Compressible Flow - TME085

Lecture 8

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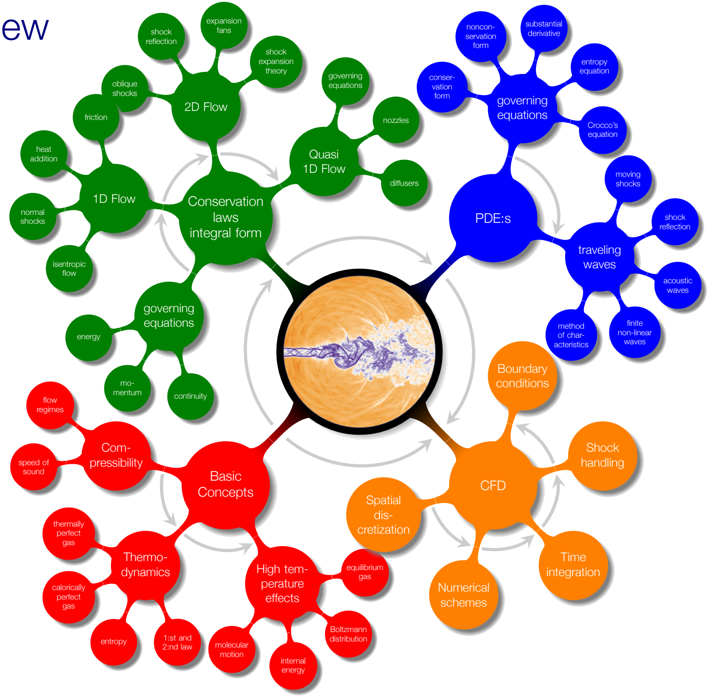
Overview



shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
Quasi 1D Flow
diffusers
heat addition
1D Flow
Conservation laws integral form
normal shocks
isentropic flow
governing equations
energy
mo-mentum
continuity
noncon-servation form
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governing equations
Crocco's equation
PDE:s
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traveling waves
acoustic waves
method of char-acteristics
finite non-linear waves
Boundary conditions
Shock handling
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Com-pressibility
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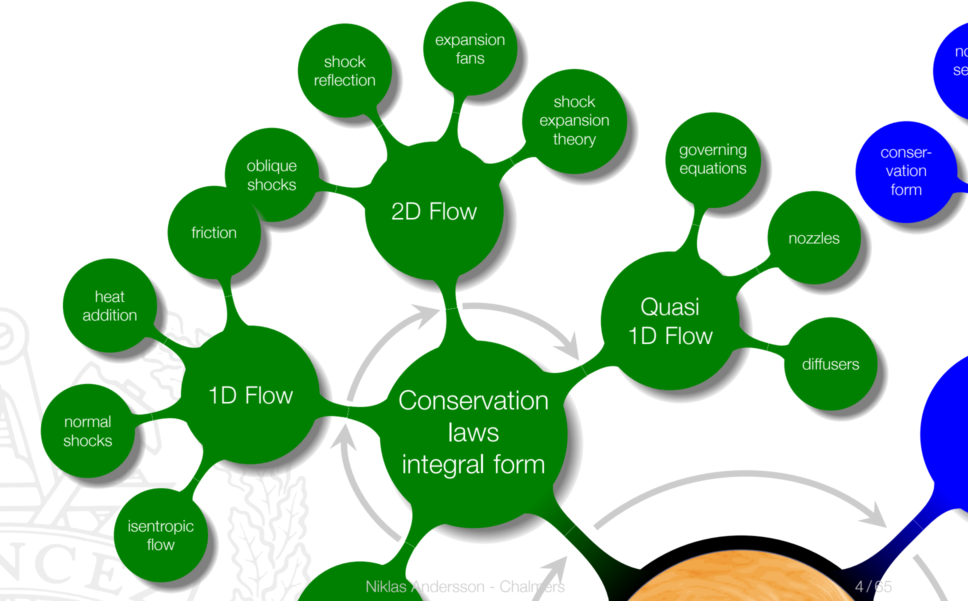
Basic Concepts
thermally perfect gas
Thermo-dynamics
calorically perfect gas
entropy
1:st and 2:nd law
High tem-perature effects
equilibrium gas
Boltzmann distribution
molecular motion
internal energy

Chapter 5
Quasi-One-Dimensional Flow

Overview



shock reflection
expansion fans
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diffusers
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normal shocks
isentropic flow
Conservation laws integral form
conser-vation form

Addressed Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

7 **Explain** why entropy is important for flow discontinuities

8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

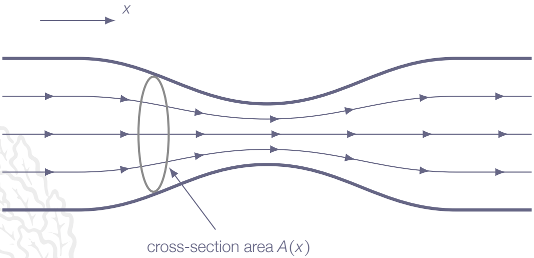
- a 1D isentropic flow*
- b normal shocks*
- i detached blunt body shocks, nozzle flows

12 **Explain** the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions

time for rocket science!

Quasi-One-Dimensional Flow

- Extension of one-dimensional flow to allow variations in streamtube area
- Steady-state flow assumption still applied



Governing Equations - Differential Form

$$d(\rho u A) = 0$$

$$dp = -\rho u du$$

$$dh + u du = 0$$

Assumptions:

- quasi-one-dimensional flow
- inviscid flow
- steady-state flow

Valid for all gases!

