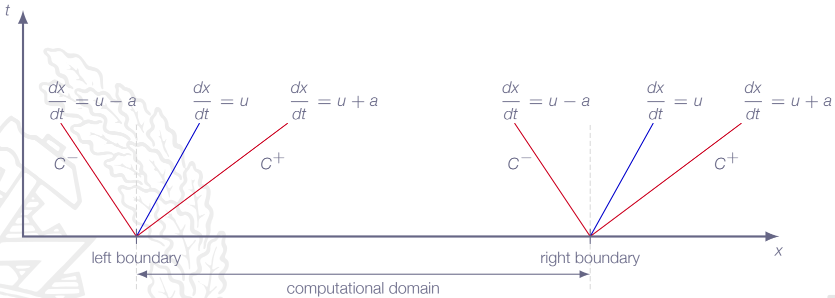
Example 1:
Finite-volume CFD code for Quasi-1D compressible flow (Time-marching procedure)

What boundary conditions should be applied at the left and right ends?

Boundary Conditions

three characteristics:

1. C^+
2. C^-
3. advection



Boundary Conditions

- C^+ and C^- characteristics describe the transport of isentropic pressure waves (often referred to as acoustics)
- The advection characteristic simply describes the transport of certain quantities with the fluid itself (for example entropy)
- In one space dimension and time, these three characteristics, together with the quantities that are known to be constant along them, give a complete description of the time evolution of the flow
- We can use the characteristics as a guide to tell us what information that should be specify at the boundaries

Left Boundary - Subsonic Inflow

we have three PDEs, and are solving for three unknowns

- Subsonic inflow: $0 < u < a$

  $$u - a < 0$$
  $$u > 0$$
  $$u + a > 0$$

  - one outgoing characteristic
  - two ingoing characteristics
- Two variables should be specified at the boundary
- The third variable must be left free

Left Boundary - Subsonic Outflow

we have three PDEs, and are solving for three unknowns

- Subsonic outflow: $-a < u < 0$

  $$u - a < 0$$
  $$u < 0$$
  $$u + a > 0$$

  - two outgoing characteristics
  - one ingoing characteristic
- One variable should be specified at the boundary
- The second and third variables must be left free

Left Boundary - Supersonic Inflow

we have three PDEs, and are solving for three unknowns

- Supersonic inflow: $u > a$

  $$u - a > 0$$
  $$u > 0$$
  $$u + a > 0$$

  - no outgoing characteristics
  - three ingoing characteristics
- All three variables should be specified at the boundary
- No variables must be left free

Left Boundary - Supersonic Outflow

we have three PDEs, and are solving for three unknowns

- Supersonic outflow: $u < -a$

  $$u - a < 0$$
  $$u < 0$$
  $$u + a < 0$$

  - three outgoing characteristics
  - no ingoing characteristics
- No variables should be specified at the boundary
- All variables must be left free

Right Boundary - Subsonic Outflow

we have three PDEs, and are solving for three unknowns

- Subsonic outflow: $0 < u < a$

  $$u - a < 0$$
  $$u > 0$$
  $$u + a > 0$$

  - one ingoing characteristic
  - two outgoing characteristics
- One variable should be specified at the boundary
- The second and third variables must be left free

Right Boundary - Subsonic Inflow

we have three PDEs, and are solving for three unknowns

- Subsonic inflow: $-a < u < 0$

  $u - a < 0$

  $u < 0$

  $u + a > 0$

  - two ingoing characteristics
  - one outgoing characteristic
- Two variables should be specified at the boundary
- The third variables must be left free

Right Boundary - Supersonic Outflow

we have three PDEs, and are solving for three unknowns

- Supersonic outflow: $u > a$

  $u - a > 0$
  $u > 0$
  $u + a > 0$

  - no ingoing characteristics
  - three outgoing characteristics
- No variables should be specified at the boundary
- All three variables must be left free

Right Boundary - Supersonic Inflow

we have three PDEs, and are solving for three unknowns

- Supersonic inflow: $u < -a$

  $$u - a < 0$$
  $$u < 0$$
  $$u + a < 0$$

  - three ingoing characteristics
  - no outgoing characteristics
- All three variables should be specified at the boundary
- No variables must be left free

Subsonic Inflow (Left Boundary) - Example

Subsonic inflow: we should specify two variables

Alt	specified variable 1	specified variable 2	well-posed	non-reflective
1	p_o	T_o	X	
2	ρu	T_o	X	
3	s	J^+	X	X

well posed:

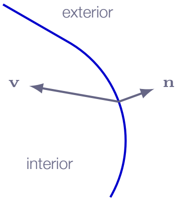
- the problem has a solution
- the solution is unique
- the solution's behaviour changes continuously with initial conditions

Subsonic Outflow (Left Boundary) - Example

Subsonic outflow: we should specify one variable

Alt	specified variable	well-posed	non-reflective
1	p	X	
2	ρu	X	
3	J^+	X	X

Subsonic Inflow 2D/3D



$\mathbf{n}$	unit normal vector
$\mathbf{v}$	fluid velocity at boundary

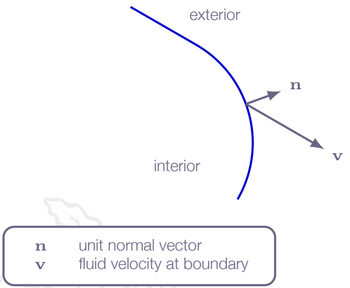
Subsonic inflow

- Assumption:

  $-a < \mathbf{v} \cdot \mathbf{n} < 0$

- Four ingoing characteristics
- One outgoing characteristic
- Specify four variables at the boundary:
  - example: p_o, T_o, flow direction (two angles)

Subsonic Outflow 2D/3D



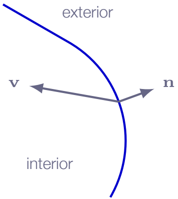
Subsonic outflow

- Assumption:

  $0 < \mathbf{v} \cdot \mathbf{n} < a$

- One ingoing characteristics
- Four outgoing characteristic
- Specify one variables at the boundary:
  - example: p

Supersonic Inflow 2D/3D



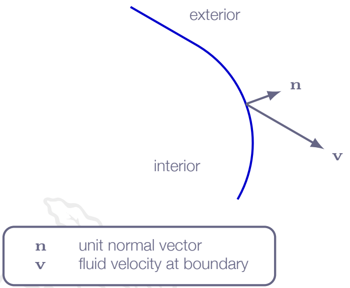
$\mathbf{n}$	unit normal vector
$\mathbf{v}$	fluid velocity at boundary

- Supersonic inflow
- Assumption:

  $\mathbf{v} \cdot \mathbf{n} < -a$

- Five ingoing characteristics
- No outgoing characteristics
- Specify five variables at the boundary:
  - all solver variables specified

Supersonic Outflow 2D/3D



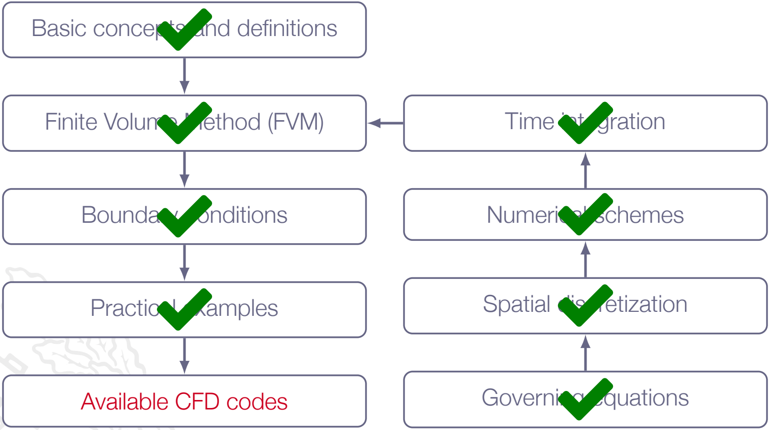
Supersonic outflow

- Assumption:

  $\mathbf{v} \cdot \mathbf{n} > a$

- No ingoing characteristics
- Five outgoing characteristics
- No variables specified at the boundary:

Roadmap - The Time-Marching Technique



Available CFD Codes

CFD Codes

List of free and commercial CFD codes:

http://www.cfd-online.com/Wiki/Codes

- Free codes are in general unsupported and poorly documented
- Commercial codes are often claimed to be suitable for all types of flows
  The reality is that the user must make sure of this!
- Industry/institute/university in-house codes not listed
  - non-commercial but proprietary
  - part of design/analysis system

CFD Codes - General Guidlines

Simulation of high-speed and/or unsteady compressible flows:

- Use correct solver options
  otherwise you may obtain completely wrong solution!
- Use a high-quality grid
  a poor grid will either not give you any solution at all (no convergence) or at best a very inaccurate solution!

