Compressible Flow - TME085

Lecture 11

Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se

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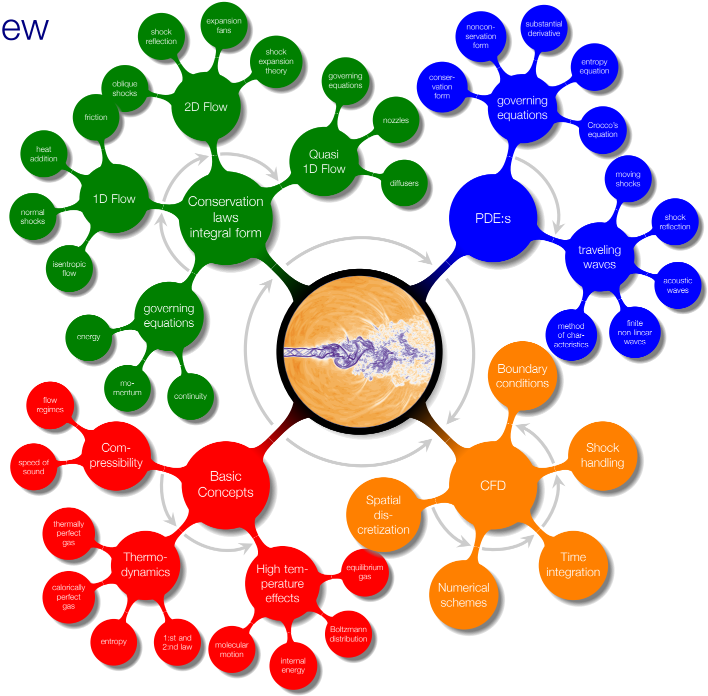
Overview



Conservation laws integral form
2D Flow
shock reflection
expansion fans
shock expansion theory
oblique shocks
friction
1D Flow
heat addition
normal shocks
isentropic flow
Quasi 1D Flow
governing equations
nozzles
diffusers
governing equations
energy
mo-mentum
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PDE:s
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noncon-servation form
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entropy equation
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Crocco's equation
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Shock handling
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Basic Concepts
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Thermo-dynamics
thermally perfect gas
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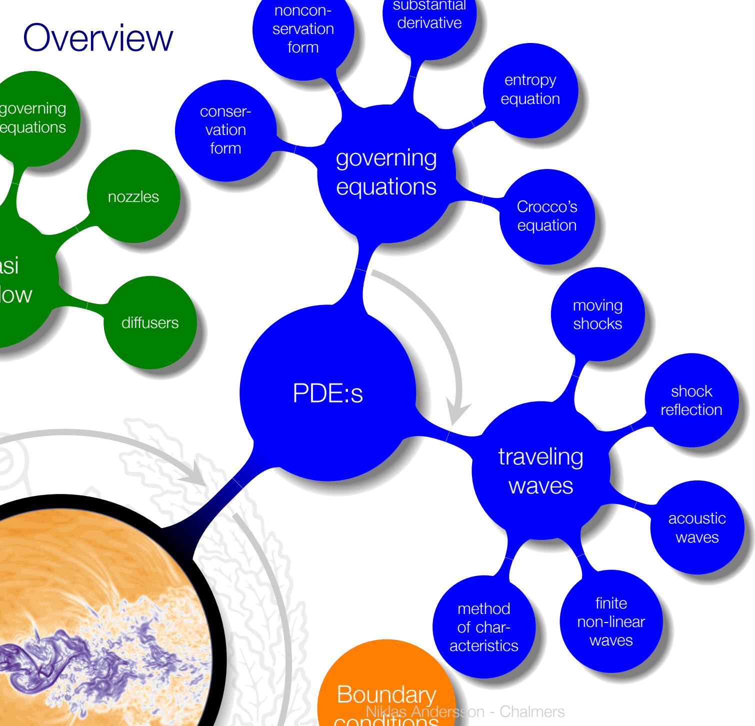
1:st and 2:nd law
High tem-perature effects
equilibrium gas
Boltzmann distribution
molecular motion
internal energy

Chapter 7
Unsteady Wave Motion

Overview



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method
of char-
acteristics
finite
non-linear
waves
Boundary

