Compressible Flow - TME085

Lecture 7

Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se

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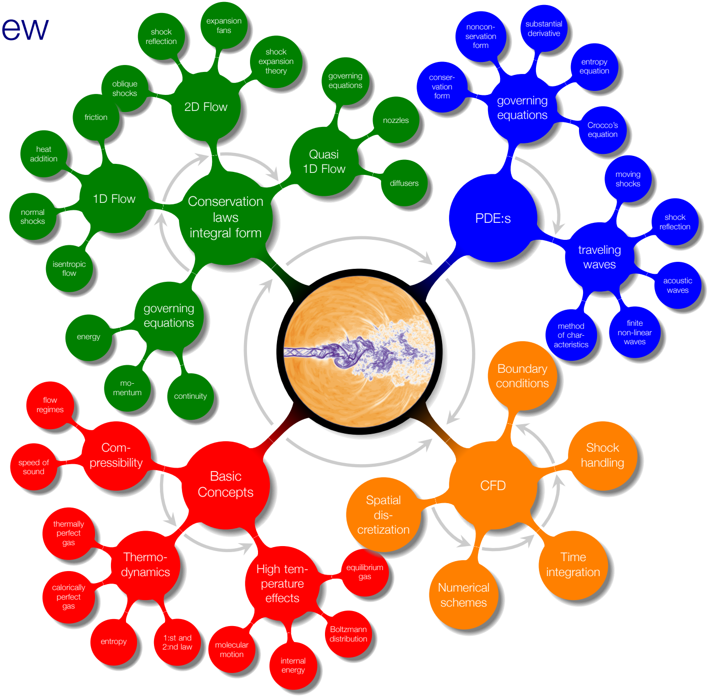
Overview



Conservation laws integral form
2D Flow
shock reflection
expansion fans
shock expansion theory
oblique shocks
friction
1D Flow
heat addition
normal shocks
isentropic flow
Quasi 1D Flow
governing equations
nozzles
diffusers
governing equations
energy
mo-mentum
continuity
governing equations
noncon-servation form
substantial derivative
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Crocco's equation
PDE:s
traveling waves
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shock reflection
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Boundary conditions
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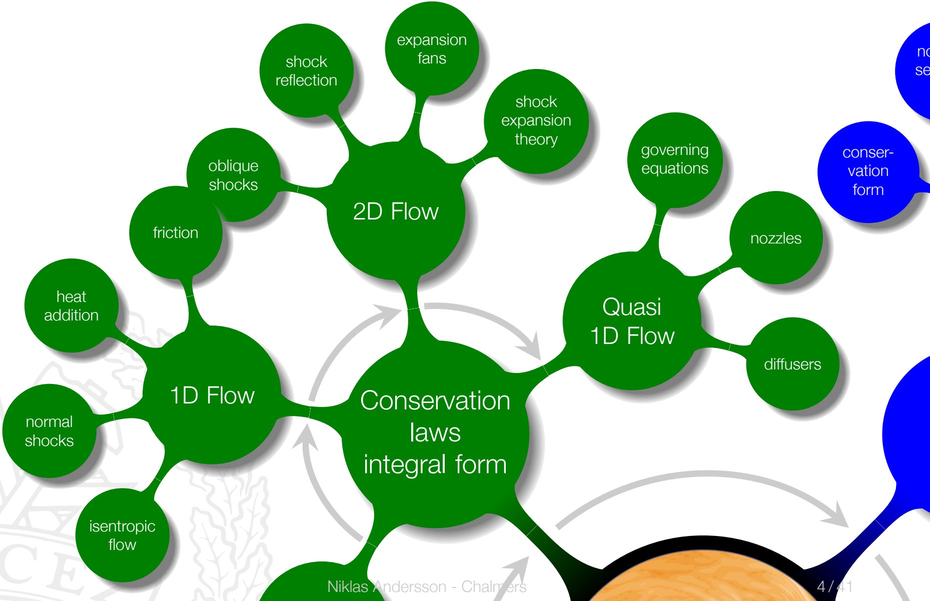
thermally perfect gas
calorically perfect gas
entropy
1:st and 2:nd law
High tem-perature effects
equilibrium gas
Boltzmann distribution
molecular motion
internal energy

