

# Compressible Flow - TME085

## Lecture 12

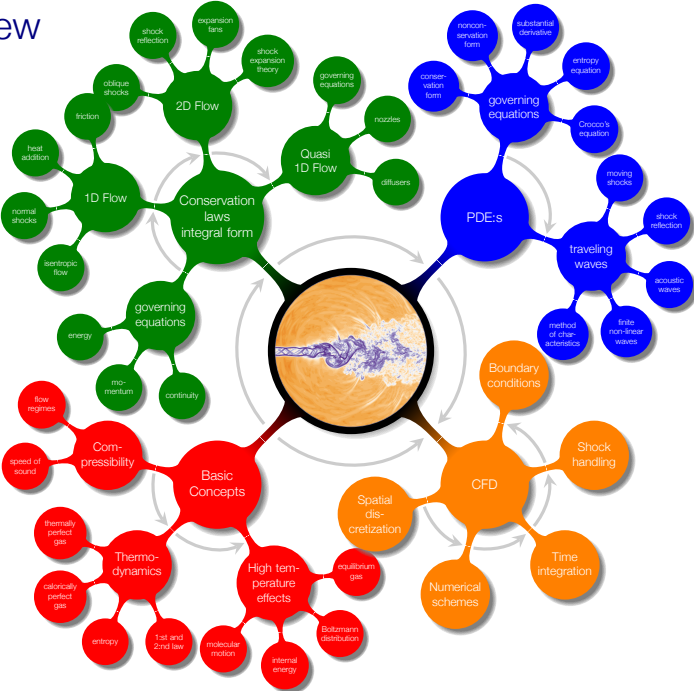
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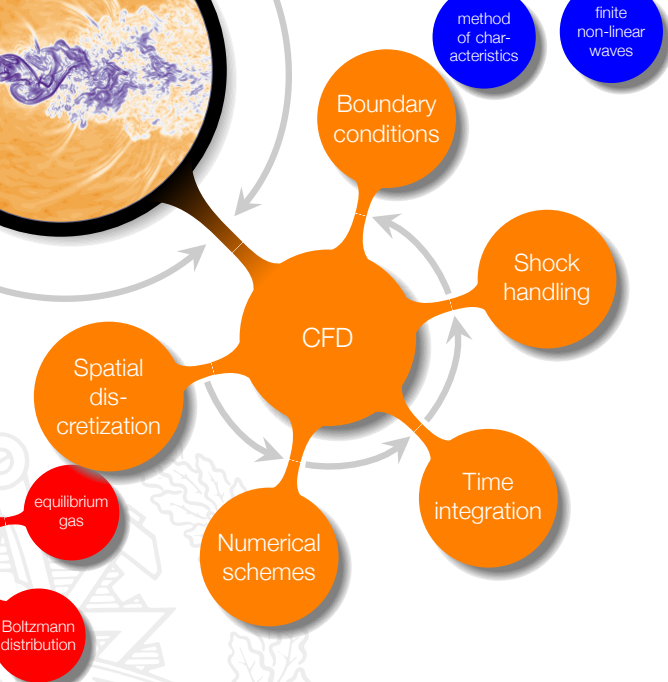
# Overview



# Chapter 12

## The Time-Marching Technique





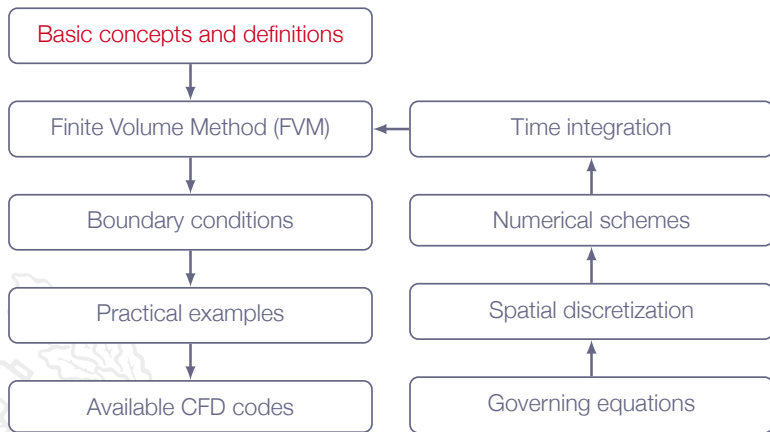
# Addressed Learning Outcomes

- 12 **Explain** the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions
- 15 **Explain** the limitations in fluid flow simulation software