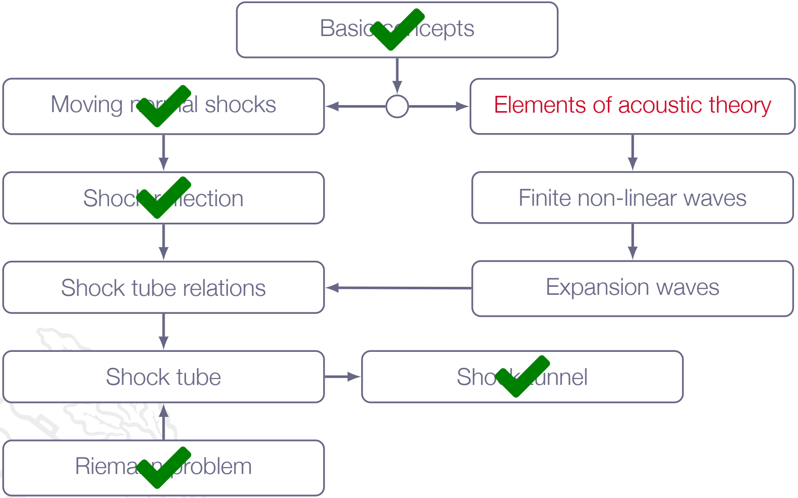
Addressed Learning Outcomes

8. Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics
   - j unsteady waves and discontinuities in 1D
   - k basic acoustics

11. Explain how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations

method of characteristics - a central element in classic compressible flow theory

Roadmap - Unsteady Wave Motion



Chapter 7.5
Elements of Acoustic Theory

Sound Waves

- Weakest audible sound wave (0 dB): Δp ∼0.00002 Pa
- Loud sound wave (94 dB): Δp ∼1 Pa
- Threshold of pain (120 dB): Δp ∼20 Pa
- Harmful sound wave (130 dB): Δp ∼60 Pa

Example:

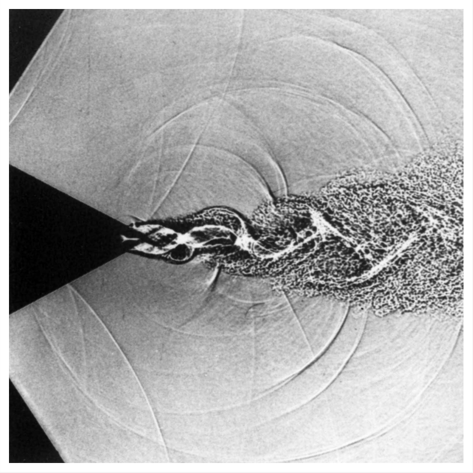
$\Delta p \sim$ 1 Pa gives $\Delta\rho$ ∼0.000009 kg/m^3 and Δu ∼0.0025 m/s

Sound Waves

Schlieren flow visualization of self-sustained oscillation of an under-expanded free jet

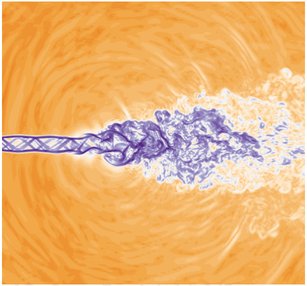
A. Hirschberg

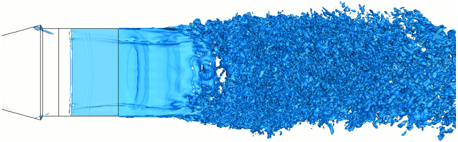
"*Introduction to aero-acoustics of internal flows*", Advances in Aeroacoustics, VKI, 12-16 March 2001



Sound Waves

Screeching rectangular supersonic jet





Elements of Acoustic Theory

PDE:s for conservation of mass and momentum are derived in Chapter 6:

	conservation form	non-conservation form
mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$
momentum	$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = 0$	$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0$

Elements of Acoustic Theory

For adiabatic inviscid flow we also have the entropy equation as

$$\frac{Ds}{Dt} = 0$$

Assume one-dimensional flow

$$\left.\begin{aligned} \rho &= \rho(x,t) \\ \mathbf{v} &= u(x,t)\mathbf{e}_x \\ p &= p(x,t) \\ \dots \end{aligned}\right\} \Rightarrow$$

continuity $\quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$

momentum $\quad \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$

s=constant

can $\frac{\partial p}{\partial x}$ be expressed in terms of density?