Chapter 5.4
Nozzles

Area-Velocity Relation

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

$M < 1$: decreasing A correlated with increasing u

$M > 1$: increasing A correlated with increasing u

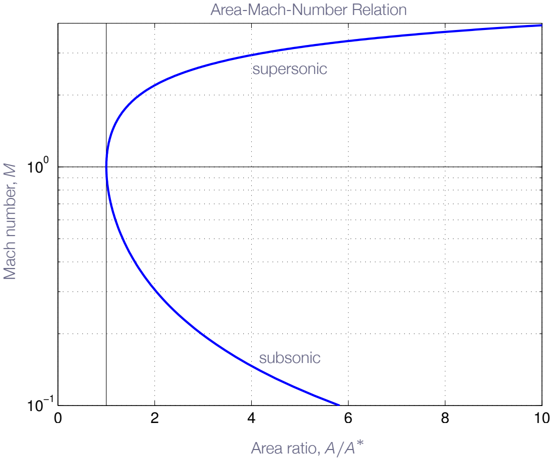
$M = 1$: $dA = 0$

Note 1: The area-velocity relation is only valid for isentropic flow

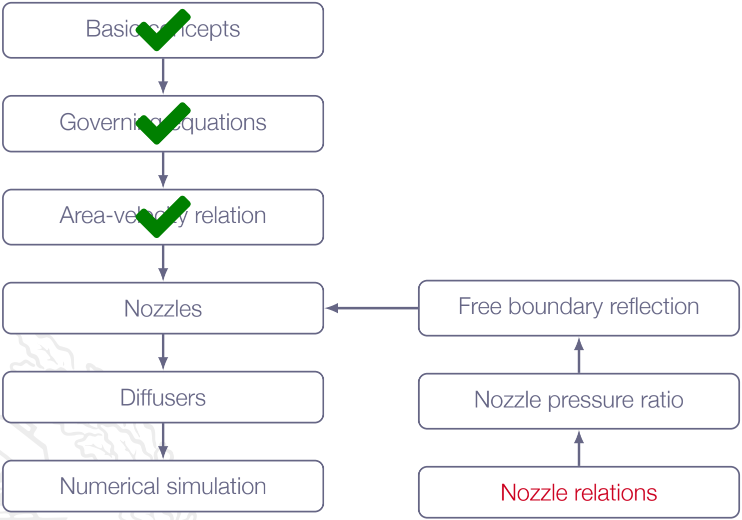
Note 2: The area-velocity relation is valid for all gases

Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}}M^2}$$



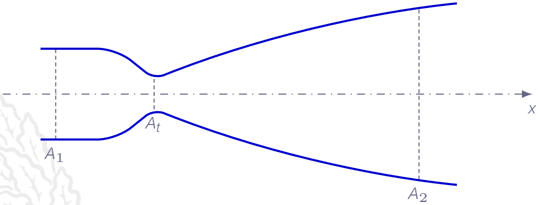
Roadmap - Quasi-One-Dimensional Flow



Nozzle Flow

Assumptions:

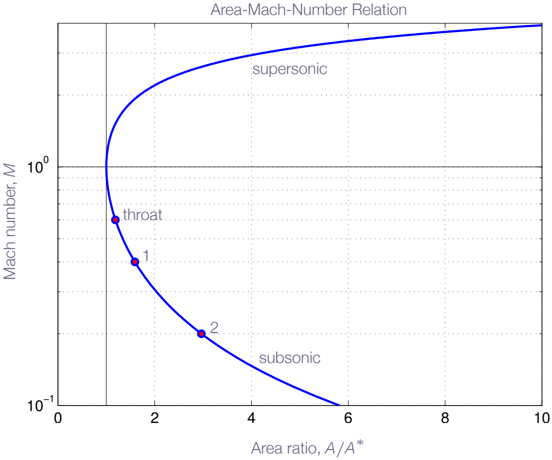
- inviscid
- steady-state
- quasi-one-dimensional
- calorically perfect gas



Nozzle Flow

Alt. 1: sub-critical (non-choked) nozzle flow

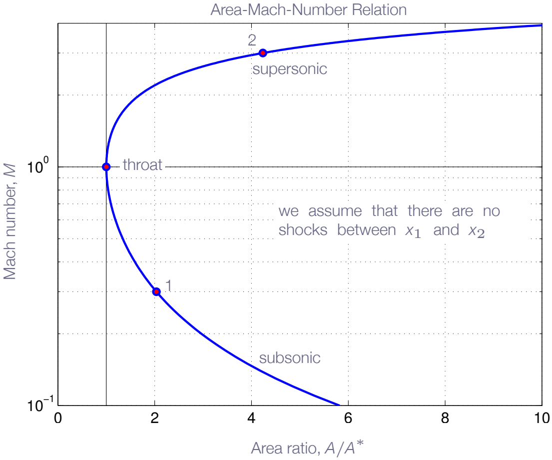
- $M < 1$ at nozzle throat
- $A_t > A^*$
- $M_1 < 1$
- $M_2 < 1$



Nozzle Flow

Alt. 2: critical (choked) nozzle flow

- $M = 1$ at nozzle throat
- $A_t = A^*$
- $M_1 < 1$
- $M_2 > 1$



Nozzle Flow

Choked nozzle flow (no shocks):

- A^* is constant throughout the nozzle
- $A_t = A^*$

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \left(\frac{A_1}{A_t}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}\left(1+\frac{1}{2}(\gamma-1)M_1^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M_2 given by the supersonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \left(\frac{A_2}{A_t}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}\left(1+\frac{1}{2}(\gamma-1)M_2^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M is uniquely determined everywhere in the nozzle, with subsonic flow upstream of throat and supersonic flow downstream of throat

Nozzle Mass Flow

For a choked nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho^* u^* A^* = \rho_2 u_2 A_2$$

$$\left.\begin{aligned} \rho^* &= \frac{\rho^*}{\rho_o}\rho_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \frac{p_o}{RT_o} \\ a^* &= \frac{a^*}{a_o}a_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \sqrt{\gamma R T_o} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

- The maximum mass flow that can be sustained through the nozzle
- Valid for quasi-one-dimensional, inviscid, steady-state flow and calorically perfect gas

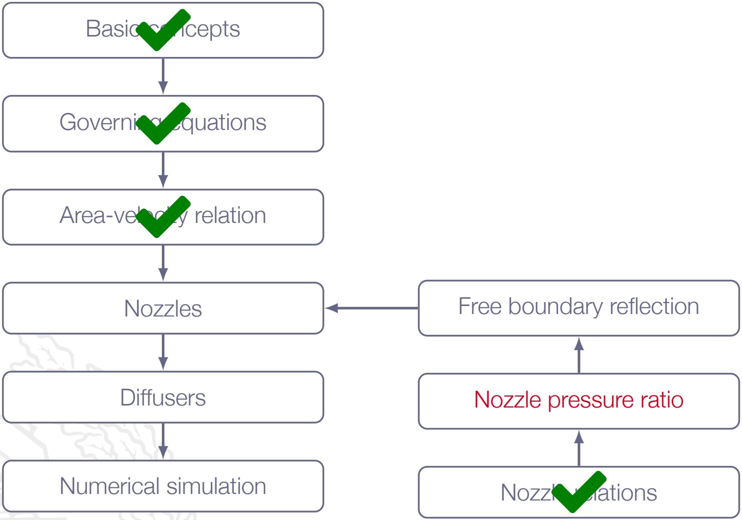
  Note: The massflow formula is valid even if there are shocks present downstream of throat!

Nozzle Mass Flow

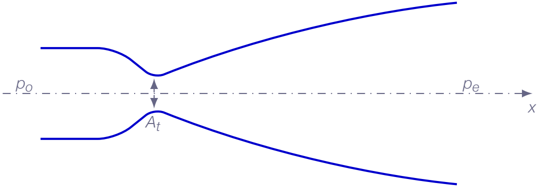
How can we increase mass flow through nozzle?