*time for CFD!*



# Roadmap - The Time-Marching Technique



# The Time-Marching Technique

## Note:

*Anderson's text is here rather out-of-date, it was written during the 70's and has not really been updated since then. The additional material covered in this lecture is an attempt to amend this.*



# The Time-Marching Technique

The problems that we like to investigate numerically within the field of compressible flows can be categorized as

steady-state  
compressible flows

unsteady  
compressible flows

The **Time-marching method** is a solver framework that addresses both problem categories





# The Time-Marching Technique

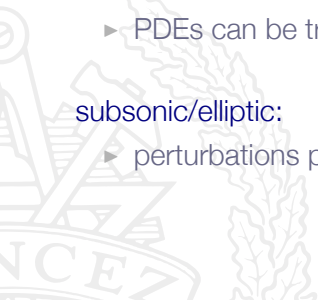
*The time-marching approach is a good alternative for simulating flows where there are both supersonic and subsonic regions*

supersonic/hyperbolic:

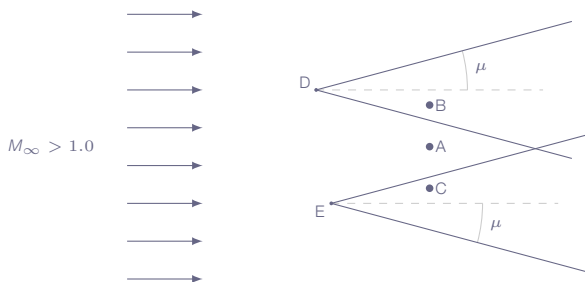
- ▶ perturbations propagate in preferred directions
- ▶ zone of influence/zone of dependence
- ▶ PDEs can be transformed into ODEs

subsonic/elliptic:

- ▶ perturbations propagate in all directions

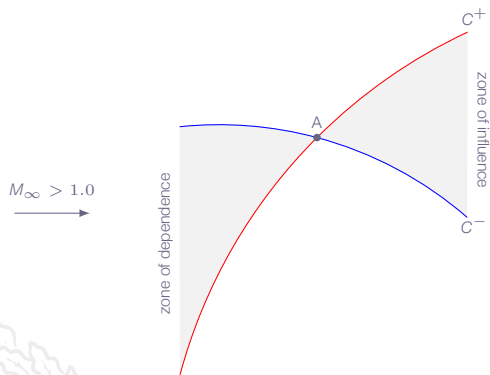


# Zone of Influence and Zone of Dependence



- ▶ A, B and C at the same axial position in the flow
- ▶ D and E are located upstream of A, B and C
- ▶ Mach waves generated at D will affect the flow in B but not in A and C
- ▶ Mach waves generated at E will affect the flow in C but not in A and B
- ▶ The flow in A is unaffected by the both D and E

# Zone of Influence and Zone of Dependence



The zone of **dependence** for point **A** and the zone of **influence** of point **A** are defined by  $C^+$  and  $C^-$  characteristic lines

# The Time-Marching Technique

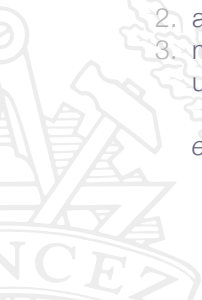
## Steady-state problems:

1. define simple initial solution
2. apply specified boundary conditions
3. march in time until steady-state solution is reached

## Unsteady problems:

1. apply specified initial solution
2. apply specified boundary conditions
3. march in time for specified total time to reach a desired unsteady solution

*establish fully developed flow before initiating data sampling*



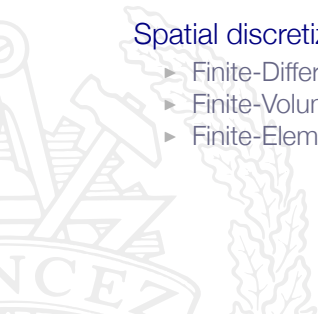
# Characterization of CFD Methods - Discretization

## Discretization in space and time:

- ▶ most common approach: Method of Lines
  1. discretize in space  $\Rightarrow$   
system of ordinary differential equations (ODEs)
  2. discretize in time  $\Rightarrow$   
time-stepping scheme for system of ODEs