Elements of Acoustic Theory

From Chapter 1: any thermodynamic state variable is uniquely defined by any tow other state variables

$$p = p(\rho, s) \Rightarrow dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

s=constant gives

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho = a^2 d\rho$$

$$\Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \\ \rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + a^2\dfrac{\partial \rho}{\partial x} = 0 \end{cases}$$

Elements of Acoustic Theory

Assume small perturbations around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

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Elements of Acoustic Theory

Speed of sound is a thermodynamic state variable
$\Rightarrow a^2 = a^2(\rho, s)$. With entropy constant $\Rightarrow a^2 = a^2(\rho)$

Taylor expansion around a_∞ with $(\Delta\rho = \rho - \rho_\infty)$ gives

$$a^2 = a_\infty^2 + \left(\frac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}(a^2)\right)_\infty (\Delta\rho)^2 + \dots$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + \left[a_\infty^2 + \left(\dfrac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + \dots\right]\dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory - Acoustic Equations

Since $\Delta\rho$ and Δu are assumed to be small ($\Delta\rho \ll \rho_\infty$, $\Delta u \ll a$)

- products of perturbations can be neglected
- higher-order terms in the Taylor expansion can be neglected

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \rho_\infty \dfrac{\partial}{\partial x}(\Delta u) = 0 \\ \rho_\infty \dfrac{\partial}{\partial t}(\Delta u) + a_\infty^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Note: Only valid for small perturbations (sound waves)

This type of derivation is based on linearization, *i.e.* the acoustic equations are linear

Acoustic equations:

> *"... describe the motion of gas induced by the passage of a sound wave ..."*

Elements of Acoustic Theory - Wave Equation

Combining linearized continuity and the momentum equations we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

(combine the time derivative of the continuity eqn. and the divergence of the momentum eqn.)

General solution:

$$\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$$

$F(x - a_\infty t)$: wave traveling in positive x-direction with speed a_∞

$G(x + a_\infty t)$: wave traveling in negative x-direction with speed a_∞

F and *G* may be arbitrary functions

Wave shape is determined by functions *F* and *G*

Elements of Acoustic Theory - Wave Equation

Spatial and temporal derivatives of F are obtained according to

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial t} = -a_\infty F' \\[2ex] \dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial x} = F' \end{cases}$$

spatial and temporal derivatives of G can of course be obtained in the same way...

Elements of Acoustic Theory - Wave Equation

with $\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$ and the derivatives of F and G we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 F'' + a_\infty^2 G''$$

and

$$\frac{\partial^2}{\partial x^2}(\Delta\rho) = F'' + G''$$

which gives

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho) = 0$$

i.e., the proposed solution fulfils the wave equation

Elements of Acoustic Theory - Wave Equation

F and G may be arbitrary functions, assume $G = 0$

$$\Delta\rho(x,t) = F(x - a_\infty t)$$

If $\Delta\rho$ is constant (constant wave amplitude), $(x - a_\infty t)$ must be a constant which implies

$$x = a_\infty t + c$$

where c is a constant

$$\frac{dx}{dt} = a_\infty$$

Elements of Acoustic Theory - Wave Equation

We want a relation between $\Delta\rho$ and Δu

$\Delta\rho(x,t) = F(x - a_\infty t)$ (wave in positive x direction) gives:

$$\frac{\partial}{\partial t}(\Delta\rho) = -a_\infty F' \qquad \text{and} \qquad \frac{\partial}{\partial x}(\Delta\rho) = F'$$

$$\underbrace{\frac{\partial}{\partial t}(\Delta\rho)}_{-a_\infty F'} + a_\infty \underbrace{\frac{\partial}{\partial x}(\Delta\rho)}_{F'} = 0$$

or

$$\frac{\partial}{\partial x}(\Delta\rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta\rho)$$

Elements of Acoustic Theory - Wave Equation

Linearized momentum equation:

$$\rho_\infty \frac{\partial}{\partial t}(\Delta u) = -a_\infty^2 \frac{\partial}{\partial x}(\Delta \rho) \Rightarrow$$

$$\frac{\partial}{\partial t}(\Delta u) = -\frac{a_\infty^2}{\rho_\infty}\frac{\partial}{\partial x}(\Delta \rho) = \left\{\frac{\partial}{\partial x}(\Delta \rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta \rho)\right\} = \frac{a_\infty}{\rho_\infty}\frac{\partial}{\partial t}(\Delta \rho)$$

$$\frac{\partial}{\partial t}\left(\Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho\right) = 0 \Rightarrow \Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho = const$$