- increase p_o
- decrease T_o
- increase A_t
- decrease R
  (increase molecular weight, without changing γ)

Roadmap - Quasi-One-Dimensional Flow

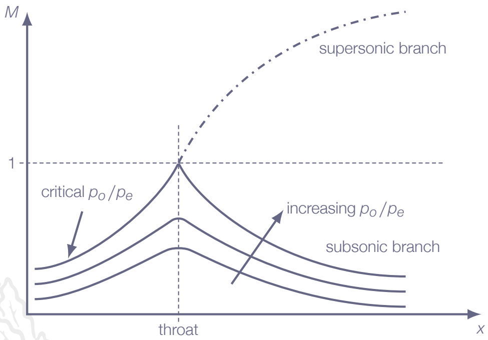


Nozzle Flow with Varying Pressure Ratio



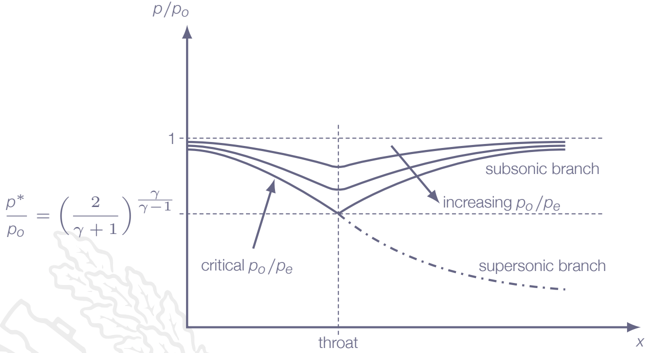
$A(x)$	area function
A_t	$\min\{A(x)\}$
p_o	inlet total pressure
p_e	outlet static pressure (ambient pressure)
p_o/p_e	pressure ratio

Nozzle Flow with Varying Pressure Ratio



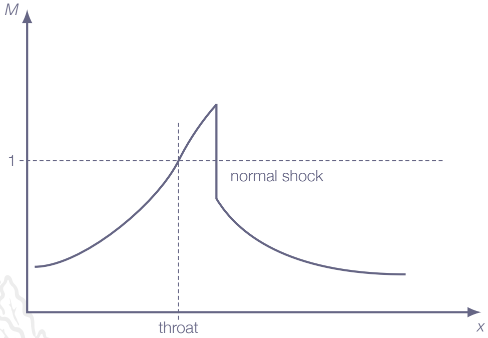
For critical p_o/p_e, a jump to supersonic solution will occur

Nozzle Flow with Varying Pressure Ratio

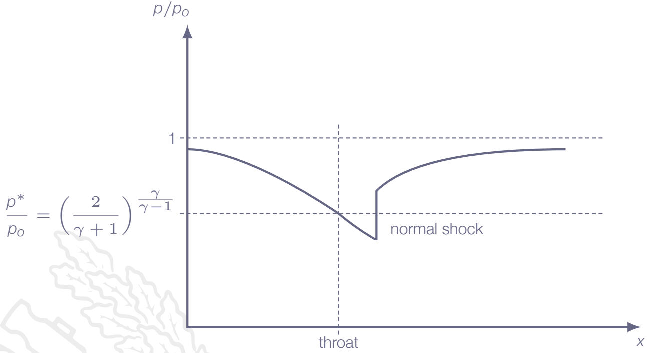


As the flow jumps to the supersonic branch downstream of the throat, a normal shock will appear in order to match the ambient pressure at the nozzle exit

Nozzle Flow with Varying Pressure Ratio



Nozzle Flow with Varying Pressure Ratio



Nozzle Flow with Varying Pressure Ratio (Summary)

$(p_o/p_e) < (p_o/p_e)_{cr}$

- the flow remains entirely subsonic
- the mass flow depends on p_e, *i.e.* the flow is not choked
- no shock is formed, therefore the flow is isentropic throughout the nozzle

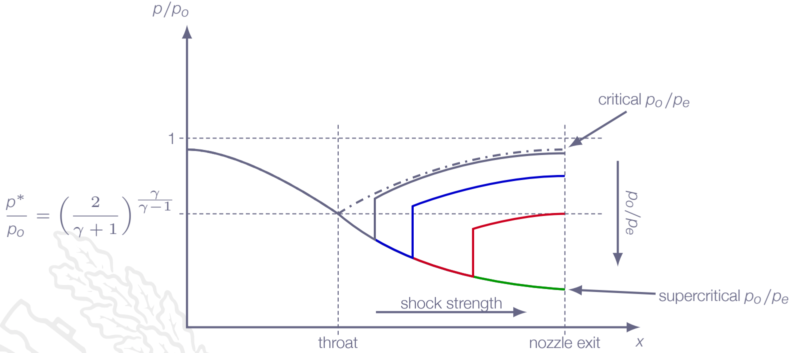
$(p_o/p_e) = (p_o/p_e)_{cr}$

- the flow just achieves $M = 1$ at the throat
- the flow will then suddenly flip to the supersonic solution downstream of the throat, for an infinitesimally small increase in (p_o/p_e)

$(p_o/p_e) > (p_o/p_e)_{cr}$

- the flow is choked (fixed mass flow), *i.e.* it does not depend on p_e
- a normal shock will appear downstream of the throat, with strength and position depending on (p_o/p_e)

Nozzle Flow with Varying Pressure Ratio



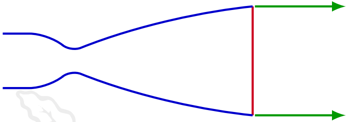
Nozzle Flow with Varying Pressure Ratio

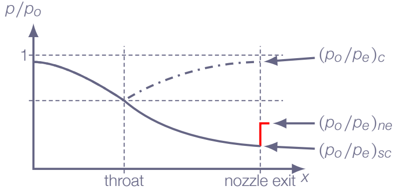
Effects of changing the pressure ratio (p_o/p_e) (*where p_e is the back pressure and p_o is the total pressure at the nozzle inlet*)

- critical value: $p_o/p_e = (p_o/p_e)_c$
  - nozzle flow reaches $M = 1$ at throat, flow becomes choked
- supercritical value: $p_o/p_e = (p_o/p_e)_{sc}$
  - nozzle flow is supersonic from throat to exit, without any interior normal shock - isentropic flow
- normal shock at exit: $(p_o/p_e) = (p_o/p_e)_{ne} < (p_o/p_e)_{sc}$
  - normal shock is still present but is located just at exit - isentropic flow inside nozzle