## Spatial discretization techniques:

- ▶ Finite-Difference Method (FDM)
- ▶ Finite-Volume Method (FVM)
- ▶ Finite-Element Method (FEM)



# Characterization of CFD Methods - Time Stepping

Temporal discretization techniques:

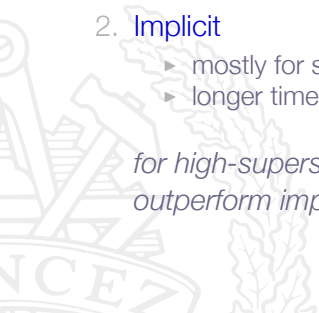
## 1. Explicit

- ▶ mostly for transonic/supersonic steady-state and unsteady flows
- ▶ short time steps
- ▶ usually very stable

## 2. Implicit

- ▶ mostly for subsonic/transonic steady-state flows
- ▶ longer time steps possible

*for high-supersonic flows, explicit solvers may very well outperform implicit solvers*



# Characterization of CFD Methods - Equations

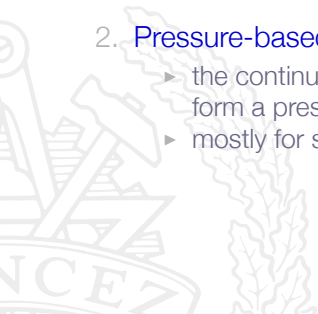
Equations solved:

## 1. Density-based

- ▶ solve for density in the continuity equation
- ▶ mostly for transonic/supersonic steady-state and unsteady flows

## 2. Pressure-based

- ▶ the continuity and momentum equations are combined to form a pressure correction equation
- ▶ mostly for subsonic/transonic steady-state flows



# Characterization of CFD Methods - Solver Approach

Solution procedure:

## 1. Fully coupled

- ▶ all equations (continuity, momentum, energy, ...) are solved simultaneously
- ▶ mostly for transonic/supersonic steady-state and unsteady flows

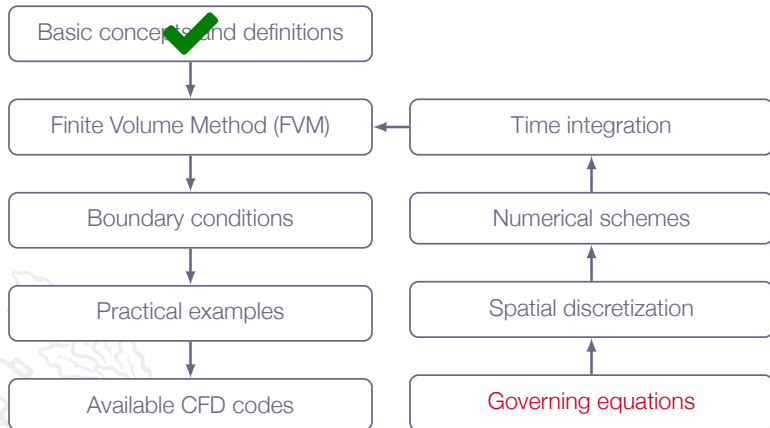
## 2. Segregated

- ▶ solve the equations in sequence
- ▶ mostly for subsonic steady-state flows





# Roadmap - The Time-Marching Technique



# Explicit Finite-Volume Method



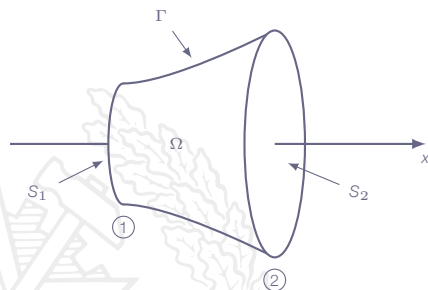
# Governing Equations



# Quasi-One-Dimensional Flow - Conceptual Idea

Introduce **cross-section-averaged flow quantities**  $\Rightarrow$   
all quantities depend on  $x$  only

$$A = A(x), \rho = \rho(x), u = u(x), p = p(x), \dots$$



- |          |                              |
|----------|------------------------------|
| $\Omega$ | control volume               |
| $S_1$    | left boundary (area $A_1$ )  |
| $S_2$    | right boundary (area $A_2$ ) |
| $\Gamma$ | perimeter boundary           |