ANSYS-FLUENT® - Typical Experiences

- Very robust solver - will almost always give you a solution
- Accuracy of solution depends a lot on grid quality
- Shocks are generally smeared more than in specialized codes
- Solver is generally very efficient for steady-state problems
- Solver is less efficient for truly unsteady problems, where both flow and acoustics must be resolved accurately

ANSYS-FLUENT® - Solver Options

- Coupled or Density-based *depends on version*
  - the continuity, momentum, energy equations are solved for simultaneously
    just like in the Quasi-1D code discussed previously
- Density = Ideal gas law
  - the calorically perfect gas assumption is activated
  - the energy equation is activated
- Explicit or Implicit time stepping
  - Explicit recommended for unsteady compressible flows
    CFL is set to 1 as default, but may be changed
  - Implicit more efficient for steady-state compressible flows
    CFL is set to 5 as default, but may be changed

ANSYS-FLUENT® - Solver Features

Spatial discretization:

- Finite-Volume Method (FVM)
- Unstructured grids
- Fully conservative, density-based scheme
- Flux approximations:
  first-order, second-order, upwind, ...
- Fully coupled solver approach

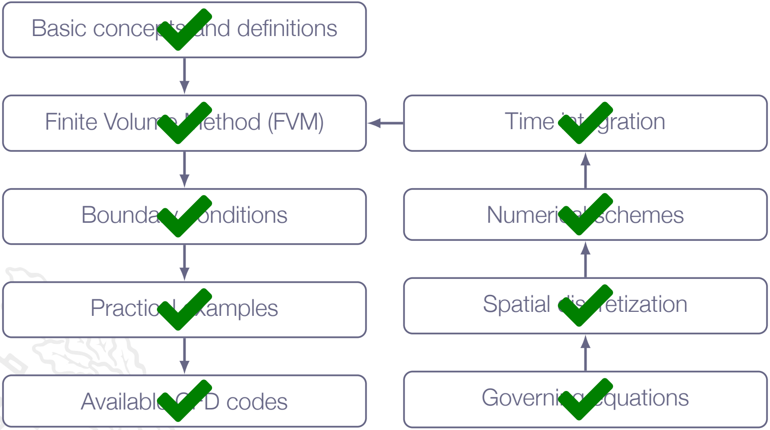
Explicit time stepping:

- Runge-Kutta time stepping

Implicit time stepping:

- Iterative solver based on Algebraic Multi-Grid (AGM)

Roadmap - The Time-Marching Technique

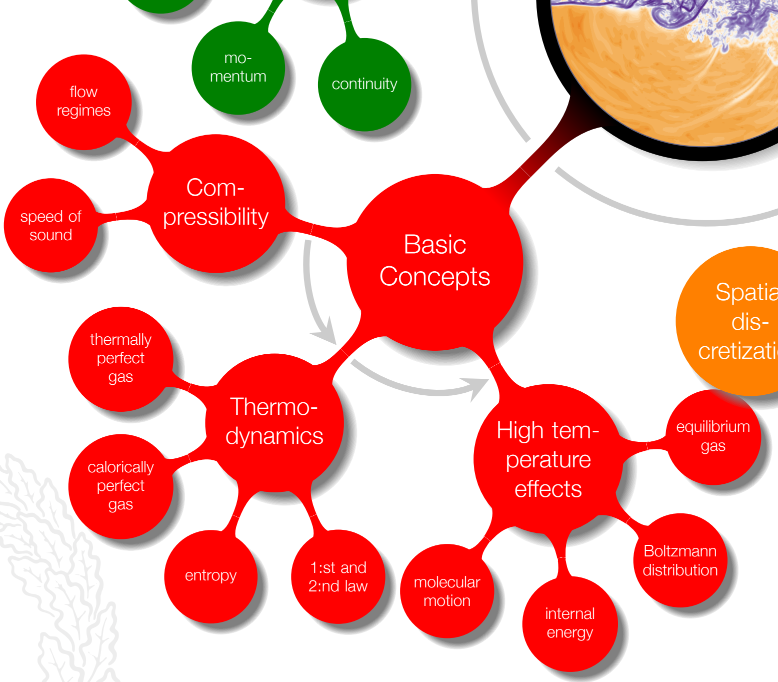


LECTURE 14

Chapter 16
Properties of High-Temperature Gases

Chapter 17
High-Temperature Flows: Basic Examples

Overview



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molecular
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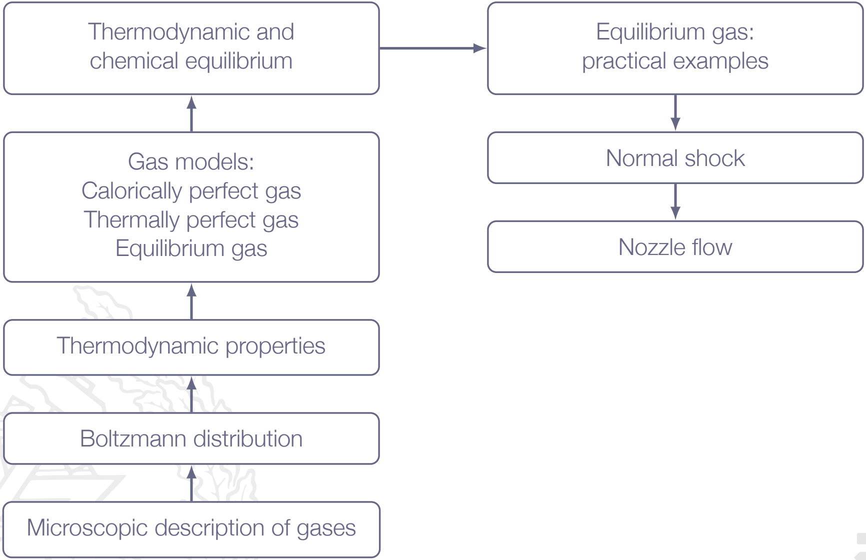
internal
energy
Boltzmann
distribution

Addressed Learning Outcomes

6 Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases

A deep dive into the theory behind the definitions of calorically perfect gas, thermally perfect gas, and other models

Roadmap - High Temperature Effects



Properties of High-Temperature Gases

Applications:

- Rocket nozzle flows
- Reentry vehicles
- Shock tubes / Shock tunnels
- Internal combustion engines

Properties of High-Temperature Gases

Example: Reentry vehicle

Mach 32.5
Air
Calorically perfect gas
$T_\infty = 283$

Table A.2 $\Rightarrow T_s/T_\infty = 206$

$T_\infty = 283 \Rightarrow T_s = 58\,300$ K

Properties of High-Temperature Gases

Example: Reentry vehicle

Mach 32.5
Air
Calorically perfect gas
$T_\infty = 283$

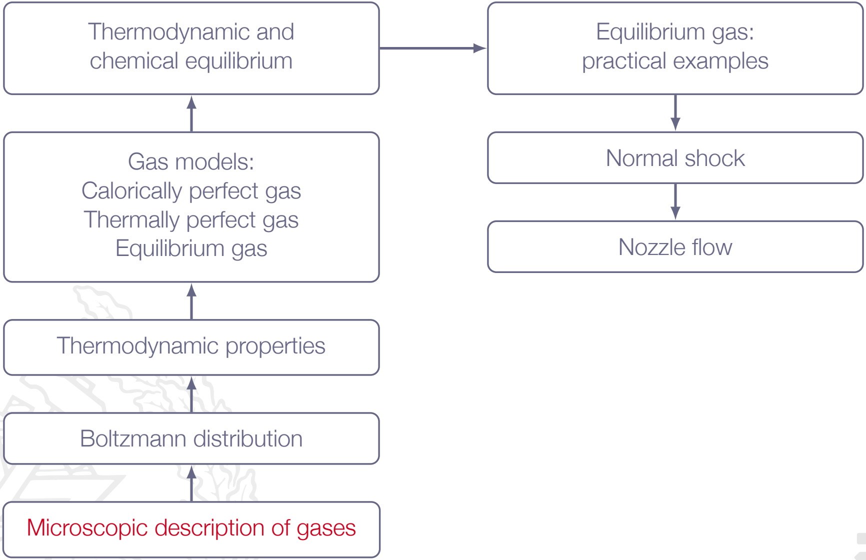
Table A.2 $\Rightarrow T_s/T_\infty = 206$

$T_\infty = 283 \Rightarrow T_s = 58\,300$ K

A more correct value is $T_s = 11\,600$ K

Something is fishy here!

Roadmap - High Temperature Effects

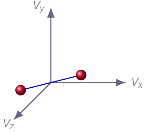


Chapter 16.2 Microscopic Description of Gases

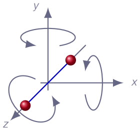
Microscopic Description of Gases

- Hard to make measurements
- Accurate, reliable theoretical models needed
- Available models do work quite well

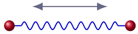
Molecular Energy



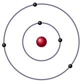
Translational kinetic energy
thermal degrees of freedom: 3



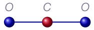
Rotational kinetic energy
thermal degrees of freedom:
2 for diatomic gases
2 for linear polyatomic gases
3 for non-linear polyatomic gases



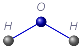
Vibrational energy
(kinetic energy + potential energy)
thermal degrees of freedom: 2



Electronic energy of electrons in orbit
(kinetic energy + potential energy)



CO_2
linear polyatomic molecule



H_2O
non-linear polyatomic molecule

- Translational energy
- Rotational energy
  (only for molecules - not for mono-atomic gases)
- Vibrational energy
- Electronic energy

Molecular Energy

The energy for one molecule can be described by

$$\varepsilon' = \varepsilon'_{trans} + \varepsilon'_{rot} + \varepsilon'_{vib} + \varepsilon'_{el}$$

Results of quantum mechanics have shown that each energy is quantized *i.e.* they can exist only at discrete values

Not continuous! Might seem unintuitive

Molecular Energy

The lowest quantum numbers defines the zero-point energy for each mode

- for rotational energy the zero-point energy is exactly zero
- $\varepsilon'_{o_{trans}}$ is very small but finite - *at absolute zero, molecules still moves but not much*

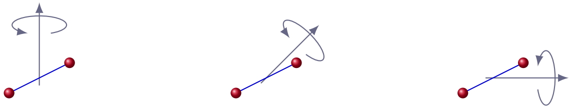
$$\varepsilon_{j_{trans}} = \varepsilon'_{j_{trans}} - \varepsilon'_{o_{trans}}$$

$$\varepsilon_{l_{vib}} = \varepsilon'_{l_{vib}} - \varepsilon'_{o_{vib}}$$

$$\varepsilon_{k_{rot}} = \varepsilon'_{k_{rot}}$$

$$\varepsilon_{m_{el}} = \varepsilon'_{m_{el}} - \varepsilon'_{o_{el}}$$

Energy States



- three cases with the same rotational energy
- different direction of angular momentum
- quantum mechanics ⇒ different distinguishable states
- a finite number of possible states for each energy level