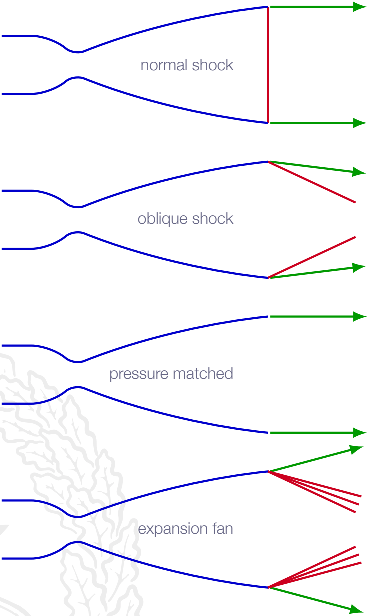
Nozzle Flow with Varying Pressure Ratio

Normal shock at exit





Nozzle Flow with Varying Pressure Ratio



$p_o/p_e = (p_o/p_e)_{ne}$
normal shock at nozzle exit

$(p_o/p_e)_{ne} < p_o/p_e < (p_o/p_e)_{sc}$
overexpanded nozzle flow

$p_o/p_e = (p_o/p_e)_{sc}$
pressure matched nozzle flow

$p_o/p_e > (p_o/p_e)_{sc}$
underexpanded nozzle flow

Nozzle Flow with Varying Pressure Ratio

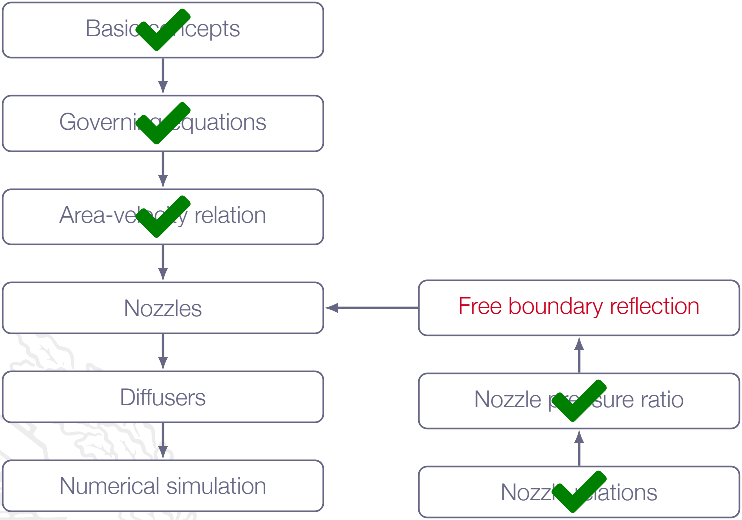
Quasi-one-dimensional theory

- When the interior normal shock is ”pushed out” through the exit (by increasing (p_o/p_e), *i.e.* lowering the back pressure), it disappears completely.
- The flow through the nozzle is then shock free (and thus also isentropic since we neglect viscosity).

Three-dimensional nozzle flow

- When the interior normal shock is ”pushed out” through the exit (by increasing (p_o/p_e)), an oblique shock is formed outside of the nozzle exit.
- For the exact supercritical value of (p_o/p_e) this oblique shock disappears.
- For (p_o/p_e) above the supercritical value an expansion fan is formed at the nozzle exit.

Roadmap - Quasi-One-Dimensional Flow

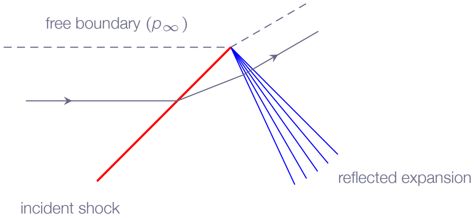


Chapter 5.6 Wave Reflection From a Free Boundary

Free-Boundary Reflection

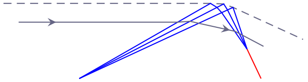
Free boundary - shear layer, interface between different fluids, etc

Free-Boundary Reflection - Shock Reflection



- No jump in pressure at the free boundary possible
- Incident shock reflects as expansion waves at the free boundary
- Reflection results in net turning of the flow

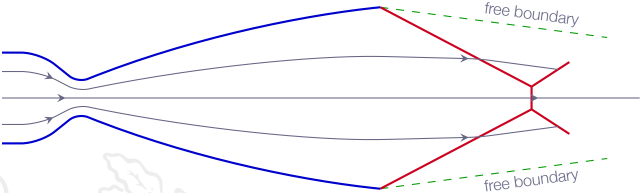
Free-Boundary Reflection - Expansion Wave Reflection



- No jump in pressure at the free boundary possible
- Incident expansion waves reflects as compression waves at the free boundary
- Finite compression waves coalesces into a shock
- Reflection results in net turning of the flow

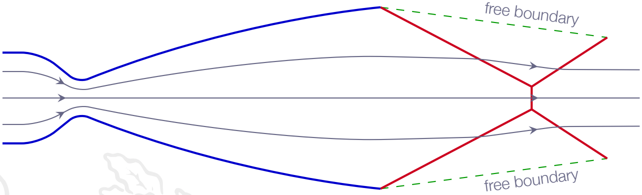
Free-Boundary Reflection - System of Reflections

overexpanded nozzle flow



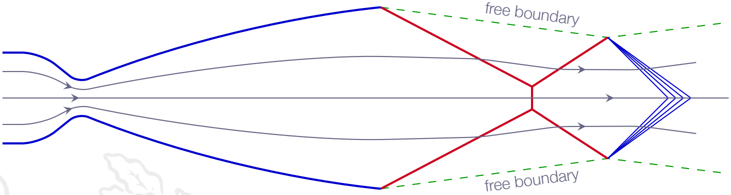
Free-Boundary Reflection - System of Reflections

shock reflection at jet centerline



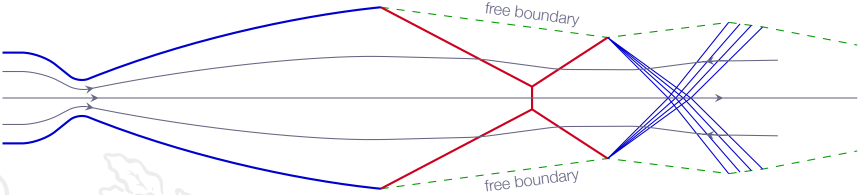
Free-Boundary Reflection - System of Reflections

shock reflection at free boundary



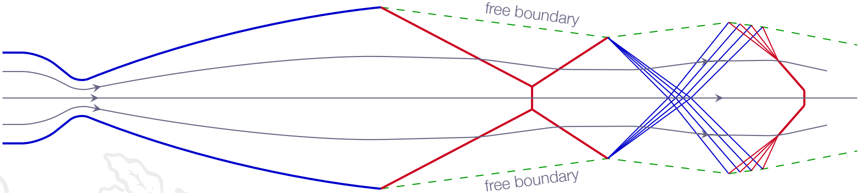
Free-Boundary Reflection - System of Reflections