$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

# Quasi-One-Dimensional Flow - Governing Equations

Governing equations (general form):

$$\frac{d}{dt} \iiint_{\Omega} \rho d\mathcal{V} + \iint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0$$

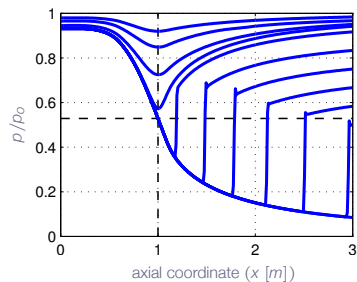
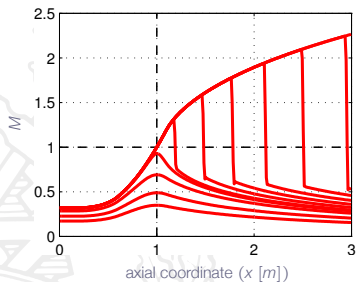
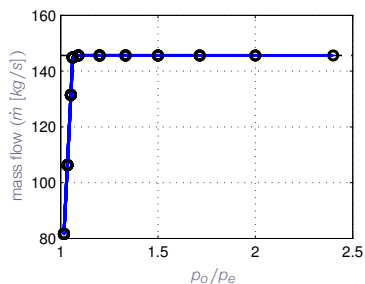
$$\frac{d}{dt} \iiint_{\Omega} \rho u d\mathcal{V} + \iint_{\partial\Omega} [\rho(\mathbf{v} \cdot \mathbf{n})u + p(\mathbf{n} \cdot \mathbf{e}_x)] dS = 0$$

$$\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V} + \iint_{\partial\Omega} \rho h_o(\mathbf{v} \cdot \mathbf{n}) dS = 0$$

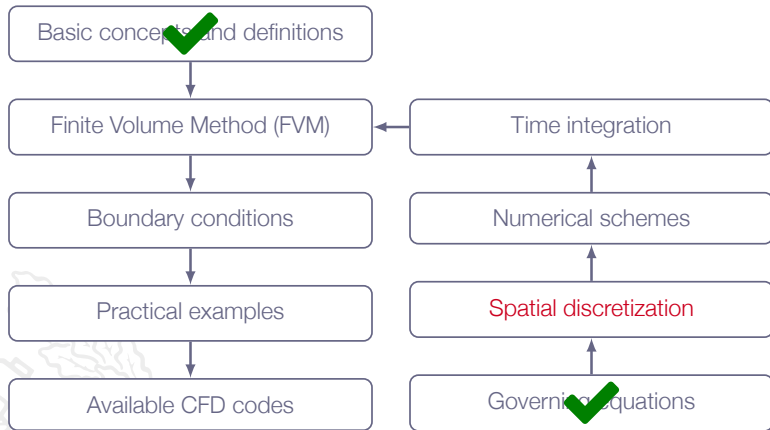


# Example: Nozzle Simulation (Back Pressure Sweep)

$\rho_o$	1.20 [bar]
$\rho_e$	0.50 [bar]
$\rho_o/\rho_e$	2.40
$\dot{m}$	145.6 [kg/s]
$M_{max}$	2.26



# Roadmap - The Time-Marching Technique



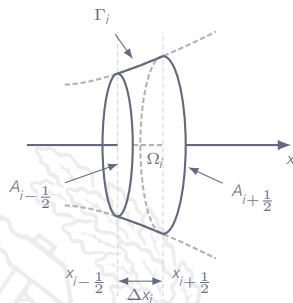
# Spatial Discretization





# Quasi-One-Dimensional Flow - Spatial Discretization

Let's look at a small tube segment with length  $\Delta x$



Streamtube with area  $A(x)$

$$A_{i-\frac{1}{2}} = A(x_{i-\frac{1}{2}})$$

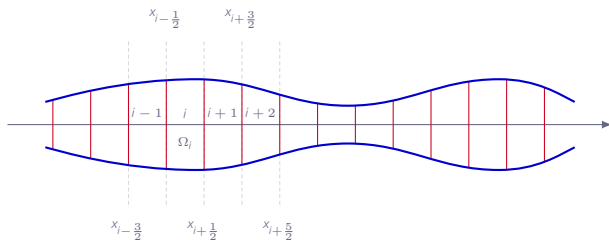
$$A_{i+\frac{1}{2}} = A(x_{i+\frac{1}{2}})$$

$$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

$\Omega_i$  - control volume enclosed by  $A_{i-\frac{1}{2}}$ ,  $A_{i+\frac{1}{2}}$ , and  $\Gamma_i$