Macrostates and Microstates

Macrostate:

- molecules collide and exchange energy $\Rightarrow$ the N_j distribution (the macrostate) will change over time
- some macrostates are more probable than other
- most probable macrostates (distribution) $\Rightarrow$ thermodynamic equilibrium

Microstate:

- same number of molecules in each energy level but different states
- the most probable macrostate is the one with the most possible microstates $\Rightarrow$ possible to find the most probable macrostate by counting microstates

Macrostates and Microstates

Macrostate I   Microstate I

ε'_o : ● ● ○ ○ ○ $(N_o = 2, g_o = 5)$

ε'_1 : ● ● ● ○ ● ● $(N_1 = 5, g_1 = 6)$

ε'_2 : ● ● ● ○ ○ $(N_2 = 3, g_2 = 5)$

$\vdots$

ε'_j : ○ ● ● $(N_j = 2, g_j = 3)$

Macrostates and Microstates

Macrostate I   Microstate II

ε'_o : ○ ● ○ ○ ● $(N_o = 2, g_o = 5)$

ε'_1 : ● ○ ● ● ● ● $(N_1 = 5, g_1 = 6)$

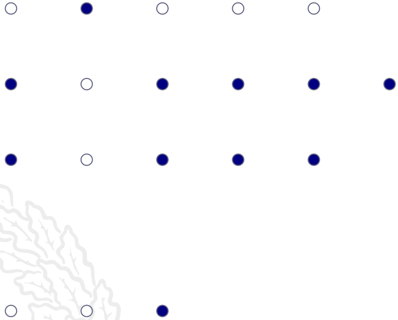
ε'_2 : ○ ○ ● ● ● $(N_2 = 3, g_2 = 5)$

$\vdots$

ε'_j : ○ ● ● $(N_j = 2, g_j = 3)$

Macrostates and Microstates

Macrostate II   Microstate I



ε_o' : $(N_o = 1, g_o = 5)$

ε_1' : $(N_1 = 5, g_1 = 6)$

ε_2' : $(N_2 = 4, g_2 = 5)$

$\vdots$

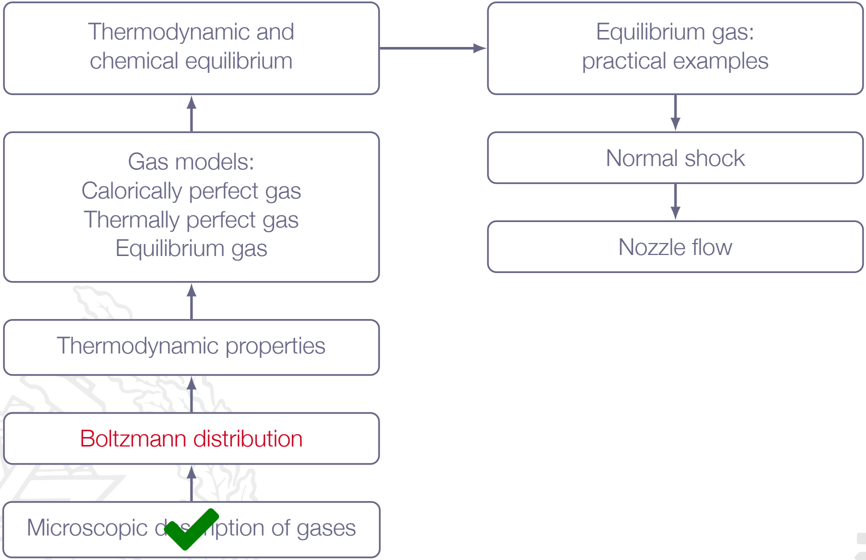
ε_j' : $(N_j = 1, g_j = 3)$

Macrostates and Microstates

$$N = \sum_j N_j$$

$$E = \sum_j \varepsilon_j' N_j$$

Roadmap - High Temperature Effects



Chapter 16.5
The Limiting Case: Boltzmann Distribution

Boltzmann Distribution

The Boltzmann distribution:

$$N_j^* = N \frac{g_j \mathrm{e}^{-\varepsilon_j/kT}}{Q}$$

where $Q = f(T, V)$ is the state sum defined as

$$Q \equiv \sum_j g_j \mathrm{e}^{-\varepsilon_j/kT}$$

g_j is the number of degenerate states, ε_j is the energy above zero-level ($\varepsilon_j = \varepsilon_j' - \varepsilon_o$), and k is the Boltzmann constant

Boltzmann Distribution

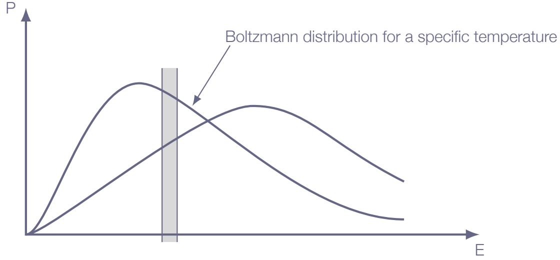
The Boltzmann distribution:

$$N_j^* = N\frac{g_j \mathrm{e}^{-\varepsilon_j/kT}}{Q}$$

For molecules or atoms of a given species, quantum mechanics says that a set of well-defined energy levels ε_j exists, over which the molecules or atoms can be distributed at any given instant, and that each energy level has a certain number of energy states, g_j.

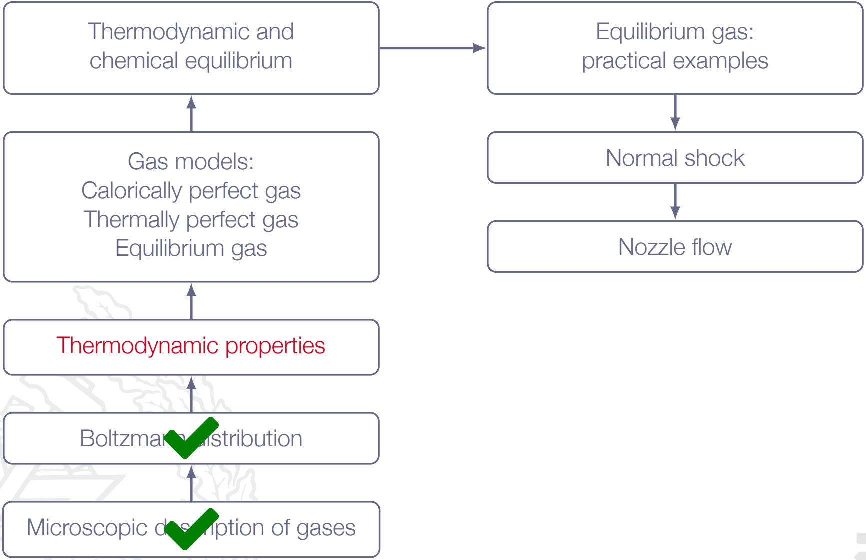
For a system of N molecules or atoms at a given T and V, N_j^* are the number of molecules or atoms in each energy level ε_j when the system is in thermodynamic equilibrium.

Boltzmann Distribution



- At temperatures above $\sim$ 5K, molecules are distributed over many energy levels, and therefore the states are generally sparsely populated ($N_j \ll g_j$)
- Higher energy levels become more populated as temperature increases

Roadmap - High Temperature Effects



Chapter 16.6 - 16.8 Evaluation of Gas Thermodynamic Properties

Internal Energy

The internal energy is calculated as

$$E = NkT^2 \left(\frac{\partial \ln Q}{\partial T}\right)_V$$

The internal energy per unit mass is obtained as

$$e = \frac{E}{M} = \frac{NkT^2}{Nm}\left(\frac{\partial \ln Q}{\partial T}\right)_V = \left\{\frac{k}{m} = R\right\} = RT^2 \left(\frac{\partial \ln Q}{\partial T}\right)_V$$

Internal Energy - Translation

$$\varepsilon'_{trans} = \frac{h^2}{8m}\left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2}\right)$$

$n_1 - n_3$	quantum numbers (1,2,3,...)
$a_1 - a_3$	linear dimensions that describes the size of the system
h	Planck's constant
m	mass of the individual molecule

$$\Rightarrow \cdots \Rightarrow$$

$$Q_{trans} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

Internal Energy - Translation

$$Q_{trans} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

$$\ln Q_{trans} = \frac{3}{2}\ln T + \frac{3}{2}\ln\frac{2\pi mk}{h^2} + \ln V \Rightarrow$$

$$\left(\frac{\partial \ln Q_{trans}}{\partial T}\right)_V = \frac{3}{2}\frac{1}{T} \Rightarrow$$

$$e_{trans} = RT^2\left(\frac{\partial \ln Q_{trans}}{\partial T}\right)_V = RT^2\frac{3}{2T} = \frac{3}{2}RT$$

Internal Energy - Rotation

$$\varepsilon'_{rot} = \frac{h^2}{8\pi^2 I} J(J+1)$$

J	rotational quantum number (0,1,2,...)
I	moment of inertia (tabulated for common molecules)
h	Planck's constant

$$\Rightarrow \cdots \Rightarrow$$

$$Q_{rot} = \frac{8\pi^2 IkT}{h^2}$$

Internal Energy - Rotation

$$Q_{rot} = \frac{8\pi^2 IkT}{h^2}$$

$$\ln Q_{rot} = \ln T + \ln \frac{8\pi^2 Ik}{h^2} \Rightarrow$$

$$\left(\frac{\partial \ln Q_{rot}}{\partial T}\right)_V = \frac{1}{T} \Rightarrow$$

$$e_{rot} = RT^2 \left(\frac{\partial \ln Q_{rot}}{\partial T}\right)_V = RT^2 \frac{1}{T} = RT$$