Chapter 5 Quasi-One-Dimensional Flow

Overview



shock reflection
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conser- vation form

Addressed Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

6 Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases

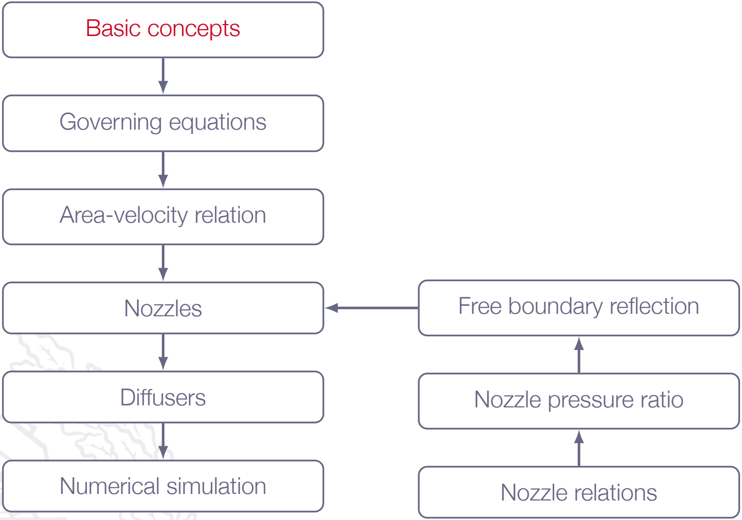
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

   a 1D isentropic flow*

      i detached blunt body shocks, nozzle flows

what does quasi-1D mean? either the flow is 1D or not, or?

Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Flow

Chapter 3 - One-dimensional steady-state flow

- overall assumption:

  one-dimensional flow
  constant cross section area
- applications:

  normal shock
  one-dimensional flow with heat addition
  one-dimensional flow with friction

Chapter 4 - Two-dimensional steady-state flow

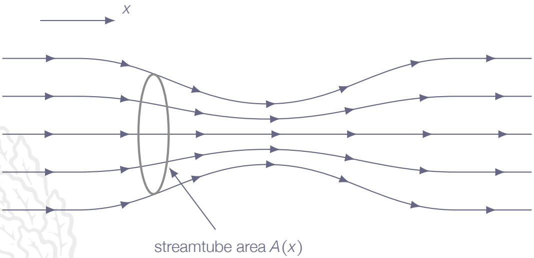
- overall assumption:

  two-dimensional flow
  uniform supersonic freestream
- applications:

  oblique shock
  expansion fan
  shock-expansion theory

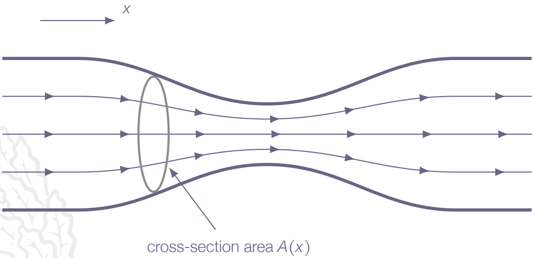
Quasi-One-Dimensional Flow

- Extension of one-dimensional flow to allow variations in streamtube area
- Steady-state flow assumption still applied