$\Rightarrow$  spatial discretization

# Quasi-One-Dimensional Flow - Spatial Discretization

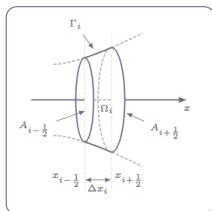


- ▶ Integer indices ( $i, i + 1, \dots$ ): control volumes or **cells**
- ▶ Fractional indices ( $i + \frac{1}{2}, i + \frac{3}{2}, \dots$ ): interfaces between control volumes or **cell faces**
- ▶ Apply control volume formulations for mass, momentum, energy to control volume  $\Omega_i$

# Quasi-One-Dimensional Flow

cell-averaged quantity

face-averaged quantity



Conservation of mass:

$$\underbrace{\frac{d}{dt} \iiint_{\Omega_i} \rho d\mathcal{V}}_{VOL_i \frac{d}{dt} \bar{\rho}_i} + \underbrace{\iint_{x_{i-1/2}} \rho \mathbf{v} \cdot \mathbf{n} dS}_{-\overline{(\rho u)}_{i-1/2} A_{i-1/2}} + \underbrace{\iint_{x_{i+1/2}} \rho \mathbf{v} \cdot \mathbf{n} dS}_{\overline{(\rho u)}_{i+1/2} A_{i+1/2}} + \underbrace{\iint_{\Gamma_i} \rho \mathbf{v} \cdot \mathbf{n} dS}_0 = 0$$

where

$$VOL_i = \iiint_{\Omega_i} d\mathcal{V}$$

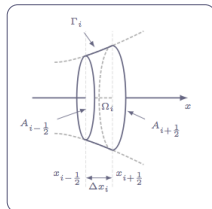
$$\bar{\rho}_i = \frac{1}{VOL_i} \iiint_{\Omega_i} \rho d\mathcal{V}$$

$$\overline{(\rho u)}_{i-1/2} = \frac{1}{A_{i-1/2}} \iint_{x_{i-1/2}} \rho u dS$$

$$\overline{(\rho u)}_{i+1/2} = \frac{1}{A_{i+1/2}} \iint_{x_{i+1/2}} \rho u dS$$

# Quasi-One-Dimensional Flow

cell-averaged quantity  
face-averaged quantity  
source term



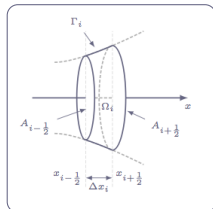
Conservation of momentum:

$$\underbrace{\frac{d}{dt} \iiint_{\Omega_i} \rho u d\mathcal{V}}_{VOL_i \frac{d}{dt} \overline{(\rho u)}} + \underbrace{\iint_{x_{i-\frac{1}{2}}} [\rho(\mathbf{v} \cdot \mathbf{n})u + p(\mathbf{n} \cdot \mathbf{e}_x)] dS}_{-\overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}}} +$$

$$+ \underbrace{\iint_{x_{i+\frac{1}{2}}} [\rho(\mathbf{v} \cdot \mathbf{n})u + p(\mathbf{n} \cdot \mathbf{e}_x)] dS}_{\overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}}} + \underbrace{\iint_{\Gamma_i} [\rho(\mathbf{v} \cdot \mathbf{n})u + p(\mathbf{n} \cdot \mathbf{e}_x)] dS}_{-\iint_{\Gamma_i} p dA} = 0$$

# Quasi-One-Dimensional Flow

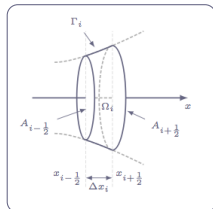
cell-averaged quantity  
face-averaged quantity