In undisturbed gas $\Delta u = \Delta \rho = 0$ which implies that the constant must be zero and thus

$$\boxed{\Delta u = \frac{a_\infty}{\rho_\infty}\Delta \rho}$$

Elements of Acoustic Theory - Wave Equation

Similarly, for $\Delta\rho(x,t) = G(x + a_\infty t)$ (wave in negative x direction) we obtain:

$$\boxed{\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho}$$

Also, since $\Delta p = a_\infty^2 \Delta\rho$ we get:

Right going wave (+x direction) $\Delta u = \frac{a_\infty}{\rho_\infty}\Delta\rho = \frac{1}{a_\infty \rho_\infty}\Delta p$

Left going wave (-x direction) $\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho = -\frac{1}{a_\infty \rho_\infty}\Delta p$

Elements of Acoustic Theory - Wave Equation

- Δu denotes induced mass motion and is positive in the positive x-direction

$$\Delta u = \pm \frac{a_\infty \Delta \rho}{\rho_\infty} = \pm \frac{\Delta p}{a_\infty \rho_\infty}$$

- condensation (the part of the sound wave where $\Delta \rho > 0$): Δu is always in the same direction as the wave motion
- rarefaction (the part of the sound wave where $\Delta \rho < 0$): Δu is always in the opposite direction as the wave motion

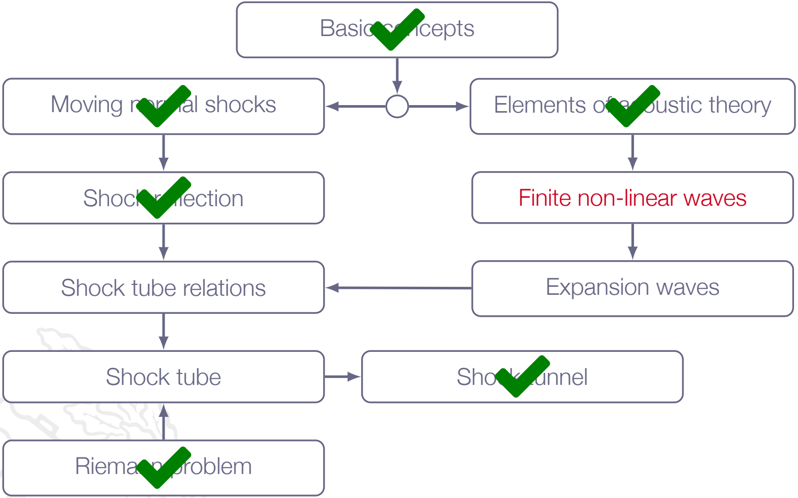
Elements of Acoustic Theory - Wave Equation *Summary*

Combining linearized continuity and the momentum equations we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

- Due to the assumptions made, the equation is not exact
- More and more accurate as the perturbations becomes smaller and smaller
- How should we describe waves with larger amplitudes?

Roadmap - Unsteady Wave Motion



Chapter 7.6
Finite (Non-Linear) Waves

Finite (Non-Linear) Waves

When $\Delta\rho,\ \Delta u,\ \Delta p,$ ... Become large, the linearized acoustic equations become poor approximations

Non-linear equations must be used

One-dimensional non-linear continuity and momentum equations

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

We still assume isentropic flow, $ds = 0$

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t} \qquad \frac{\partial \rho}{\partial x} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x}$$

Inserted in the continuity equation this gives:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

Add $1/(\rho a)$ times the continuity equation to the momentum equation:

$$\left[\frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right] = 0$$

If we instead subtraction $1/(\rho a)$ times the continuity equation from the momentum equation, we get:

$$\left[\frac{\partial u}{\partial t} + (u - a)\frac{\partial u}{\partial x}\right] - \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u - a)\frac{\partial p}{\partial x}\right] = 0$$

Finite (Non-Linear) Waves

Since $u = u(x, t)$, we have:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt$$