expansion wave reflection at jet centerline



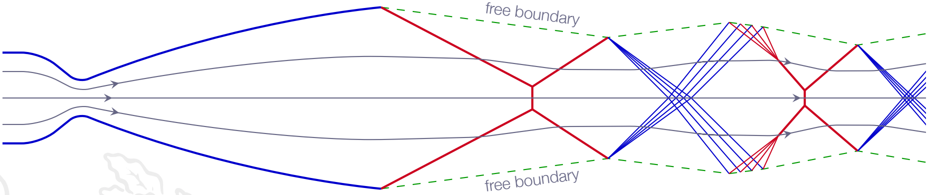
Free-Boundary Reflection - System of Reflections

expansion wave reflection at free boundary



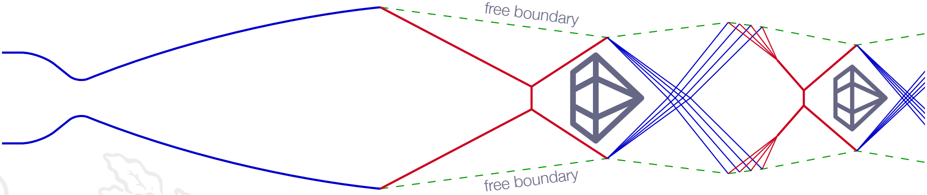
Free-Boundary Reflection - System of Reflections

repeated shock/expansion system



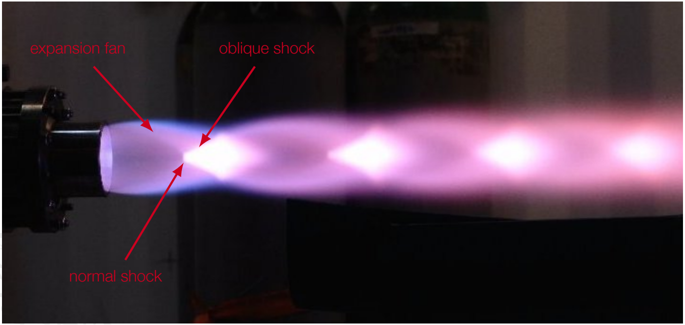
Free-Boundary Reflection - System of Reflections

shock diamonds



Free-Boundary Reflection - System of Reflections

underexpanded jet



Free-Boundary Reflection - Summary

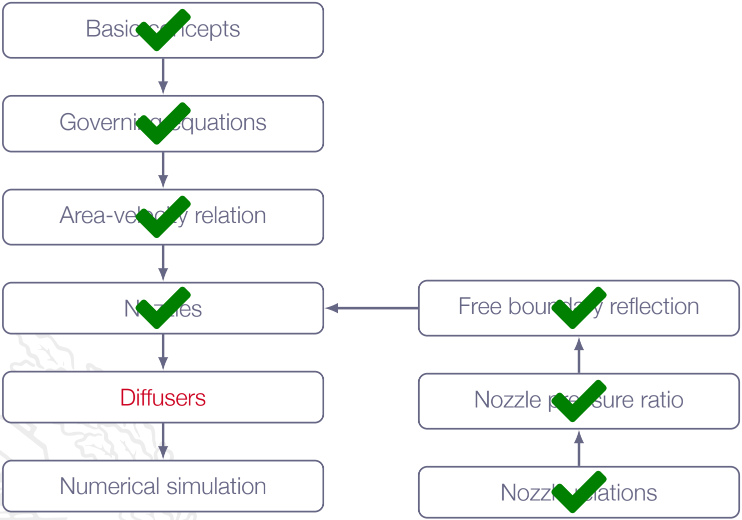
Solid-wall reflection

Compression waves reflects as compression waves
Expansion waves reflects as expansion waves

Free-boundary reflection

Compression waves reflects as expansion waves
Expansion waves reflects as compression waves

Roadmap - Quasi-One-Dimensional Flow

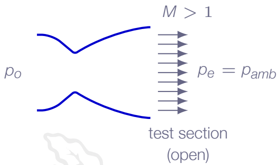


Chapter 5.5
Diffusers

Supersonic Wind Tunnel

wind tunnel with supersonic test section

open test section



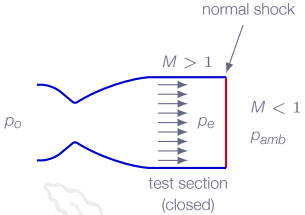
$p_o/p_e = (p_o/p_e)_{sc}$

$M = 3.0$ in test section $\Rightarrow$ $p_o/p_e = 36.7$!!!

Supersonic Wind Tunnel

wind tunnel with supersonic test section

enclosed test section, normal shock at exit



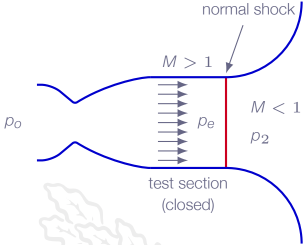
$p_o/p_{amb} = (p_o/p_e)(p_e/p_{amb}) < (p_o/p_e)_{sc}$

$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33 = 3.55$

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add subsonic diffuser after normal shock



$\left(p_{o_2} = p_{amb}\right)$

$p_o/p_{amb} = (p_o/p_e)(p_e/p_2)(p_2/p_{o_2})$

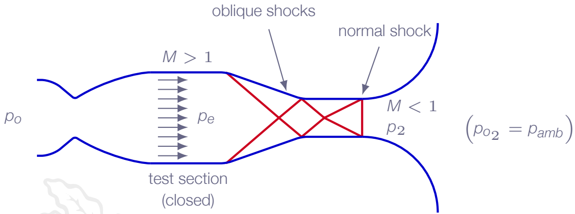
$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33/1.17 = 3.04$

Note: this corresponds exactly to total pressure loss across normal shock

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add supersonic diffuser before normal shock



well-designed supersonic + subsonic diffuser ⇒

1. decreased total pressure loss
2. decreased p_o and power to drive wind tunnel

Supersonic Wind Tunnel

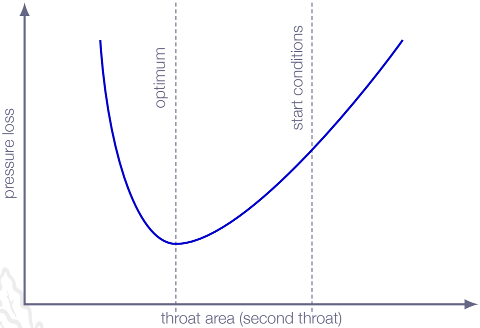
Main problems:

1. Design is extremely difficult due to complex 3D flow in diffuser
   - viscous effects
   - oblique shocks
   - separations

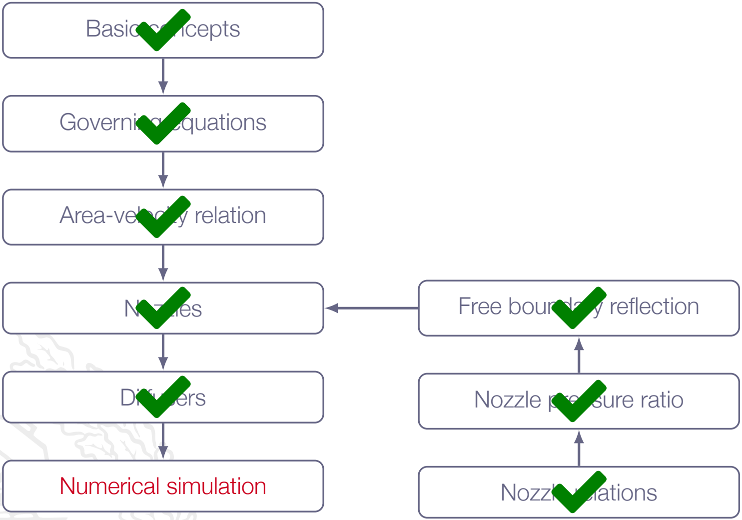
2. Starting requirements: second throat must be significantly larger than first throat

   solution:
   - variable geometry diffuser
   - second throat larger during startup procedure
   - decreased second throat to optimum value after flow is established

Supersonic Wind Tunnel



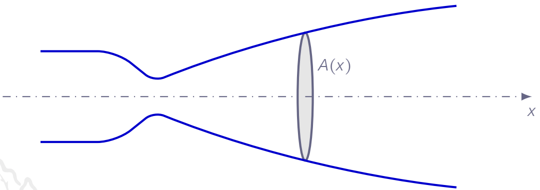
Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Euler Equations

Quasi-One-Dimensional Euler Equations

Example: choked flow through a convergent-divergent nozzle



Assumptions: inviscid, $Q = Q(x, t)$

Quasi-One-Dimensional Euler Equations

$$A(x)\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}\left[A(x)E\right] = A'(x)H$$

where $A(x)$ is the cross section area and

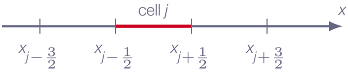
$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho e_o \end{bmatrix},\ E(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho h_o u \end{bmatrix},\ H(Q) = \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}$$

Numerical Approach

- Finite-Volume Method
- Method of lines, three-stage Runge-Kutta time stepping
- 3^{rd}-order characteristic upwinding scheme
- Subsonic inflow boundary condition at min(x)
  - T_o, p_o given
- Subsonic outflow boundary condition at max(x)
  - p given

Finite-Volume Spatial Discretization

$$\left(\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}\right)$$



Integration over cell j gives:

$$\frac{1}{2}\left[A(x_{j-\frac{1}{2}}) + A(x_{j+\frac{1}{2}})\right]\Delta x_j \frac{d}{dt}\bar{Q}_j +$$
$$\left[A(x_{j+\frac{1}{2}})\hat{E}_{j+\frac{1}{2}} - A(x_{j-\frac{1}{2}})\hat{E}_{j-\frac{1}{2}}\right] =$$
$$\left[A(x_{j+\frac{1}{2}}) - A(x_{j-\frac{1}{2}})\right]\hat{H}_j$$

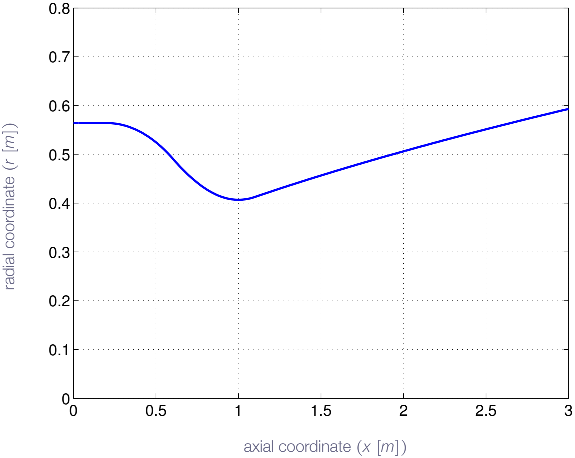
Finite-Volume Spatial Discretization

$$\bar{Q}_j = \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} QA(x)dx \right) \Bigg/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A(x)dx \right)$$

$$\hat{E}_{j+\frac{1}{2}} \approx E\left(Q\left(x_{j+\frac{1}{2}}\right)\right)$$

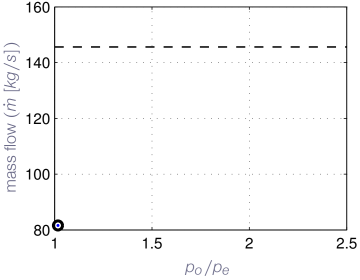
$$\hat{H}_j \approx \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} HA'(x)dx \right) \Bigg/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A'(x)dx \right)$$

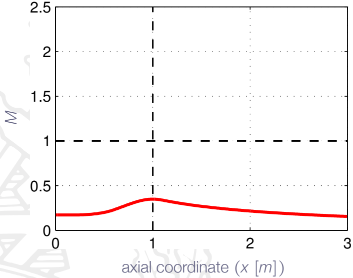
Nozzle Simulation - Back Pressure Sweep

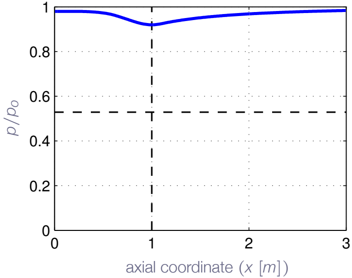


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.18 [bar]
p_o/p_e	1.02
$\dot{m}$	81.61 [kg/s]
M_{max}	0.35