Conservation of energy:

$$\begin{aligned}
 & \underbrace{\frac{d}{dt} \iiint_{\Omega_i} \rho e_o d\mathcal{V}}_{VOL_i \frac{d}{dt} \overline{(\rho e_o)}_i} + \underbrace{\iint_{x_{i-1/2}} \rho h_o (\mathbf{v} \cdot \mathbf{n}) dS}_{-\overline{(\rho h_o)}_{i-1/2} A_{i-1/2}} + \\
 & + \underbrace{\iint_{x_{i+1/2}} \rho h_o (\mathbf{v} \cdot \mathbf{n}) dS}_{\overline{(\rho h_o)}_{i+1/2} A_{i+1/2}} + \underbrace{\iint_{\Gamma_i} \rho h_o (\mathbf{v} \cdot \mathbf{n}) dS}_0 = 0
 \end{aligned}$$

# Quasi-One-Dimensional Flow



Lower order term due to varying stream tube area:

$$\iint_{\Gamma_i} p dA \approx \bar{p}_i \left( A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right)$$

where  $\bar{p}_i$  is **calculated from cell-averaged quantities** (DOFs)

$$\left\{ \bar{p}, \overline{(\rho u)}, \overline{(\rho e_o)} \right\}_i$$

as

$$\bar{p}_i = (\gamma - 1) \left( \overline{(\rho e_o)}_i - \frac{1}{2} \bar{\rho}_i \bar{u}_i \right), \quad \bar{u}_i = \frac{\overline{(\rho u)}_i}{\bar{\rho}_i}$$

# Quasi-One-Dimensional Flow - Spatial Discretization

cell-averaged quantity

face-averaged quantity

source term

$$VOL_i \frac{d}{dt} \bar{\rho}_i - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

$$\begin{aligned} VOL_i \frac{d}{dt} \overline{(\rho u)}_i - \overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} &= \\ = \bar{p}_i \left( A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right) \end{aligned}$$

$$VOL_i \frac{d}{dt} \overline{(\rho e_o)}_i - \overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

Application of these equations to all cells  $i \in \{1, 2, \dots, N\}$  of the computational domain results in a system of ODEs

# Spatial Discretization - Summary

Steps to achieve spatial discretization:

1. Choose primary variables (Degrees of Freedom or DOFs)
2. Approximate all other quantities in terms of DOFs

⇒ System of ordinary differential equations (ODEs)

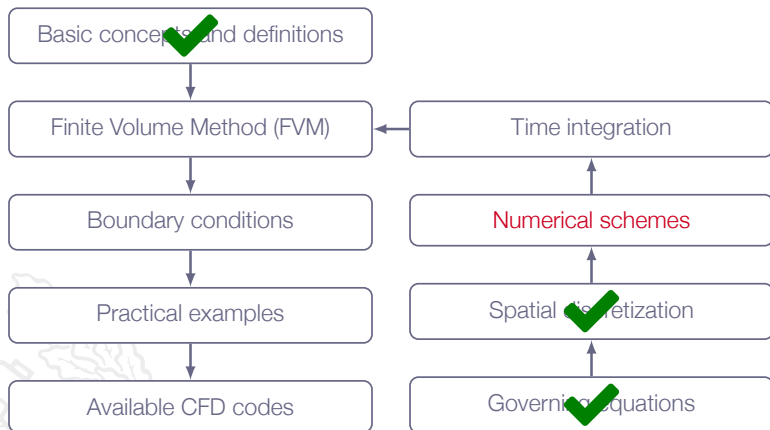
Degrees of freedom:

- ▶ Choose  $\left\{ \bar{\rho}, \overline{(\rho U)}, \overline{(\rho e_o)} \right\}_i$  in all control volumes  $\Omega_i$ ,  $i \in \{1, 2, \dots, N\}$  as degrees of freedom, or primary variables
- ▶ Note that these are **cell-averaged quantities**

What about the face values?



# Roadmap - The Time-Marching Technique



# Numerical Schemes



# Flux Term Approximation

$$\left\{ \begin{array}{c} \overline{(\rho u)} \\ \overline{(\rho u^2 + p)} \\ \overline{(\rho u h_o)} \end{array} \right\}_{i+\frac{1}{2}} = f \left( \left\{ \begin{array}{c} \bar{\rho} \\ \overline{(\rho u)} \\ \overline{(\rho e_o)} \end{array} \right\}_i, \left\{ \begin{array}{c} \bar{\rho} \\ \overline{(\rho u)} \\ \overline{(\rho e_o)} \end{array} \right\}_{i+1}, \dots \right)$$

cell face values

cell-averaged values

Simple example:

$$\overline{(\rho u)}_{i+\frac{1}{2}} \approx \frac{1}{2} \left[ \overline{(\rho u)}_i + \overline{(\rho u)}_{i+1} \right]$$