Internal Energy - Vibration

$$\varepsilon'_{vib} = h\nu \left(n + \frac{1}{2} \right)$$

n	vibrational quantum number (0,1,2,...)
ν	fundamental vibrational frequency (tabulated for common molecules)
h	Planck's constant

$$\Rightarrow \cdots \Rightarrow$$

$$Q_{vib} = \frac{1}{1 - e^{-h\nu/kT}}$$

Internal Energy - Vibration

$$Q_{vib} = \frac{1}{1 - e^{-h\nu/kT}}$$

$$\ln Q_{vib} = -\ln(1 - e^{-h\nu/kT}) \Rightarrow$$

$$\left(\frac{\partial \ln Q_{vib}}{\partial T}\right)_V = \frac{h\nu/kT^2}{e^{h\nu/kT} - 1} \Rightarrow$$

$$e_{vib} = RT^2 \left(\frac{\partial \ln Q_{vib}}{\partial T}\right)_V = RT^2 \frac{h\nu/kT^2}{e^{h\nu/kT} - 1} = \frac{h\nu/kT}{e^{h\nu/kT} - 1} RT$$

$$\lim_{T \to \infty} \frac{h\nu/kT}{e^{h\nu/kT} - 1} = 1 \Rightarrow e_{vib} \leq RT$$

Specific Heat

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el}$$

$$e = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

$$C_v \equiv \left(\frac{\partial e}{\partial T}\right)_V$$

Specific Heat

Molecules with only translational and rotational energy

$$e = \frac{3}{2}RT + RT = \frac{5}{2}RT \Rightarrow C_v = \frac{5}{2}R$$

$$C_p = C_v + R = \frac{7}{2}R$$

$$\gamma = \frac{C_p}{C_v} = \frac{7}{5} = 1.4$$

Specific Heat

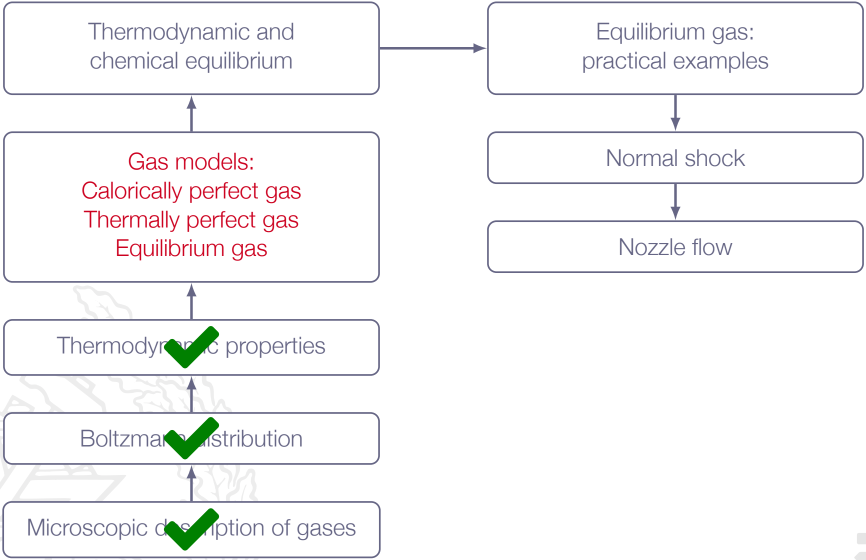
Mono-atomic gases with only translational and rotational energy

$$e = \frac{3}{2}RT \Rightarrow C_v = \frac{3}{2}R$$

$$C_p = C_v + R = \frac{5}{2}R$$

$$\gamma = \frac{C_p}{C_v} = \frac{5}{3} = 1\frac{2}{3} \simeq 1.67$$

Roadmap - High Temperature Effects



Calorically Perfect Gas

- In general, only translational and rotational modes of molecular excitation
- Translational and rotational energy levels are sparsely populated, according to Boltzmann distribution (the Boltzmann limit)
- Vibrational energy levels are practically unpopulated (except for the zero level)
- Characteristic values of γ for each type of molecule, *e.g.* mono-atomic gas, di-atomic gas, tri-atomic gas, etc
  - *He*, *Ar*, *Ne*, ... - mono-atomic gases ($\gamma = 5/3$)
  - H_2, O_2, N_2, ... - di-atomic gases ($\gamma = 7/5$)
  - H_2O (gaseous), CO_2, ... - tri-atomic gases ($\gamma < 7/5$)

Calorically Perfect Gas

$$p = \rho RT \quad \begin{aligned} e &= C_v T \\ h &= C_p T \\ h &= e + p/\rho \end{aligned}$$

$$C_p - C_v = R$$

$$\gamma = C_p / C_v$$

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

γ, R, C_v, and C_p are constants

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$$

Thermally Perfect Gas

- In general, only translational, rotational and vibrational modes of molecular excitation
- Translational and rotational energy levels are sparsely populated, according to Boltzmann distribution (the Boltzmann limit)
- The population of the vibrational energy levels approaches the Boltzmann limit as temperature increases
- Temperature dependent values of γ for all types of molecules except mono-atomic (no vibrational modes possible)

Thermally Perfect Gas

$$p = \rho RT \qquad \begin{array}{l} e = e(T) \\ h = h(T) \\ h = e + p/\rho \end{array} \qquad \begin{array}{l} C_v = de/dT \\ C_p = dh/dT \end{array}$$

$$C_p - C_v = R$$

$$\gamma = C_p/C_v$$

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

R is constant

γ, C_v, and C_p are variable (functions of T)

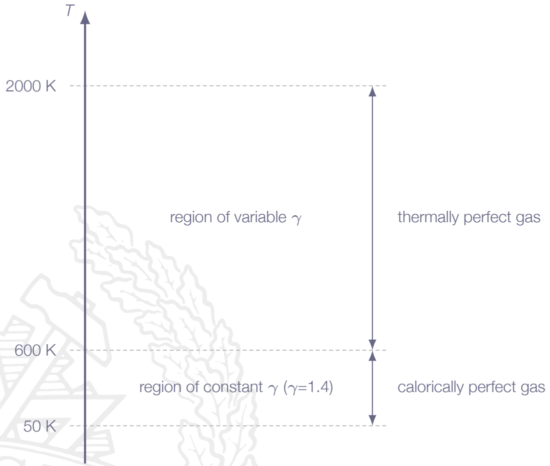
$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$$

High-Temperature Effects

Example: properties of air

Thermally perfect gas:
e and *h* are non-linear functions of *T*

the temperatur range represents standard atmospheric pressure (lower pressure gives lower temperatures)



High-Temperature Effects

For cases where the vibrational energy is not negligible (high temperatures)

$$\lim_{T \to \infty} e_{vib} = RT \Rightarrow C_v = \frac{7}{2}R$$

However, chemical reactions and ionization will take place long before that

- Translational and rotational energy fully excited above ∼5 K
- Vibrational energy is non-negligible above 600 K
- Chemical reactions begin to occur above ∼2000 K

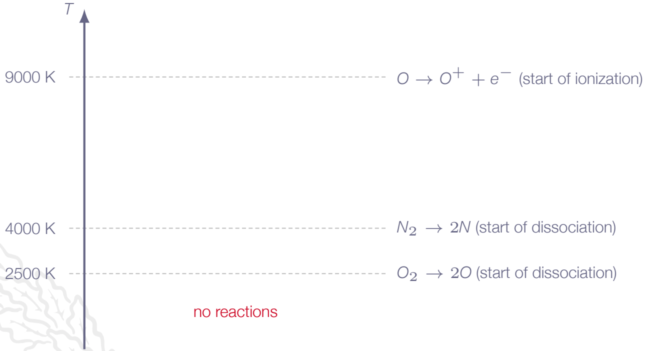
High-Temperature Effects

As temperature increase further vibrational energy becomes less important

Why is that so?

High-Temperature Effects

Example: properties of air (continued)



With increasing temperature, the gas becomes more and more mono-atomic which means that vibrational modes becomes less important

Equilibrium Gas

For temperatures $T >\sim 2500K$

- Air may be described as being in thermodynamic and chemical equilibrium (Equilibrium Gas)
  - reaction rates (time scales) low compared to flow time scales
  - reactions in both directions (example: $O_2 \rightleftharpoons 2O$)

- Tables must be used (Equilibrium Air Data) or special functions which have been made to fit the tabular data

Equilibrium Gas

How do we obtain a thermodynamic description?

$$p = p(R, T) \qquad e = e(\nu, T) \qquad C_v = \left(\frac{\partial e}{\partial T}\right)_\nu$$

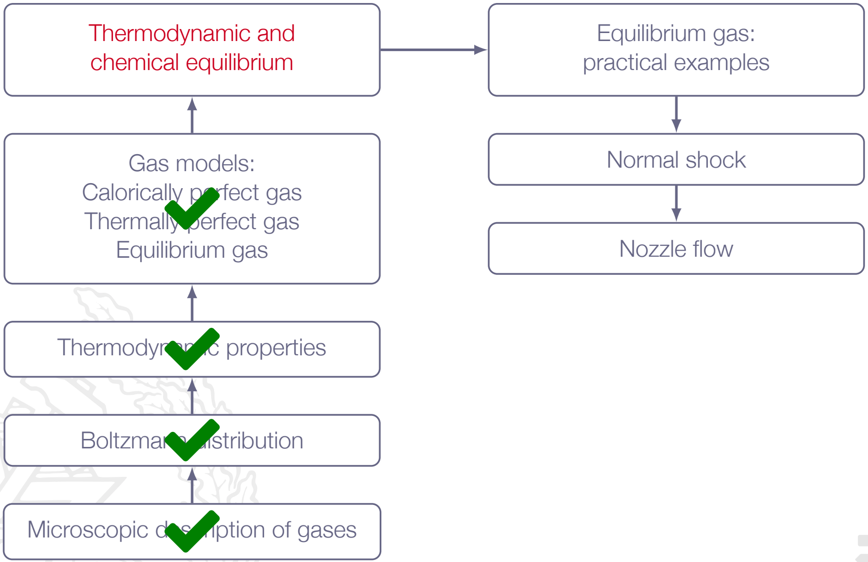
$$h = h(p, T)$$

$$h = e + \frac{p}{\rho} \qquad C_p = \left(\frac{\partial h}{\partial T}\right)_p$$

$$a_e^2 = \gamma RT \frac{1 + \frac{1}{p}\left(\frac{\partial e}{\partial \nu}\right)_T}{1 - \rho\left(\frac{\partial h}{\partial p}\right)_T} \qquad \gamma = \frac{C_p}{C_v} = \frac{\left(\frac{\partial h}{\partial T}\right)_p}{\left(\frac{\partial e}{\partial T}\right)_\nu} \qquad RT = \frac{p}{\rho}$$