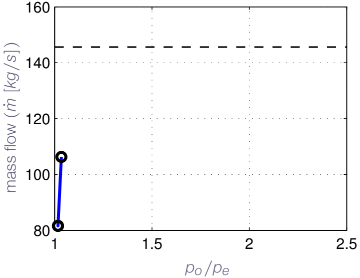


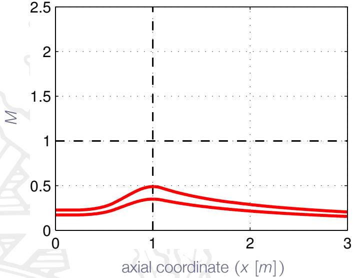


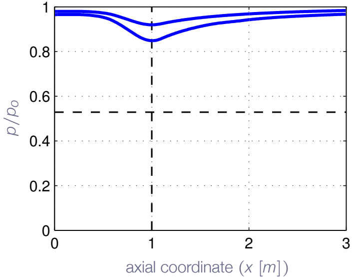


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.16 [bar]
p_o/p_e	1.03
$\dot{m}$	106.27 [kg/s]
M_{max}	0.49

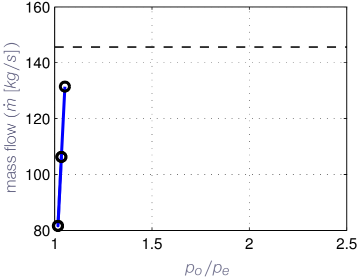


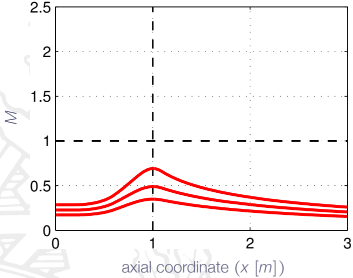


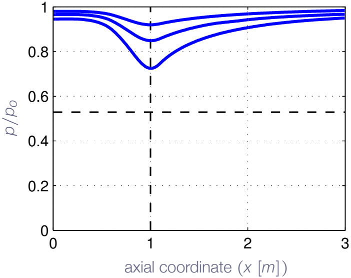


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.14 [bar]
p_o/p_e	1.05
$\dot{m}$	131.45 [kg/s]
M_{max}	0.69

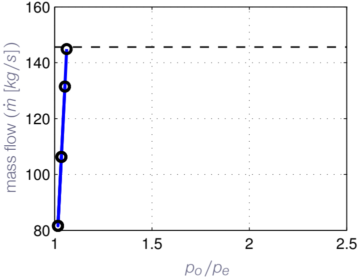


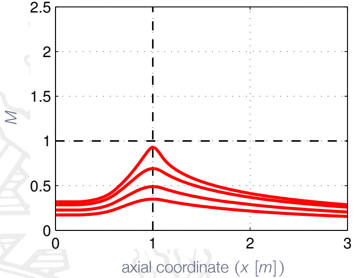


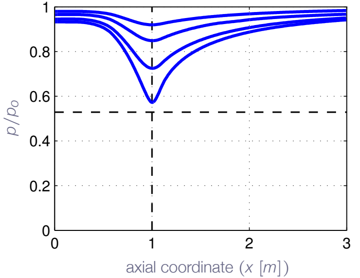


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_e	1.13	[bar]
p_o/p_e	1.06	
$\dot{m}$	144.88	[kg/s]
M_{max}	0.93	

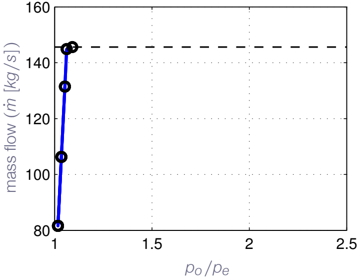


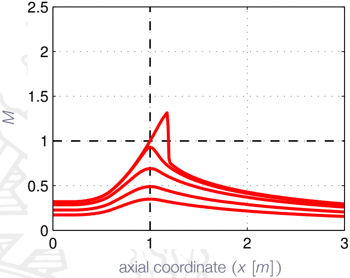


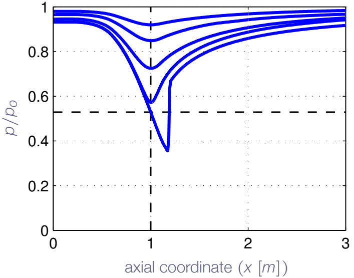


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.10 [bar]
p_o/p_e	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

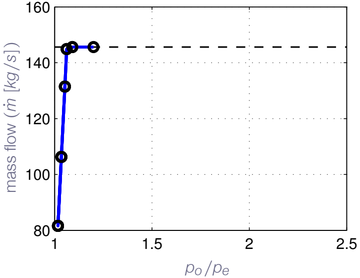


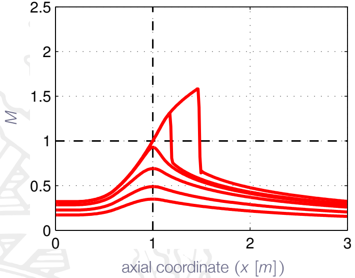


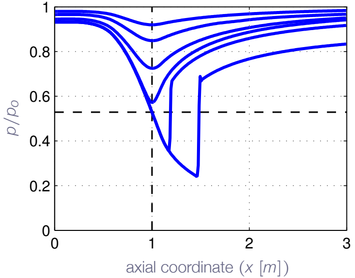


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.00 [bar]
p_o/p_e	1.20
$\dot{m}$	145.6 [kg/s]
M_{max}	1.58

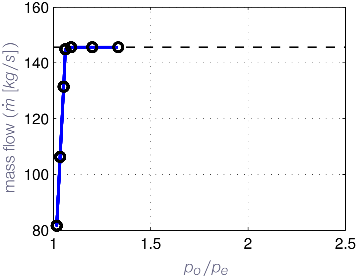


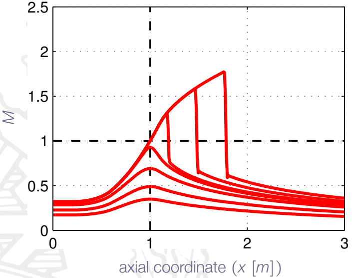


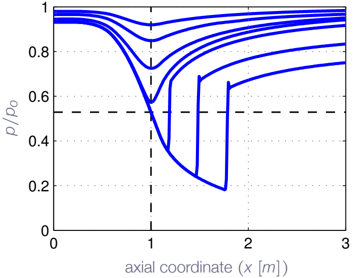


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_e	0.90	[bar]
p_o/p_e	1.33	
$\dot{m}$	145.6	[kg/s]
M_{max}	1.77	

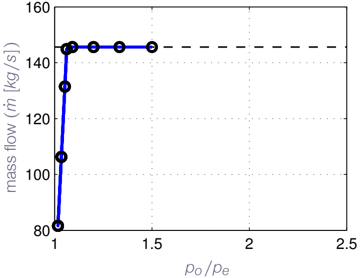


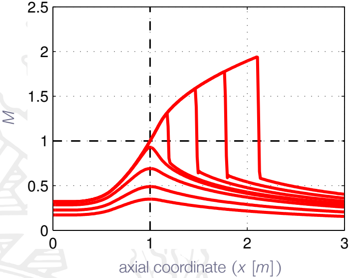


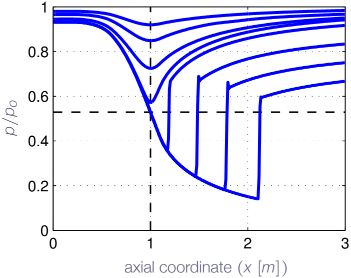


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.80 [bar]
p_o/p_e	1.50
$\dot{m}$	145.6 [kg/s]
M_{max}	1.94