Let $\dfrac{dx}{dt} = u + a$ gives

$$du = \left[\frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x}\right] dt$$

Interpretation: change of u in the direction of line $\dfrac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

In the same way we get:

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}\frac{dx}{dt}dt$$

and thus

$$dp = \left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right]dt$$

Finite (Non-Linear) Waves

Now, if we combine

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

$$du = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] dt$$

$$dp = \left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] dt$$

we get

$$\boxed{\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0}$$

Characteristic Lines

Thus, along a line $dx = (u + a)dt$ we have

$$du + \frac{dp}{\rho a} = 0$$

In the same way we get along a line where $dx = (u - a)dt$

$$du - \frac{dp}{\rho a} = 0$$

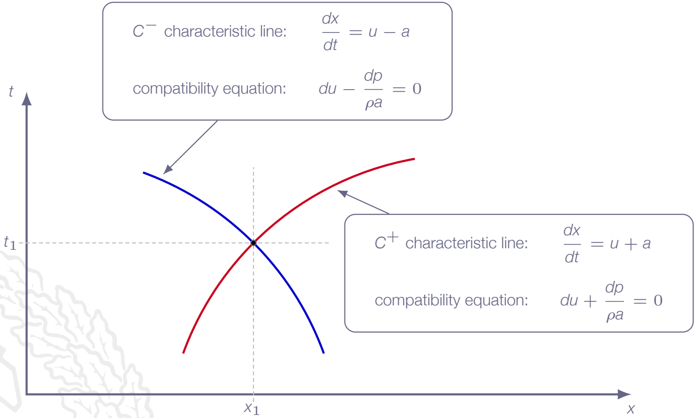
Characteristic Lines

- We have found a path through a point (x_1, t_1) along which the governing partial differential equations reduces to ordinary differential equations

- These paths or lines are called characteristic lines

- The C^+ and C^- characteristic lines are physically the paths of right- and left-running sound waves in the xt-plane

Characteristic Lines



Characteristic Lines

summary:

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^- \text{ characteristic}$$

or

$$du + \frac{dp}{\rho a} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$du - \frac{dp}{\rho a} = 0 \quad \text{along } C^- \text{ characteristic}$$

Riemann Invariants

Integration gives:

$$J^+ = u + \int \frac{dp}{\rho a} = \text{constant along } C^+ \text{ characteristic}$$

$$J^- = u - \int \frac{dp}{\rho a} = \text{constant along } C^- \text{ characteristic}$$

We need to rewrite $\dfrac{dp}{\rho a}$ to be able to perform the integrations

Riemann Invariants

Isentropic processes:

$$p = c_1 T^{\gamma/(\gamma-1)} = c_2 a^{2\gamma/(\gamma-1)}$$

where c_1 and c_2 are constants

$$\Rightarrow dp = c_2 \left(\frac{2\gamma}{\gamma - 1}\right) a^{[2\gamma/(\gamma-1)-1]} da$$

Assume calorically perfect gas:

$$a^2 = \frac{\gamma p}{\rho} \Rightarrow \rho = \frac{\gamma p}{a^2}$$

with $p = c_2 a^{2\gamma/(\gamma-1)}$ we get

$$\rho = c_2 \gamma a^{[2\gamma/(\gamma-1)-2]}$$

Riemann Invariants

$$J^+ = u + \int \frac{dp}{\rho a} = u + \int \frac{c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]}}{c_2 \gamma a^{[2\gamma/(\gamma-1)-1]}} da = u + \int \frac{2da}{\gamma - 1}$$

$$J^+ = u + \frac{2a}{\gamma - 1}$$

$$J^- = u - \frac{2a}{\gamma - 1}$$

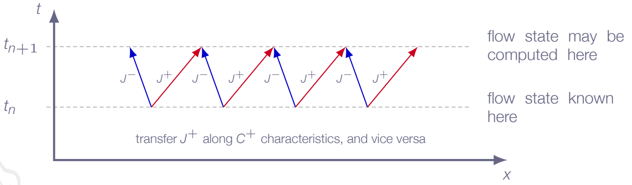
Riemann Invariants

If J^+ and J^- are known at some point (x, t), then

$$\begin{cases} J^+ + J^- = 2u \\ J^+ - J^- = \dfrac{4a}{\gamma - 1} \end{cases} \Rightarrow \begin{cases} u = \dfrac{1}{2}(J^+ + J^-) \\ a = \dfrac{\gamma - 1}{4}(J^+ - J^-) \end{cases}$$

Flow state is uniquely defined!