Note: R is not a constant here *i.e.* this is not the ideal gas law

Roadmap - High Temperature Effects



Chapter 17.1 Thermodynamic and Chemical Equilibrium

Thermodynamic Equilibrium

Molecules are distributed among their possible energy states according to the Boltzmann distribution (which is a statistical equilibrium) for the given temperature of the gas

- extremely fast process (time and length scales of the molecular processes)
- much faster than flow time scales in general (not true inside shocks)

Thermodynamic Equilibrium

Global thermodynamic equilibrium:

- there are no gradients of p, T, ρ, $\mathbf{v}$, species concentrations
- "true thermodynamic equilibrium"

Local thermodynamic equilibrium:

- gradients can be neglected locally
- this requirement is fulfilled in most cases (hard not to get)

Chemical Equilibrium

Composition of gas (species concentrations) is fixed in time

- forward and backward rates of all chemical reactions are equal
- zero net reaction rates
- chemical reactions may be either slow or fast in comparison to flow time scale depending on the case studied

Chemical Equilibrium

Global chemical equilibrium:

- there are no gradients of species concentrations
- together with global thermodynamic equilibrium $\Rightarrow$ all gradients are zero

Local chemical equilibrium

- gradients of species concentrations can be neglected locally
- not always true - depends on reaction rates and flow time scales

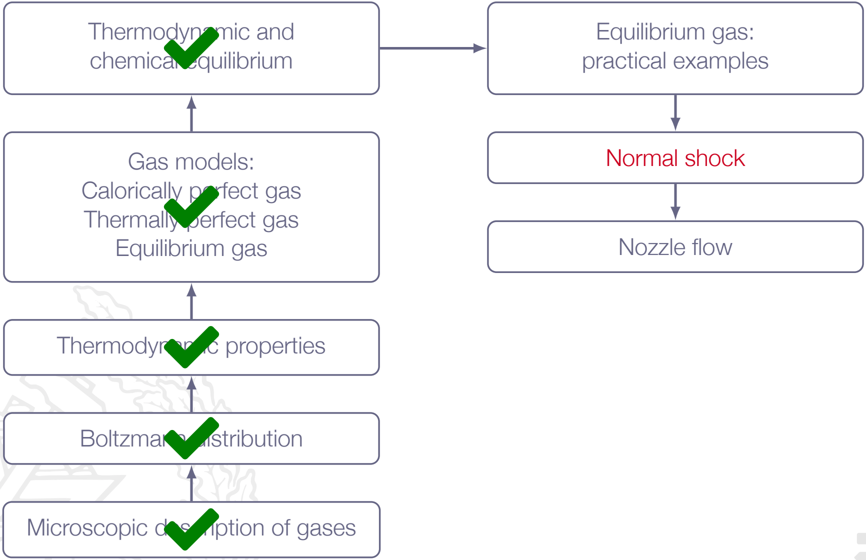
Thermodynamic and Chemical Equilibrium

Most common cases:

	Thermodynamic Equilibrium	Chemical Equilibrium	Gas Model
1	local thermodynamic equilibrium	local chemical equilibrium	equilibrium gas
2	local thermodynamic equilibrium	chemical non-equilibrium	finite rate chemistry
3	local thermodynamic equilibrium	frozen composition	frozen flow
4	thermodynamic non-equilibrium	frozen composition	vibrationally frozen flow

- length and time scales of flow decreases from 1 to 4
- Frozen composition $\Rightarrow$ no (or slow) reactions
- vibrationally frozen flow gives the same gas relations as calorically perfect gas!
  - no chemical reactions and unchanged vibrational energy
  - example: small nozzles with high-speed flow

Roadmap - High Temperature Effects



Chapter 17.2 Equilibrium Normal Shock Wave Flows

Equilibrium Normal Shock Wave Flows

Question: Is the equilibrium gas assumption OK?

Answer:

- for hypersonic flows with very little ionization in the shock region, it is a fair approximation
- not perfect, since the assumption of local thermodynamic and chemical equilibrium is not really true around the shock
- however, it gives a significant improvement compared to the calorically perfect gas assumption

Equilibrium Normal Shock Wave Flows

Basic relations (for all gases), stationary normal shock:

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\ h_1 \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2 \end{cases}$$

For equilibrium gas we have:

$$\begin{cases} \rho = \rho(p, h) \\ T = T(\rho, h) \end{cases}$$

(we are free to choose any two states as independent variables)

Equilibrium Normal Shock Wave Flows

Assume that ρ_1, u_1, p_1, T_1, and h_1 are known

$$u_2 = \frac{\rho_1 u_1}{\rho_2} \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 \left(\frac{\rho_1}{\rho_2} u_1\right)^2 + p_2 \Rightarrow$$

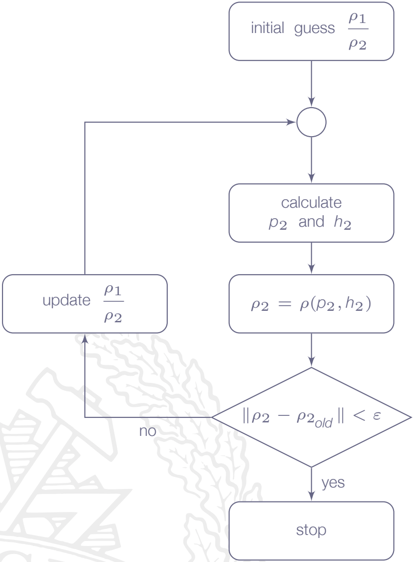
$$p_2 = p_1 + \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

Also

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} \left(\frac{\rho_1}{\rho_2} u_1\right)^2 \Rightarrow$$

$$h_2 = h_1 + \frac{1}{2} u_1^2 \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right)$$

Equilibrium Normal Shock Wave Flows



when converged:

$$\left.\begin{aligned} \rho_2 &= \rho(p_2, h_2) \\ T_2 &= T(\rho_2, h_2) \end{aligned}\right\} \Rightarrow$$

ρ_2, u_2, p_2, T_2, h_2 known

Equilibrium Air - Normal Shock

Tables of thermodynamic properties for different conditions are available

For a very strong shock case ($M_1 = 32$), the table below (Table 17.1) shows some typical results for equilibrium air

	calorically perfect gas ($\gamma = 1.4$)	equilibrium air
p_2/p_1	1233	1387
ρ_2/ρ_1	5.97	15.19
h_2/h_1	206.35	212.80
T_2/T_1	206.35	41.64

Equilibrium Air - Normal Shock

Analysis:

- Pressure ratio is comparable
- Density ratio differs by factor of 2.5
- Temperature ratio differs by factor of 5

Explanation:

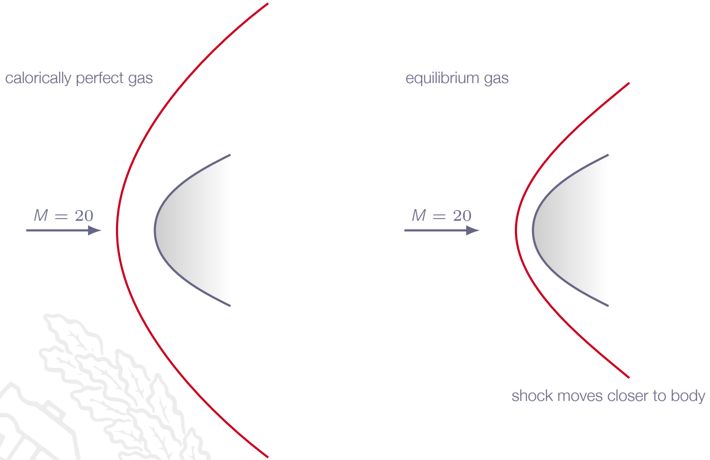
- Using equilibrium gas means that vibration, dissociation and chemical reactions are accounted for
- The chemical reactions taking place in the shock region lead to an ”absorption” of energy into chemical energy
  - drastically reducing the temperature downstream of the shock
  - this also explains the difference in density after the shock

Equilibrium Air - Normal Shock

Additional notes:

- For a normal shock in an equilibrium gas, the pressure ratio, density ratio, enthalpy ratio, temperature ratio, etc all depend on three upstream variables, *e.g.* u_1, p_1, T_1
- For a normal shock in a thermally perfect gas, the pressure ratio, density ratio, enthalpy ratio, temperature ratio, etc all depend on two upstream variables, *e.g.* M_1, T_1
- For a normal shock in a calorically perfect gas, the pressure ratio, density ratio, enthalpy ratio, temperature ratio, etc all depend on one upstream variable, *e.g.* M_1

Equilibrium Gas - Detached Shock



What's the reason for the difference in predicted shock position?