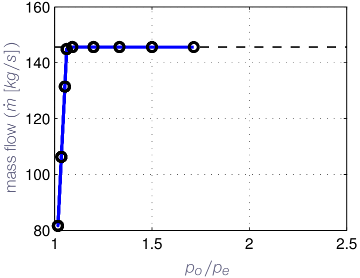


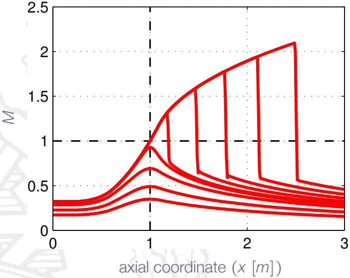


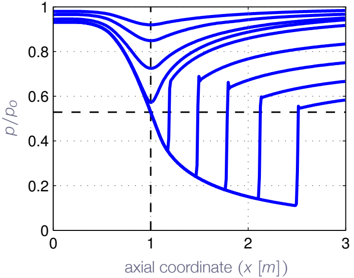


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.70 [bar]
p_o/p_e	1.71
$\dot{m}$	145.6 [kg/s]
M_{max}	2.10

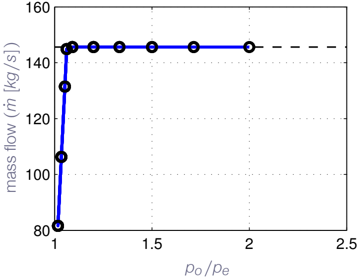


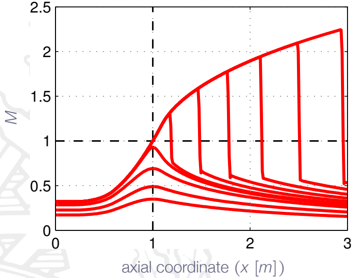


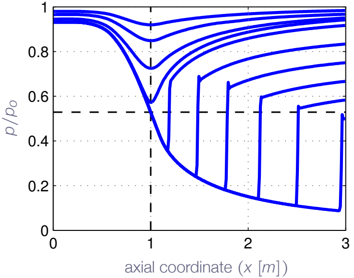


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.60 [bar]
p_o/p_e	2.00
$\dot{m}$	145.6 [kg/s]
M_{max}	2.24

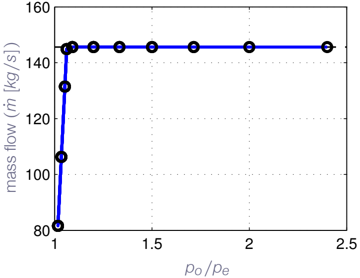


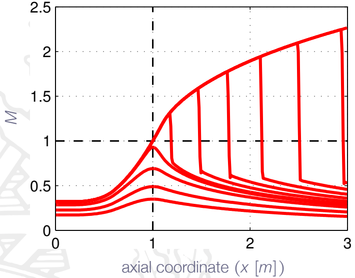


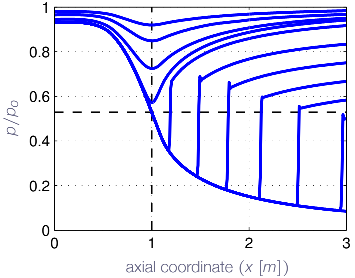


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	0.50 [bar]
p_o/p_e	2.40
$\dot{m}$	145.6 [kg/s]
M_{max}	2.26

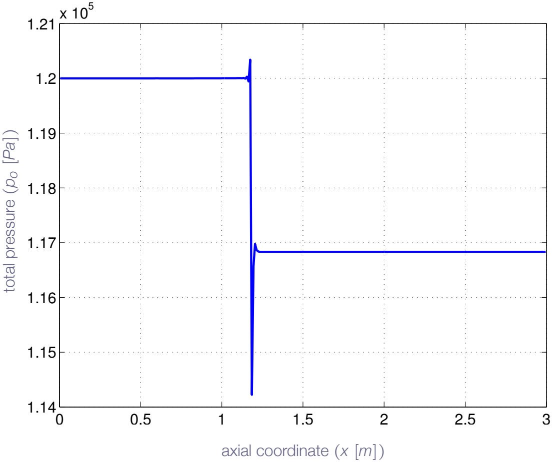






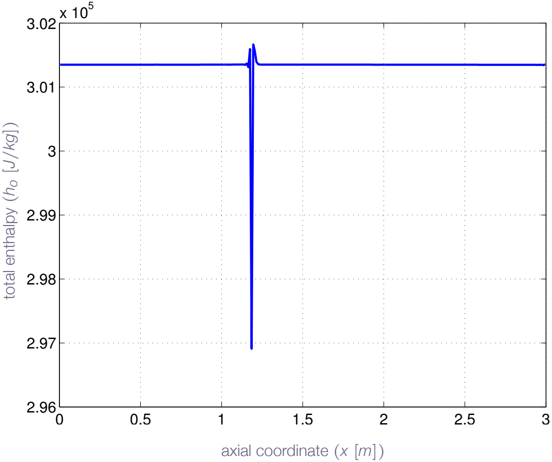
Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.10 [bar]
p_o/p_e	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

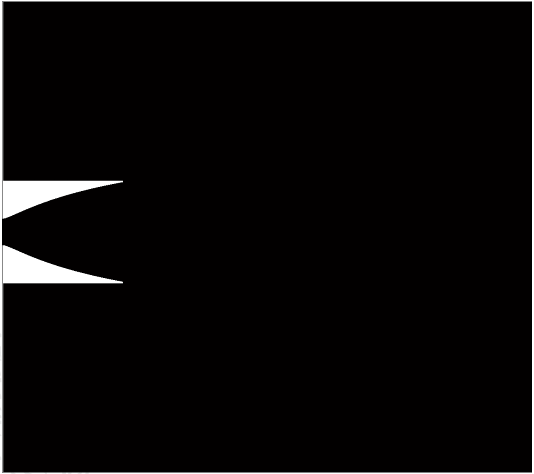


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_e	1.10 [bar]
p_o/p_e	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31



Modern Compressible Flow



Roadmap - Quasi-One-Dimensional Flow