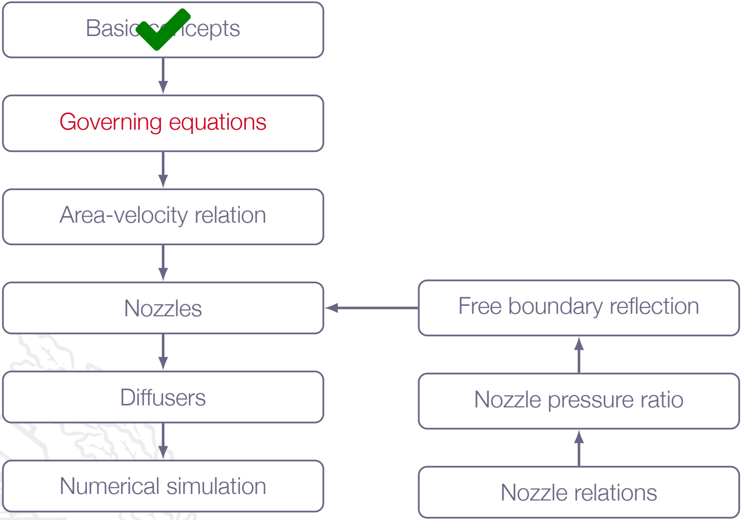


Quasi-One-Dimensional Flow

Example: tube with variable cross-section area



Roadmap - Quasi-One-Dimensional Flow

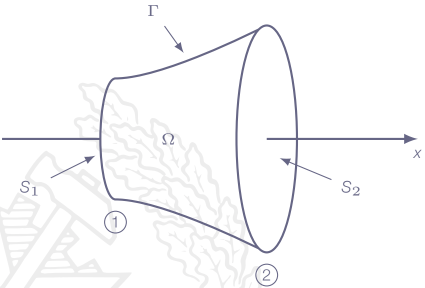


Chapter 5.2
Governing Equations

Governing Equations

Introduce cross-section-averaged flow quantities $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \ldots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Governing Equations - Mass Conservation

- steady-state
- no flow through Γ

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{=0} + \underbrace{\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{-\rho_1 u_1 A_1+\rho_2 u_2 A_2} = 0$$

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

Governing Equations - Momentum Conservation

- steady-state
- no flow through Γ

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}}_{=0}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=0$$

$$\oiint\limits_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=-\rho_1u_1^2A_1+\rho_2u_2^2A_2$$

$$\oiint\limits_{\partial\Omega}p\mathbf{n}dS=-p_1A_1+p_2A_2-\int_{A_1}^{A_2}pdA$$

$$\boxed{(\rho_1u_1^2+p_1)A_1+\int_{A_1}^{A_2}pdA=(\rho_2u_2^2+p_2)A_2}$$

Governing Equations - Energy Conservation

- steady-state
- no flow through Γ

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V}}_{=0} + \oiint\limits_{\partial\Omega} \left[\rho h_o(\mathbf{v}\cdot\mathbf{n})\right] dS = 0$$

which gives

$$\rho_1 u_1 A_1 h_{o_1} = \rho_2 u_2 A_2 h_{o_2}$$

from continuity we have that $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow$

$$\boxed{h_{o_1} = h_{o_2}}$$

Governing Equations - Summary

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

$$h_{o_1} = h_{o_2}$$

Governing Equations - Differential Form

Continuity equation:

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

or

$$\rho u A = c$$

where c is a constant $\Rightarrow$

$$\boxed{d(\rho u A) = 0}$$

Governing Equations - Differential Form

Momentum equation:

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2 \Rightarrow$$

$$d\left[(\rho u^2 + p)A\right] = pdA \Rightarrow$$

$$d(\rho u^2 A) + d(pA) = pdA \Rightarrow$$

$$u \underbrace{d(\rho u A)}_{=0} + \rho u A du + A dp + p dA = p dA \Rightarrow$$

$$\rho u A du + A dp = 0 \Rightarrow$$

$$\boxed{dp = -\rho u du}$$

Euler's equation

Governing Equations - Differential Form

Energy equation:

$$h_{o_1} = h_{o_2} \Rightarrow$$

$$dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{dh + udu = 0}$$

Governing Equations - Differential Form

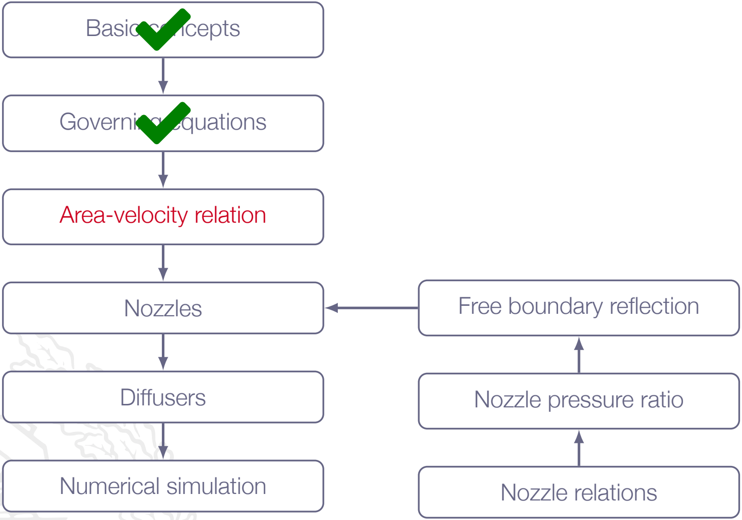
Summary (valid for all gases):

$$\boxed{\begin{gathered} d(\rho uA) = 0 \\[2em] dp = -\rho u du \\[2em] dh + u du = 0 \end{gathered}}$$