Equilibrium Gas - Detached Shock

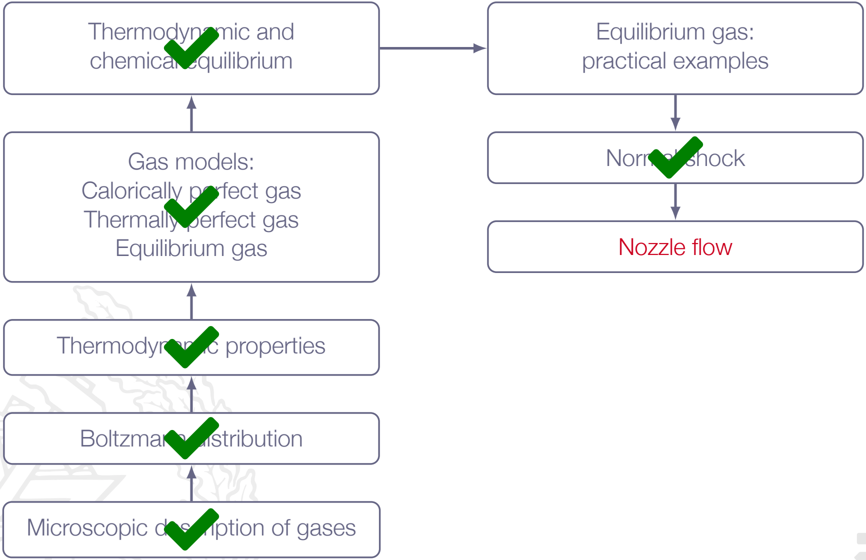
Calorically perfect gas:

- all energy ends up in translation and rotation $\Rightarrow$ increased temperature

Equilibrium gas:

- energy is absorbed by reactions $\Rightarrow$ does not contribute to the increase of gas temperature

Roadmap - High Temperature Effects



Chapter 17.3
Equilibrium Quasi-One-Dimensional Nozzle Flows

Equilibrium Quasi-1D Nozzle Flows

First question: Is chemically reacting gas also isentropic (for inviscid and adiabatic case)?

entropy equation: $Tds = dh - \nu dp$

Quasi-1D equations in differential form (all gases):

momentum equation: $dp = -\rho udu$

energy equation: $dh + udu = 0$

Equilibrium Quasi-1D Nozzle Flows

$$udu = -\frac{dp}{\rho} = -\nu dp$$

$$Tds = -udu - \nu dp = -udu + udu = 0 \Rightarrow$$

$$ds = 0$$

Isentropic flow!

Equilibrium Quasi-1D Nozzle Flows

Second question: Does the area-velocity relation also hold for a chemically reacting gas?

Isentropic process gives

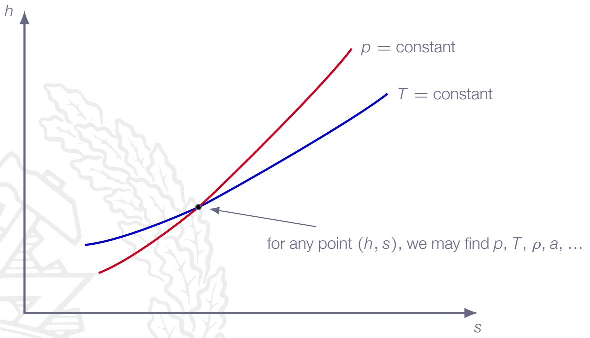
$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

$M = 1$ at nozzle throat still holds

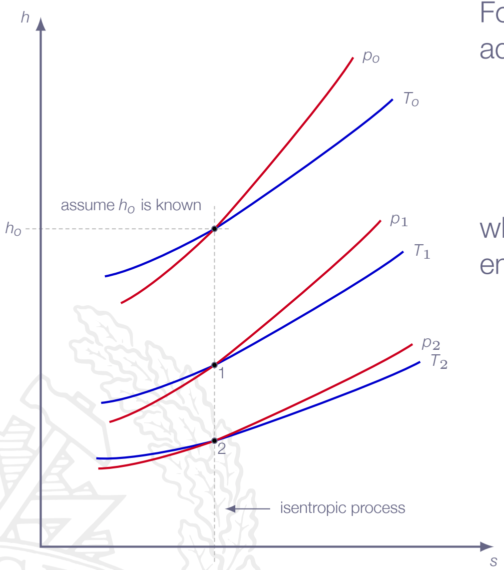
Equilibrium Quasi-1D Nozzle Flows

For general gas mixture in thermodynamic and chemical equilibrium, we may find tables or graphs describing relations between state variables.

Example: Mollier diagram



Equilibrium Quasi-1D Nozzle Flows



For steady-state inviscid adiabatic nozzle flow we have:

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 = h_o$$

where h_o is the reservoir enthalpy

Equilibrium Quasi-1D Nozzle Flows

At point 1 in Mollier diagram we have:

$$\frac{1}{2}u_1^2 = h_o - h_1 \Rightarrow u_1 = \sqrt{2(h_o - h_1)}$$

Assume that $u_1 = a_1$ (sonic conditions) gives

$$\rho_1 u_1 A_1 = \rho^* a^* A^*$$

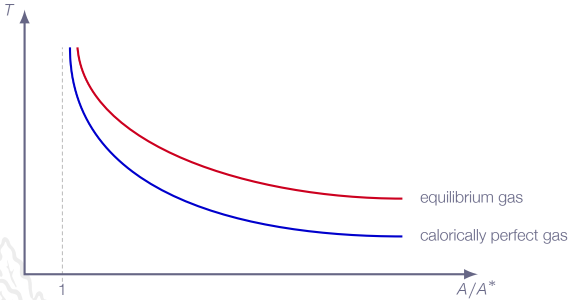
At any point along isentropic line, we have $u = \sqrt{2(h_o - h)}$ and ρ, p, T, a etc are all given which means that ρu is given

$$\frac{A}{A^*} = \frac{\rho^* a^*}{\rho u}$$

may be computed for any point along isentropic line

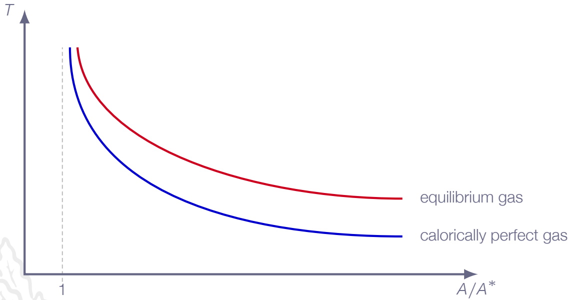
Equilibrium Quasi-1D Nozzle Flows

- Equilibrium gas gives higher T and more thrust
- During the expansion chemical energy is released due to shifts in the equilibrium composition



Equilibrium Quasi-1D Nozzle Flows

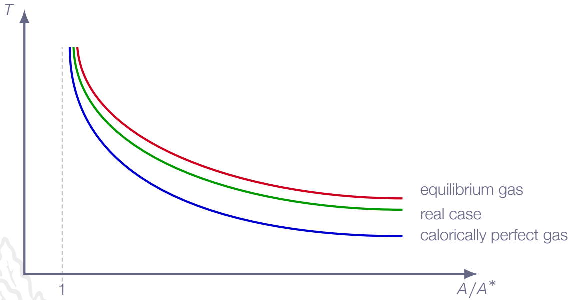
- Equilibrium gas gives higher T and more thrust
- During the expansion chemical energy is released due to shifts in the equilibrium composition



- Chemical and vibrational energy transfered to translation and rotation $\Rightarrow$ increased temperature

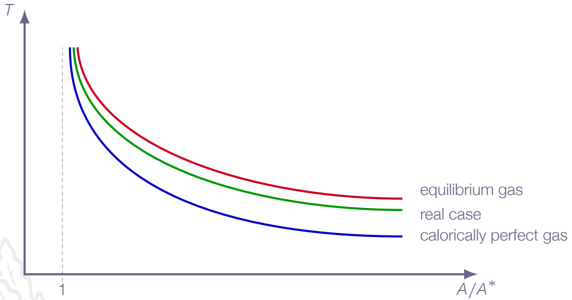
Equilibrium Quasi-1D Nozzle Flows - Reacting Mixture

Real nozzle flow with reacting gas mixture:



Equilibrium Quasi-1D Nozzle Flows - Reacting Mixture

Real nozzle flow with reacting gas mixture:



- Space nozzle applications: $u_e \approx 4000$ m/s
- Required prediction accuracy 5 m/s

Equilibrium Quasi-1D Nozzle Flows - Reacting Mixture

Equilibrium gas:

- very fast chemical reactions
- local thermodynamic and chemical equilibrium

Vibrationally frozen gas:

- very slow chemical reactions
  (no chemical reactions $\Rightarrow$ frozen gas)
- vibrational energy of molecules have no time to change
- calorically perfect gas!