# Flux Term Approximation

More complex approximations usually needed

## High-order schemes:

- ▶ increased accuracy
- ▶ more cell values involved (*wider flux molecule*)
- ▶ boundary conditions more difficult to implement

## Optimized numerical dissipation:

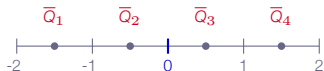
- ▶ upwind type of flux scheme

## Shock handling:

- ▶ non-linear treatment needed (e.g. TVD schemes)
- ▶ artificial damping



# Flux Term Approximation

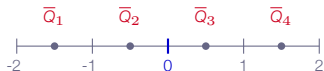


$$Q(x) = A + Bx + Cx^2 + Dx^3$$

Assume constant area:  $A(x) = 1.0$



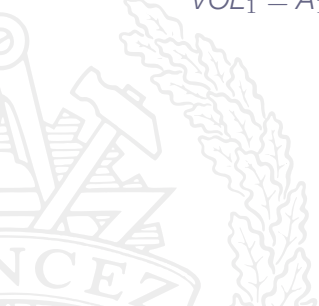
# Flux Term Approximation



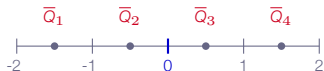
$$\bar{Q}_1 = \frac{1}{VOL_1} \int_{-2}^{-1} Q(x) dx$$

$$VOL_1 = A_1 \Delta x_1 = \{A_1 = 1.0, \Delta x_1 = 1.0\} = 1.0$$

$$\Rightarrow \bar{Q}_1 = \int_{-2}^{-1} Q(x) dx$$



# Flux Term Approximation



$$\bar{Q}_1 = \int_{-2}^{-1} Q(x) dx = \left[ Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4 \right]_{-2}^{-1}$$

$$\bar{Q}_2 = \int_{-1}^0 Q(x) dx = \left[ Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4 \right]_{-1}^0$$

$$\bar{Q}_3 = \int_0^1 Q(x) dx = \left[ Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4 \right]_0^1$$

$$\bar{Q}_4 = \int_1^2 Q(x) dx = \left[ Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4 \right]_1^2$$

# Flux Term Approximation



$$\bar{Q}_1 = A - \frac{3}{2}B + \frac{7}{3}C - \frac{15}{4}D$$

$$\bar{Q}_2 = A - \frac{1}{2}B + \frac{1}{3}C - \frac{1}{4}D$$

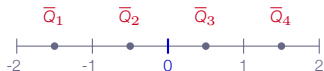
$$\bar{Q}_3 = A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D$$

$$\bar{Q}_4 = A + \frac{3}{2}B + \frac{7}{3}C + \frac{15}{4}D$$





# Flux Term Approximation



$$A = \frac{1}{12} \left[ -\bar{Q}_1 + 7\bar{Q}_2 + 7\bar{Q}_3 - \bar{Q}_4 \right]$$

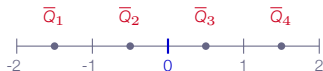
$$B = \frac{1}{12} \left[ \bar{Q}_1 - 15\bar{Q}_2 + 15\bar{Q}_3 - \bar{Q}_4 \right]$$

$$C = \frac{1}{4} \left[ \bar{Q}_1 - \bar{Q}_2 - \bar{Q}_3 + \bar{Q}_4 \right]$$

$$D = \frac{1}{6} \left[ -\bar{Q}_1 + 3\bar{Q}_2 - 3\bar{Q}_3 + \bar{Q}_4 \right]$$



# Flux Term Approximation

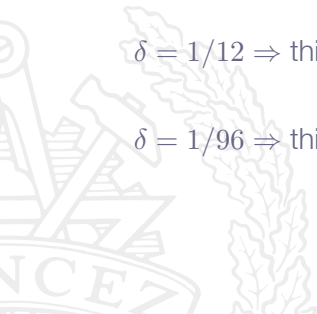


$$Q_0 = Q(0) + \delta Q'''(0) \Rightarrow Q_0 = A + 6\delta D$$

$\delta = 0 \Rightarrow$  fourth-order central scheme

$\delta = 1/12 \Rightarrow$  third-order upwind scheme

$\delta = 1/96 \Rightarrow$  third-order low-dissipation upwind scheme



# Flux Term Approximation



$$Q_0 = A + 6\delta D = \{\delta = 1/12\} = -\frac{1}{6}\bar{Q}_1 + \frac{5}{6}\bar{Q}_2 + \frac{1}{3}\bar{Q}_3$$