Method of Characteristics



Summary

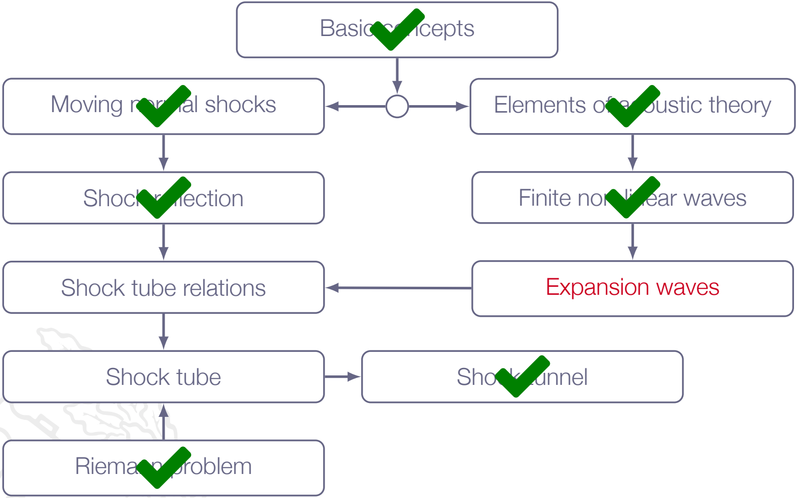
Acoustic waves

- $\Delta\rho$, Δu, etc - very small
- All parts of the wave propagate with the same velocity a_∞
- The wave shape stays the same
- The flow is governed by linear relations

Finite (non-linear) waves

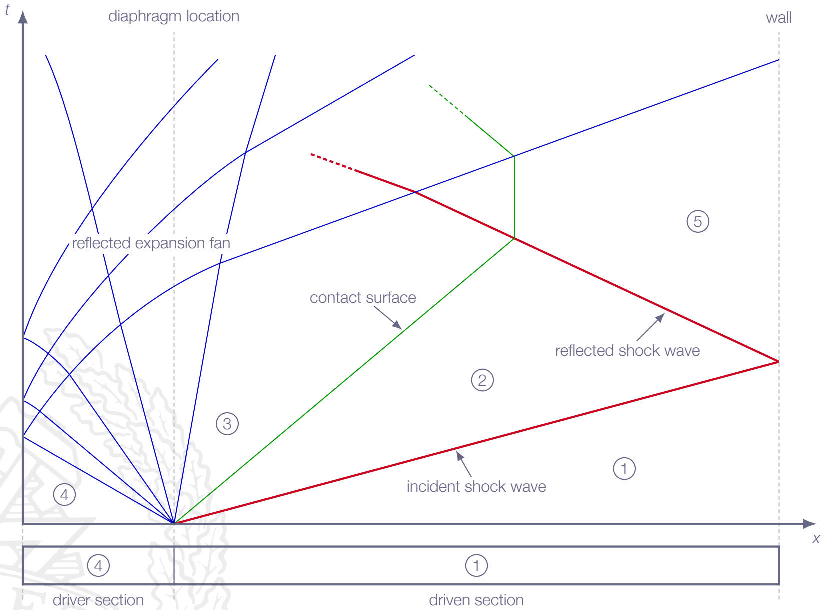
- $\Delta\rho$, Δu, etc - can be large
- Each local part of the wave propagates at the local velocity $(u + a)$
- The wave shape changes with time
- The flow is governed by non-linear relations

Roadmap - Unsteady Wave Motion



Chapter 7.7
Incident and Reflected Expansion Waves

Expansion Waves



Expansion Waves

Properties of a left-running expansion wave

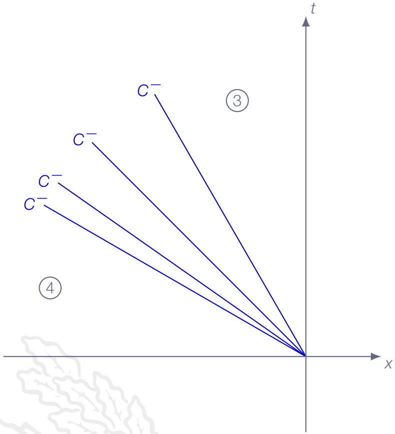
1. All flow properties are constant along C^- characteristics
2. The wave head is propagating into region 4 (high pressure)
3. The wave tail defines the limit of region 3 (lower pressure)
4. Regions 3 and 4 are assumed to be constant states