Large Nozzles

High T_o, high p_o, high reactivity

Real case is close to equilibrium gas results

Example: Ariane 5 launcher, main engine (Vulcain 2)

- $H_2 + O_2 \rightarrow H_2O$ in principle, but many different radicals and reactions involved (at least ∼10 species, ∼20 reactions)
- $T_o \sim 3600\ K$, $p_o \sim 120\ bar$
- Length scale ∼ a few meters
- Gas mixture is quite close to equilibrium conditions all the way through the expansion

Small Nozzles

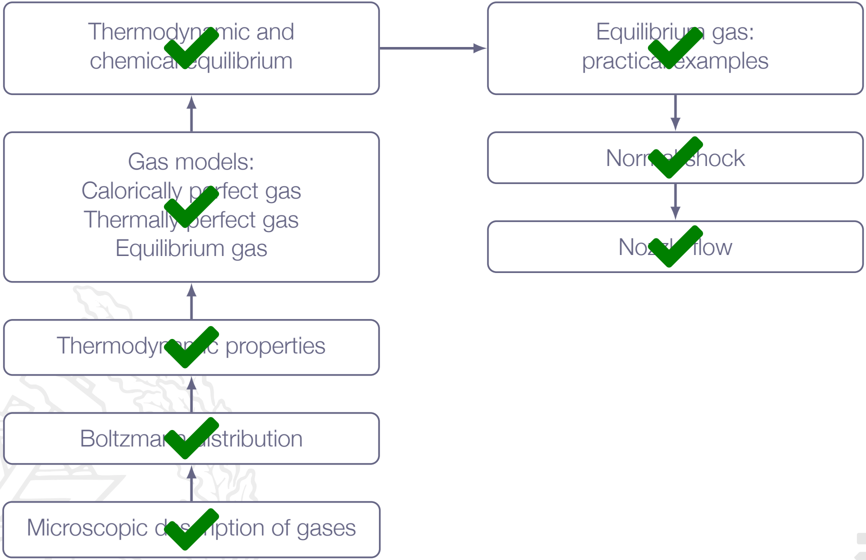
Low T_o, low p_o, lower reactivity

Real case is close to frozen flow results

Example:

Small rockets on satellites (for maneuvering, orbital adjustments, etc)

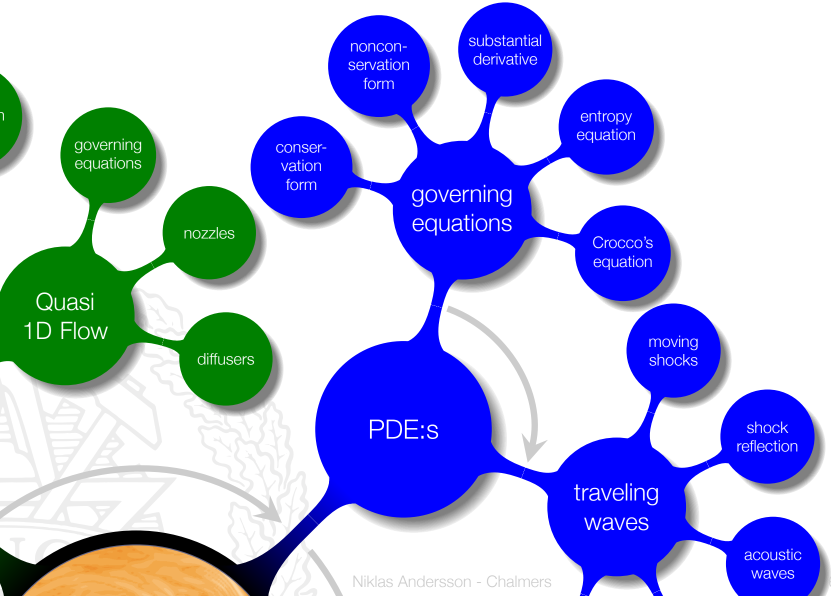
Roadmap - High Temperature Effects



LECTURE 15

Chapter 6
Differential Conservation Equations for Inviscid Flows

Overview



governing equations
nozzles
Quasi 1D Flow
diffusers
noncon-servation form
substantial derivative
entropy equation
conser-vation form
governing equations
Crocco's equation
moving shocks
PDE:s
shock reflection
traveling waves
acoustic waves

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

the governing equations for compressible flows on differential form - finally ...

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

PDE:s on conservation form

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on non-conservation form

The entropy equation

Crocco's theorem

Chapter 6.2
Differential Equations in Conservation Form

Differential Equations in Conservation Form

Basic principle to derive PDE:s in conservation form:

- Start with control volume formulation
- Convert to volume integral via Gauss Theorem
- Arbitrary control volume implies that integrand equals to zero everywhere

Continuity Equation

Mass conservation:

Control volume formulation

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V} + \oiint\limits_{\partial\Omega} \rho \mathbf{v}\cdot\mathbf{n}dS = 0$$

where Ω is a fixed control volume

Applying Gauss Theorem gives

$$\oiint\limits_{\partial\Omega} \rho \mathbf{v}\cdot\mathbf{n}dS = \iiint\limits_{\Omega} \nabla\cdot(\rho\mathbf{v})d\mathscr{V}$$

Also,

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V} = \iiint\limits_{\Omega} \frac{\partial\rho}{\partial t}d\mathscr{V}$$

Continuity Equation

Therefore

$$\iiint\limits_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimally small and thus

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

which is the continuity equation

Momentum Equation

Momentum conservation:

Control volume formulation

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}$$

where Ω is a fixed control volume

Applying Gauss Theorem gives

$$\oiint\limits_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=\iiint\limits_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})d\mathscr{V}\ ;\ \oiint\limits_{\partial\Omega}p\mathbf{n}dS=\iiint\limits_{\Omega}\nabla p d\mathscr{V}$$

Also,

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}=\iiint\limits_{\Omega}\frac{\partial}{\partial t}(\rho\mathbf{v})d\mathscr{V}$$

Momentum Equation

Therefore

$$\iiint\limits_{\Omega} \left[\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p - \rho \mathbf{f}\right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimally small and thus

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}}$$

which is the momentum equation

Momentum Equation

In cartesian form ($\mathbf{v} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$):

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \mathbf{v}) + \frac{\partial p}{\partial x} = \rho f_x$$
$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v \mathbf{v}) + \frac{\partial p}{\partial y} = \rho f_y$$
$$\frac{\partial}{\partial t}(\rho w) + \nabla \cdot (\rho w \mathbf{v}) + \frac{\partial p}{\partial z} = \rho f_z$$

Momentum Equation

or expanded:

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) + \frac{\partial p}{\partial x} &= \rho f_x \\
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho vw) + \frac{\partial p}{\partial y} &= \rho f_y \\
\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww) + \frac{\partial p}{\partial z} &= \rho f_z
\end{aligned}
$$

Momentum Equation

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}$$

$$\begin{bmatrix} (\rho uu + p) & \rho uv & \rho uw \\ \rho vu & (\rho vv + p) & \rho vw \\ \rho wu & \rho wv & (\rho ww + p) \end{bmatrix} = \rho \mathbf{v}\mathbf{v} + p\mathbf{I}$$

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = \rho \mathbf{f}}$$

Energy Equation

Energy conservation:

Control volume formulation

$$\frac{d}{dt}\iiint_{\Omega}\rho e_o d\mathscr{V} + \oiint_{\partial\Omega}\rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

where Ω is a fixed control volume

Applying Gauss Theorem gives

$$\oiint_{\partial\Omega}\rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint_{\Omega}\nabla\cdot(\rho h_o\mathbf{v})d\mathscr{V}$$

Also,

$$\frac{d}{dt}\iiint_{\Omega}\rho e_o d\mathscr{V} = \iiint_{\Omega}\frac{\partial}{\partial t}(\rho e_o)d\mathscr{V}$$

Energy Equation

Therefore

$$\iiint\limits_{\Omega} \left[\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) - \rho(\mathbf{f} \cdot \mathbf{v})\right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimally small and thus

$$\boxed{\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v})}$$

which is the energy equation

Partial Differential Equations in Conservation Form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}$$

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v})$$

These equations are referred to as PDE:s on conservation form since they stem directly from the integral conservation equations applied to a fixed control volume

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on conservation form

PDE:s on non-conservation form

The entropy equation

Crocco's theorem

Chapter 6.4
Differential Equations in Non-Conservation Form

The Substantial Derivative

Introducing the substantial derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

"... the time rate of change of any quantity associated with a particular moving fluid element is given by the substantial derivative ..."

"... the properties of the fluid element are changing as it moves past a point in a flow because the flowfield itself may be fluctuating with time (the local derivative) and because the fluid element is simply on its way to another point in the flowfield where the properties are different (the convective derivative) ..."

Non-Conservation Form of Continuity Equation

Applying the substantial derivative operator to density gives

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{v}) = 0 \Rightarrow$$

$$\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0}$$

Non-Conservation Form of Continuity Equation

$$\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0}$$

"... the mass of a fluid element made up of a fixed set of particles (molecules or atoms) is constant as the fluid element moves through space ..."

Non-Conservation Form of Momentum Equation

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = \rho \mathbf{f} \Rightarrow$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v}(\nabla \cdot \rho \mathbf{v}) + \nabla p = \rho \mathbf{f} \Rightarrow$$

$$\underbrace{\rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right]}_{=\frac{D\mathbf{v}}{Dt}} + \mathbf{v} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}\right]}_{=0} + \nabla p = \rho \mathbf{f}$$

$$\boxed{\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f}}$$

Non-Conservation Form of Energy Equation

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q}$$

$$h_o = e_o + \frac{p}{\rho} \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho e_o \mathbf{v}) + \nabla \cdot (p\mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q} \Rightarrow$$

$$\rho \frac{\partial e_o}{\partial t} + e_o \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla e_o + e_o \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (p\mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q} \Rightarrow$$

$$\rho \underbrace{\left[\frac{\partial e_o}{\partial t} + \mathbf{v} \cdot \nabla e_o\right]}_{=\frac{De_o}{Dt}} + e_o \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right]}_{=0} + \nabla \cdot (p\mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q}$$

Non-Conservation Form of Energy Equation

$$\rho\frac{De_o}{Dt} + \nabla\cdot(p + \mathbf{v}) = \rho\mathbf{f}\cdot\mathbf{v} + \rho\dot{q}$$

$$e_o = e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} \Rightarrow$$

$$\rho\frac{De}{Dt} + \rho\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt} + \nabla\cdot(p\mathbf{v}) = \rho\mathbf{f}\cdot\mathbf{v} + \rho\dot{q}$$

Using the momentum equation, $\left(\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f}\right)$, gives

$$\rho\frac{De}{Dt} - \mathbf{v}\cdot\nabla p + \rho\mathbf{f}\cdot\mathbf{v} + \mathbf{v}\cdot\nabla p + p(\nabla\cdot\mathbf{v}) = \rho\mathbf{f}\cdot\mathbf{v} + \rho\dot{q} \Rightarrow$$

$$\boxed{\frac{De}{Dt} + \frac{p}{\rho}(\nabla\cdot\mathbf{v}) = \dot{q}}$$

Non-Conservation Form of Energy Equation

$$\frac{De}{Dt} + \frac{p}{\rho}(\nabla \cdot \mathbf{v}) = \dot{q}$$

From the continuity equation we get

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \Rightarrow \nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{D\rho}{Dt} \Rightarrow$$