Assumptions:

- quasi-one-dimensional flow
- inviscid flow
- steady-state flow

Roadmap - Quasi-One-Dimensional Flow



Chapter 5.3
Area-Velocity Relation

Area-Velocity Relation

$$d(\rho u A) = 0 \Rightarrow uAd\rho + \rho A du + \rho u dA = 0$$

divide by $\rho u A$ gives

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation:

$$dp = -\rho u du \Rightarrow \frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -udu$$

Assuming adiabatic, reversible (isentropic) process and the definition of speed of sound gives

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \Rightarrow a^2\frac{d\rho}{\rho} = -udu \Rightarrow \frac{d\rho}{\rho} = -M^2\frac{du}{u}$$

Area-Velocity Relation

Now, inserting the expression for $\frac{d\rho}{\rho}$ in the rewritten continuity equation gives

$$(1 - M^2)\frac{du}{u} + \frac{dA}{A} = 0$$

or

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

which is the area-velocity relation

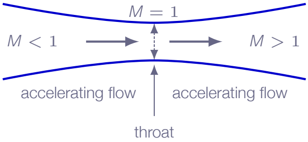
Area-Velocity Relation

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

$M < 1$: decreasing A correlated with increasing u

$M > 1$: increasing A correlated with increasing u

$M = 1$: $dA = 0$

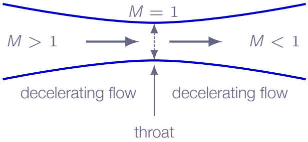


converging-diverging nozzle only possibility to obtain supersonic flow!

Area-Velocity Relation

Alternative:

Slowing down from supersonic to subsonic flow (supersonic diffuser)



in practice:
difficult to obtain completely shock-free flow in this case

Area-Velocity Relation

$$M \rightarrow 0 \Rightarrow \frac{dA}{A} = -\frac{du}{u}$$

$$\frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow$$

$$\frac{1}{Au}\left[udA + Adu\right] = 0 \Rightarrow$$

$$d(uA) = 0 \Rightarrow Au = c$$

where c is a constant

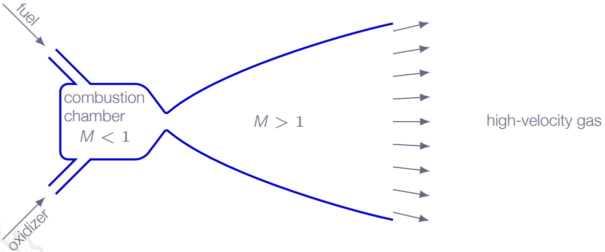
Area-Velocity Relation

Note 1: The area-velocity relation is only valid for isentropic flow

- not valid across a compression shock
  (due to entropy increase)

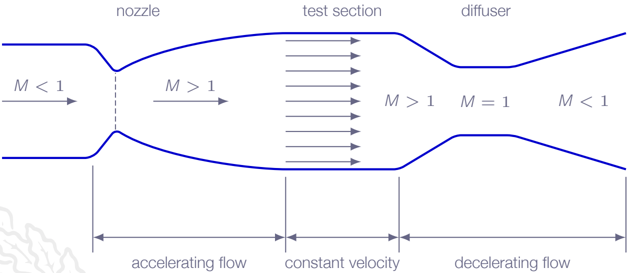
Note 2: The area-velocity relation is valid for all gases