$$Q_{0_{left}} = -\frac{1}{6}\bar{Q}_1 + \frac{5}{6}\bar{Q}_2 + \frac{1}{3}\bar{Q}_3$$

$$Q_{0_{right}} = -\frac{1}{6}\bar{Q}_4 + \frac{5}{6}\bar{Q}_3 + \frac{1}{3}\bar{Q}_2$$

method of characteristics used in order to decide whether left- or right-upwinded flow quantities should be used

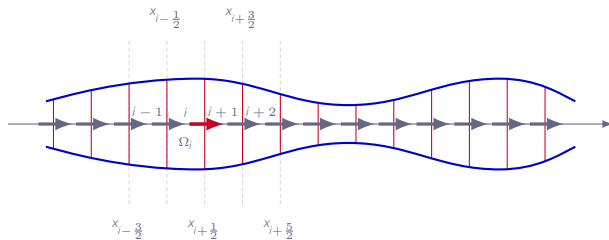
# Flux Term Approximation

High-order numerical schemes:

- ▶ low numerical dissipation (smearing due to amplitudes errors)
- ▶ low dispersion errors (wiggles due to phase errors)



# Conservative Scheme



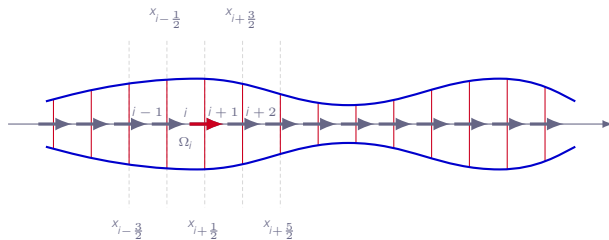
mass conservation:

$$\text{cell } (i): \quad \text{VOL}_i \frac{d}{dt} \bar{\rho}_i + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} = 0$$

$$\text{cell } (i+1): \quad \text{VOL}_{i+1} \frac{d}{dt} \bar{\rho}_{i+1} + \overline{(\rho u)}_{i+\frac{3}{2}} A_{i+\frac{3}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

(similarly for momentum and energy conservation)

# Conservative Scheme



mass conservation:

cell (i):

$$VOL_i \frac{d}{dt} \bar{\rho}_i + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} = 0$$

cell (i + 1):

$$VOL_{i+1} \frac{d}{dt} \bar{\rho}_{i+1} + \overline{(\rho u)}_{i+\frac{3}{2}} A_{i+\frac{3}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

(similarly for momentum and energy conservation)

# Conservative Scheme

## Conservative scheme

*"The flux leaving one control volume equals the flux entering neighbouring control volume"*

Conservation property for mass, momentum and energy is crucial for the correct prediction of shocks\*

\* correct prediction of shocks:  
strength  
position  
velocity

# Spatial Discretization - Summary

1. Primary variables defined for all cells

$$\left\{ \bar{\rho}, \overline{(\rho u)}, \overline{(\rho e_o)} \right\}_i, i \in \{1, 2, \dots, N\}$$

2. Flux terms and lower-order terms may be computed

3. **Temporal derivatives of the primary variables** are defined for all cells

$$\left\{ \frac{d}{dt} \bar{\rho}, \frac{d}{dt} \overline{(\rho u)}, \frac{d}{dt} \overline{(\rho e_o)} \right\}_i$$



# Spatial Discretization - Summary

cell-averaged quantity

face-averaged quantity

source term

$$\frac{d}{dt} \bar{\rho}_i = \frac{1}{VOL_i} \left\{ \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} \right\}$$

$$\frac{d}{dt} \overline{(\rho u)}_i = \frac{1}{VOL_i} \left\{ \overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} - \overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} + \right.$$

$$\left. \bar{\rho}_i \left( A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right) \right\}$$

$$\frac{d}{dt} \overline{(\rho e_o)}_i = \frac{1}{VOL_i} \left\{ \overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} - \overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} \right\}$$

# Roadmap - The Time-Marching Technique