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \dot{q} \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$\boxed{\frac{De}{Dt} = \dot{q} - p\frac{D\nu}{Dt}}$$

where $\nu = 1/\rho$

Compare with first law of thermodynamics: $de = \delta q - \delta W$

Non-Conservation Form of Energy Equation

If we instead express the energy equation in terms of enthalpy:

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$h = e + \frac{p}{\rho} \Rightarrow \frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow$$

$$\boxed{\frac{Dh}{Dt} = \dot{q} + \frac{1}{\rho}\frac{Dp}{Dt}}$$

Non-Conservation Form of Energy Equation

and total enthalpy ...

$$h_o = h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} \Rightarrow \frac{Dh_o}{Dt} = \frac{Dh}{Dt} + \mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}$$

From the momentum equation we get

$$\rho\frac{D\mathbf{v}}{Dt} + \nabla p = \mathbf{f} \Rightarrow \frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{f} \Rightarrow$$

$$\frac{Dh_o}{Dt} = \underbrace{\frac{Dh}{Dt}}_{\dot{q}+\frac{1}{\rho}\frac{Dp}{Dt}} - \frac{1}{\rho}\mathbf{v}\cdot\nabla p + \mathbf{f}\cdot\mathbf{v} \Rightarrow$$

$$\frac{Dh_o}{Dt} = \dot{q} + \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v}\cdot\nabla p\right] + \mathbf{f}\cdot\mathbf{v}$$

Non-Conservation Form of Energy Equation

$$\frac{Dh_o}{Dt} = \dot{q} + \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v}\cdot\nabla p\right] + \mathbf{f}\cdot\mathbf{v}$$

Expanding the substantial derivative $\frac{Dp}{Dt}$ gives

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p \Rightarrow$$

$$\boxed{\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{f}\cdot\mathbf{v}}$$

Let's examine the above relation ...

Non-Conservation Form of Energy Equation

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{f} \cdot \mathbf{v}$$

The total enthalpy of a moving fluid element in an inviscid flow can change due to

- unsteady flow: $\partial p/\partial t \neq 0$
- heat transfer: $\dot{q} \neq 0$
- body forces: $\mathbf{f} \cdot \mathbf{v} \neq 0$

Non-Conservation Form of Energy Equation

Adiabatic flow and without body forces ⇒

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

Steady-state adiabatic flow without body forces ⇒

$$\frac{Dh_o}{Dt} = 0$$

h_o is constant along streamlines!

Additional Form of Energy Equation

Start from

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

Calorically perfect gas:

$$e = C_v T \; ; \; C_v = \frac{R}{\gamma - 1} \; ; \; p = \rho R T \; ; \; \gamma, R = const$$

$$\frac{De}{Dt} = C_v \frac{DT}{Dt} = \frac{R}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho R}\right) = \frac{1}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho}\right) \Rightarrow$$

$$\frac{1}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow$$

Additional Form of Energy Equation

$$\frac{1}{\gamma - 1}\left[p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)\frac{Dp}{Dt}\right] = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)\frac{Dp}{Dt} = (\gamma - 1)\dot{q} - (\gamma - 1)p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$\gamma p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)\frac{Dp}{Dt} = (\gamma - 1)\dot{q}$$

Additional Form of Energy Equation

Continuity:

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v}) \Rightarrow \frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}(\nabla \cdot \mathbf{v}) \Rightarrow$$

$$\frac{\gamma p}{\rho}(\nabla \cdot \mathbf{v}) + \left(\frac{1}{\rho}\right)\frac{Dp}{Dt} = (\gamma - 1)\dot{q}$$

Additional Form of Energy Equation

$$\frac{Dp}{Dt} + \gamma p(\nabla \cdot \mathbf{v}) = (\gamma - 1)\rho\dot{q}$$

Adiabatic flow (no added heat):

$$\boxed{\frac{Dp}{Dt} + \gamma p(\nabla \cdot \mathbf{v}) = 0}$$

Non-conservation form (calorically perfect gas)

Conservation Form

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0$$

where $Q(x, y, z, t)$, $E(x, y, z, t)$, ... may be scalar or vector fields

Example: the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

If an equation cannot be written in this form, it is said to be in non-conservation form

Euler Equations - Conservation Form

Continuity, momentum and energy equations in Cartesian coordinates, velocity components u, v, w (no body forces, no added heat)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu + p) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = 0$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv + p) + \frac{\partial}{\partial z}(\rho vw) = 0$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww + p) = 0$$

$$\frac{\partial}{\partial t}(\rho e_o) + \frac{\partial}{\partial x}(\rho h_o u) + \frac{\partial}{\partial y}(\rho h_o v) + \frac{\partial}{\partial z}(\rho h_o w) = 0$$

Euler Equations - Non-Conservation Form

Continuity, momentum and energy equations in Cartesian coordinates, velocity components u, v, w (no body forces, no added heat), calorically perfect gas

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + \gamma p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Conservation and Non-Conservation Form

The governing equations on non-conservation form are not, although the name might give that impression, less physically accurate than the equations on conservation form. The nomenclature comes from CFD where the equations on conservation form are preferred.

Conservation and Non-Conservation Form

Conservation forms are useful for:

1. Numerical methods for compressible flow
2. Theoretical understanding of non-linear waves (shocks etc)
3. Provide link between integral forms (control volume formulations) and PDE:s

Non-conservation forms are useful for:

1. Theoretical understanding of behavior of numerical methods
2. Theoretical understanding of boundary conditions
3. Analysis of linear waves (aero-acoustics)

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

PDE:s on conservation form

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on non-conservation form

The entropy equation

Crocco's theorem

Chapter 6.5
The Entropy Equation

The Entropy Equation

From the first and second law of thermodynamics we have

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

which is called the entropy equation

The Entropy Equation

Compare the entropy equation

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

with the energy equation (inviscid flow):

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

we see that

$$T\frac{Ds}{Dt} = \dot{q}$$

The Entropy Equation

If $\dot{q} = 0$ (adiabatic flow) then

$$\frac{Ds}{Dt} = 0$$

i.e., entropy is constant for moving fluid element

Furthermore, if the flow is steady we have

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla)s = (\mathbf{v} \cdot \nabla)s = 0$$

i.e., entropy is constant along streamlines

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

→ PDE:s on conservation form ✔

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$ ✔

→ PDE:s on non-conservation form ✔

→ The entropy equation ✔

→ Crocco's theorem

Chapter 6.6
Crocco's Theorem

Crocco's Theorem

"... a relation between gradients of total enthalpy, gradients of entropy, and flow rotation ..."

Crocco's Theorem

Momentum equation (no body forces)

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p$$

Writing out the substantial derivative gives

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p \Rightarrow \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p$$

First and second law of thermodynamics (energy equation)

$$dh = Tds + \frac{1}{\rho} dp$$

Replace differentials with a gradient operator

$$\nabla h = T \nabla s + \frac{1}{\rho} \nabla p \Rightarrow T \nabla s = \nabla h - \frac{1}{\rho} \nabla p$$

Crocco's Theorem

With pressure derivative from the momentum equation inserted in the energy equation we get

$$T\nabla s = \nabla h + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$h = h_o - \frac{1}{2}\mathbf{v} \cdot \mathbf{v} \Rightarrow \nabla h = \nabla h_o - \nabla(\frac{1}{2}\mathbf{v} \cdot \mathbf{v})$$

$$\nabla(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \times (\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\mathbf{A} = \mathbf{B} = \mathbf{v} \Rightarrow$$

$$\nabla(\mathbf{v} \cdot \mathbf{v}) = 2[\mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v})]$$

Crocco's Theorem

$$T\nabla s = \nabla h_o - \mathbf{v} \times (\nabla \times \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$\boxed{T\nabla s = \nabla h_o + \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v})}$$

Note: $\nabla \times \mathbf{v}$ is the vorticity of the fluid

the rotational motion of the fluid is described by the angular velocity $\omega = \frac{1}{2}(\nabla \times \mathbf{v})$

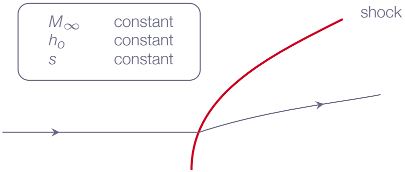
Crocco's Theorem

$$T\nabla s = \nabla h_o + \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v})$$

"... when a steady flow field has gradients of total enthalpy and/or entropy Crocco's theorem dramatically shows that it is rotational ..."

Crocco's Theorem - Example

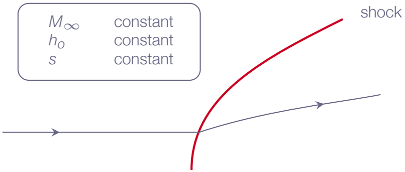
Curved stationary shock (steady-state flow)



- s is constant upstream of shock
- jump in s across shock depends on local shock angle
- s will vary from streamline to streamline downstream of shock
- $\nabla s \neq 0$ downstream of shock

Crocco's Theorem - Example

Curved stationary shock (steady-state flow)



- Total enthalpy upstream of shock
  - h_o is constant along streamlines
  - h_o is uniform
- Total enthalpy downstream of shock
  - h_o is uniform

$$\nabla h_o = 0$$

Crocco's Theorem - Example

Crocco's equation for steady-state flow:

$$T\nabla s = \nabla h_o - \mathbf{v} \times (\nabla \times \mathbf{v})$$

- $\mathbf{v} \times (\nabla \times \mathbf{v}) \neq 0$ downstream of a curved shock
- the rotation $\nabla \times \mathbf{v} \neq 0$ downstream of a curved shock

Explains why it is difficult to solve such problems by analytic means!

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on conservation form

PDE:s on non-conservation form

The entropy equation

Crocco's theorem