Area-Velocity Relation Examples - Rocket Engine

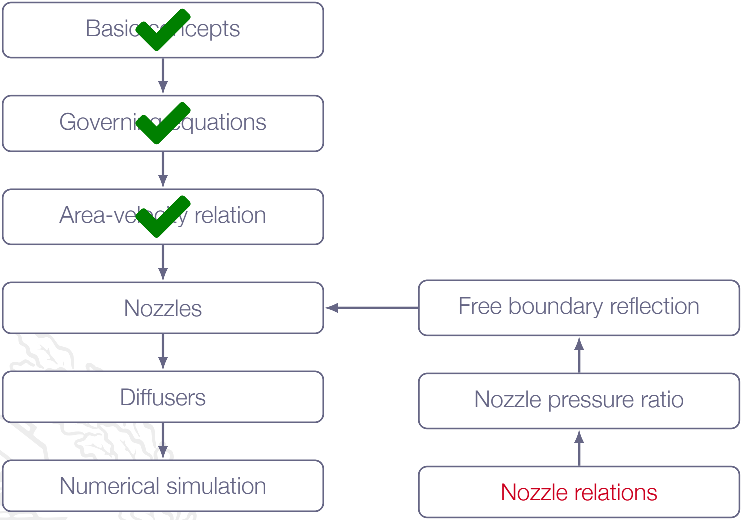


High-temperature, high-pressure gas in combustion chamber expand through the nozzle to very high velocities. Typical figures for a LH^2/LOx rocket engine: $p_o \sim 120$ [bar], $T_o \sim 3600$ [K], exit velocity ~ 4000 [m/s]

Area-Velocity Relation Examples - Wind Tunnel



Roadmap - Quasi-One-Dimensional Flow



Chapter 5.4
Nozzles

Nozzle Flow - Relations

Calorically perfect gas assumed:

From Chapter 3:

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

Critical conditions:

$$\frac{T_o}{T^*} = \left(\frac{a_o}{a^*}\right)^2 = \frac{1}{2}(\gamma + 1)$$

$$\frac{p_o}{p^*} = \left(\frac{T_o}{T^*}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

$$M^{*^2} = \frac{u^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*2}} \Rightarrow$$

$$M^{*^2} = M^2 \frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}$$

Nozzle Flow - Relations

For nozzle flow we have

$$\rho u A = c$$

where c is a constant and therefore

$$\rho^* u^* A^* = \rho u A$$

or, since at critical conditions $u^* = a^*$

$$\rho^* a^* A^* = \rho u A$$

which gives

$$\frac{A}{A*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

Nozzle Flow - Relations

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

$$\left.\begin{aligned}
\frac{\rho_o}{\rho} &= \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}} \\
\frac{\rho^*}{\rho_o} &= \left(\frac{T_o}{T^*}\right)^{\frac{-1}{\gamma-1}} \\
\frac{a^*}{u} &= \frac{1}{M^*}
\end{aligned}\right\} \Rightarrow \frac{A}{A^*} = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{1}{\gamma-1}} M^*}$$

Nozzle Flow - Relations

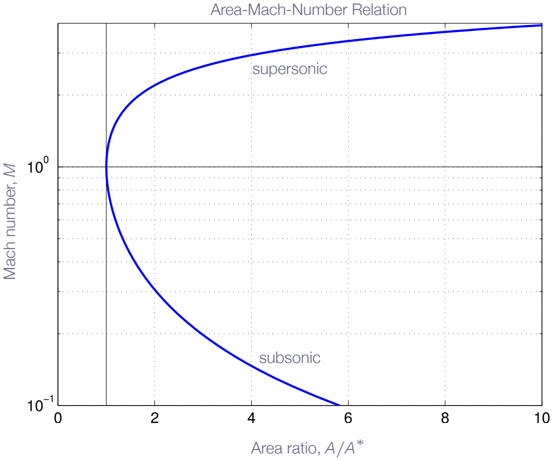
$$\left.\begin{aligned}
\left(\frac{A}{A^*}\right)^2 &= \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{2}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{2}{\gamma-1}} {M^*}^2} \\
{M^*}^2 &= M^2 \frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}
\end{aligned}\right\} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$

which is the area-Mach-number relation

Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$



Area-Mach-Number Relation

Note 1: Critical conditions used here are those corresponding to isentropic flow. Do not confuse these with the conditions in the cases of one-dimensional flow with heat addition and friction

Note 2: For quasi-one-dimensional flow, assuming inviscid steady-state flow, both total and critical conditions are constant along streamlines unless shocks are present (then the flow is no longer isentropic)

Note 3: The derived area-Mach-number relation is only valid for calorically perfect gas and for isentropic flow. It is not valid across a compression shock

Roadmap - Quasi-One-Dimensional Flow