For calorically perfect gas:

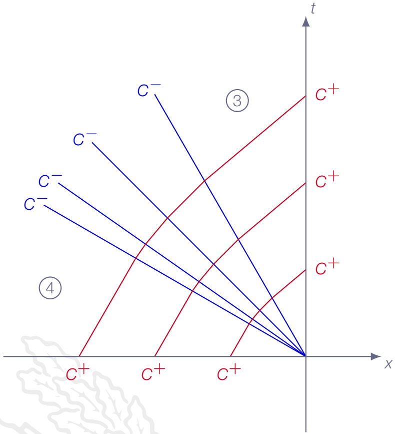
$$J^+ = u + \frac{2a}{\gamma - 1} \quad \text{is constant along } C^+ \text{ lines}$$

$$J^- = u - \frac{2a}{\gamma - 1} \quad \text{is constant along } C^- \text{ lines}$$

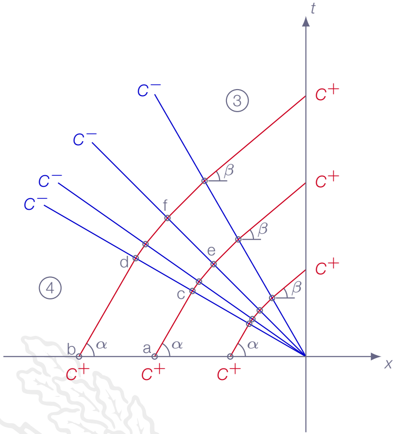
Expansion Waves



Expansion Waves



Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

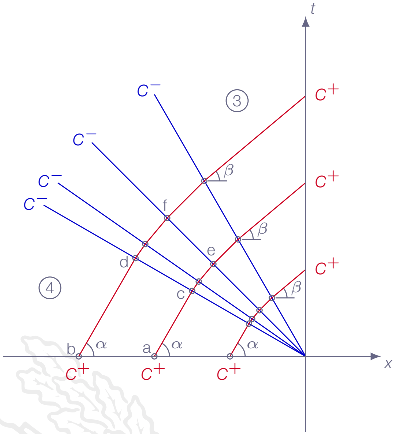
since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

Expansion Waves



constant flow properties in region 4: $J^+_a = J^+_b$

J^+ invariants constant along C^+ characteristics:

$$J^+_a = J^+_c = J^+_e$$

$$J^+_b = J^+_d = J^+_f$$

since $J^+_a = J^+_b$ this also implies $J^+_e = J^+_f$

J^- invariants constant along C^- characteristics:

$$J^-_c = J^-_d$$

$$J^-_e = J^-_f$$

$$u_e = \frac{1}{2}(J^+_e + J^-_e),\ u_f = \frac{1}{2}(J^+_f + J^-_f),\ \Rightarrow u_e = u_f$$

$$a_e = \frac{\gamma - 1}{4}(J^+_e - J^-_e),\ a_f = \frac{\gamma - 1}{4}(J^+_f - J^-_f),\ \Rightarrow a_e = a_f$$

Expansion Waves

Along each C^- line u and a are constants which means that

$$\frac{dx}{dt} = u - a = const$$

C^- characteristics are straight lines in xt-space

Expansion Waves

The start and end conditions are the same for all C^+ lines

J^+ invariants have the same value for all C^+ characteristics

C^- characteristics are straight lines in xt-space

Simple expansion waves centered at $(x, t) = (0, 0)$

Expansion Waves

In a left-running expansion fan:

- J^+ is constant throughout expansion fan, which implies:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = u_3 + \frac{2a_3}{\gamma - 1}$$

- J^- is constant along C^- lines, but varies from one line to the next, which means that

$$u - \frac{2a}{\gamma - 1}$$

is constant along each C^- line

Expansion Waves

Since $u_4 = 0$ we obtain:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = \frac{2a_4}{\gamma - 1} \Rightarrow$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}$$

with $a = \sqrt{\gamma R T}$ we get

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

Expansion Wave Relations

Isentropic flow $\Rightarrow$ we can use the isentropic relations

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

$$\frac{p}{p_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2}{\gamma-1}}$$

complete description in terms of u/a_4

Expansion Wave Relations

Since C^- characteristics are straight lines, we have:

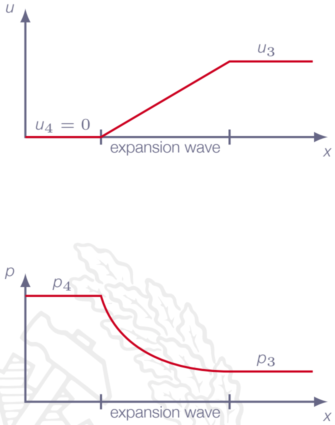
$$\frac{dx}{dt} = u - a \Rightarrow x = (u - a)t$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4} \Rightarrow a = a_4 - \frac{1}{2}(\gamma - 1)u \Rightarrow$$

$$x = \left[u - a_4 + \frac{1}{2}(\gamma - 1)u\right]t = \left[\frac{1}{2}(\gamma - 1)u - a_4\right]t \Rightarrow$$

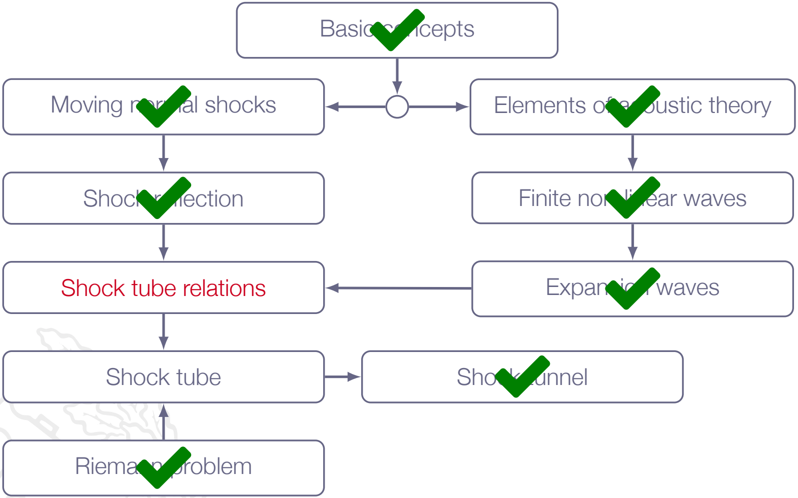
$$u = \frac{2}{\gamma + 1}\left[a_4 + \frac{x}{t}\right]$$

Expansion Wave Relations



- Expansion wave head is advancing to the left with speed a_4 into the stagnant gas
- Expansion wave tail is advancing with speed $u_3 - a_3$, which may be positive or negative, depending on the initial states

Roadmap - Unsteady Wave Motion



Chapter 7.8
Shock Tube Relations

Shock Tube Relations

$$u_p = u_2 = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma_1}{\gamma_1+1}}{\frac{p_2}{p_1} + \frac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2}$$

$$\frac{p_3}{p_4} = \left[1 - \frac{\gamma_4-1}{2}\left(\frac{u_3}{a_4}\right)\right]^{2\gamma_4/(\gamma_4-1)}$$

solving for u_3 gives

$$u_3 = \frac{2a_4}{\gamma_4-1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

But, $p_3 = p_2$ and $u_3 = u_2$ (no change in velocity and pressure over contact discontinuity)

$$\Rightarrow u_2 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

We have now two expressions for u_2 which gives us

$$\frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma_1}{\gamma_1+1}}{\frac{p_2}{p_1} + \frac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2} = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

Rearranging gives:

$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left\{1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1\left[2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1)\right]}}\right\}^{-2\gamma_4/(\gamma_4 - 1)}$$

- p_2/p_1 as implicit function of p_4/p_1
- for a given p_4/p_1, p_2/p_1 will increase with decreased a_1/a_4

$$a = \sqrt{\gamma R T} = \sqrt{\gamma (R_u/M) T}$$

- the speed of sound in a light gas is higher than in a heavy gas
    - driver gas: low molecular weight, high temperature
    - driven gas: high molecular weight, low temperature

Roadmap - Unsteady Wave Motion