Compressible Flow - TME085

Lecture 6

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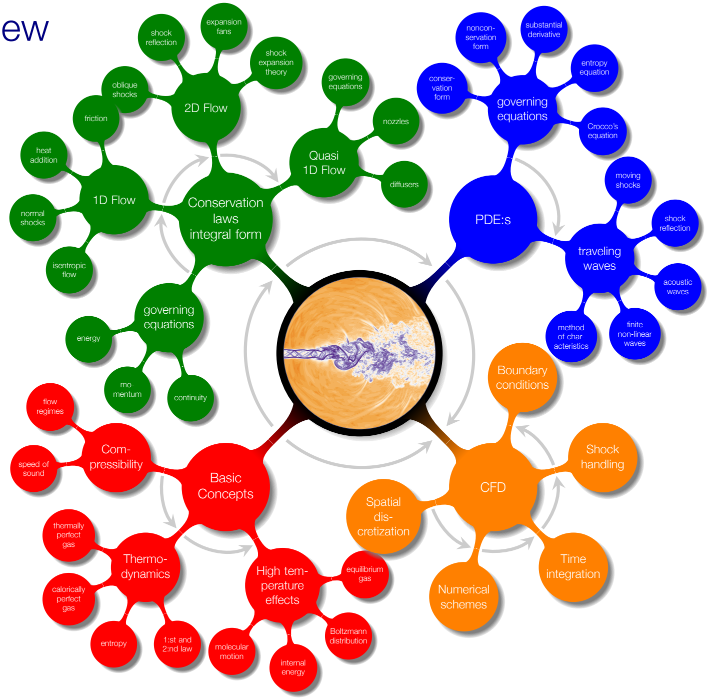
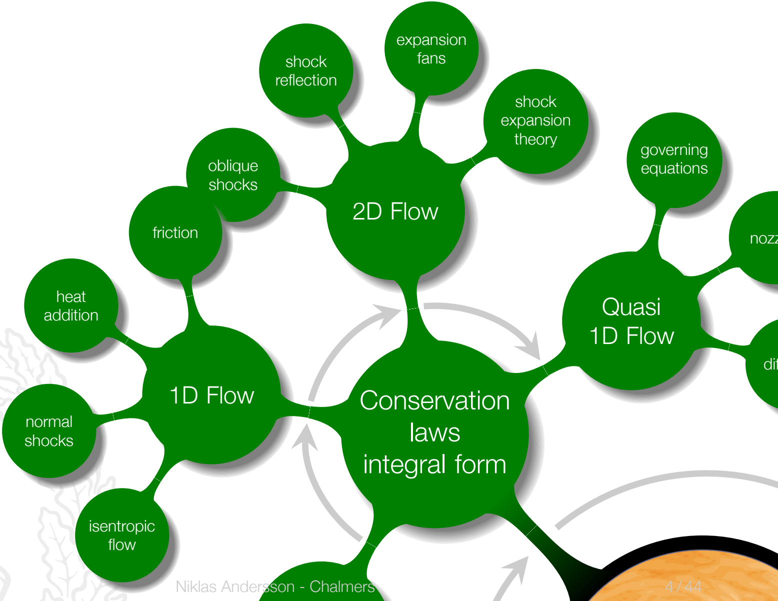
Overview



Conservation laws integral form
2D Flow
shock reflection
expansion fans
shock expansion theory
oblique shocks
friction
1D Flow
heat addition
normal shocks
isentropic flow
Quasi 1D Flow
governing equations
nozzles
diffusers
governing equations
energy
mo-mentum
continuity
PDE:s
governing equations
noncon-servation form
substantial derivative
entropy equation
conser-vation form
Crocco's equation
traveling waves
moving shocks
shock reflection
acoustic waves
method of char-acteristics
finite non-linear waves
CFD
Boundary conditions
Shock handling
Spatial dis-cretization
Numerical schemes
Time integration
Basic Concepts
Com-pressibility
flow regimes
speed of sound
Thermo-dynamics
thermally perfect gas
calorically perfect gas
entropy
1:st and 2:nd law
High tem-perature effects
equilibrium gas
Boltzmann distribution
molecular motion
internal energy

Chapter 4
Oblique Shocks and Expansion Waves

Overview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Conservation
laws
integral form
isentropic
flow

Addressed Learning Outcomes

7 **Explain** why entropy is important for flow discontinuities

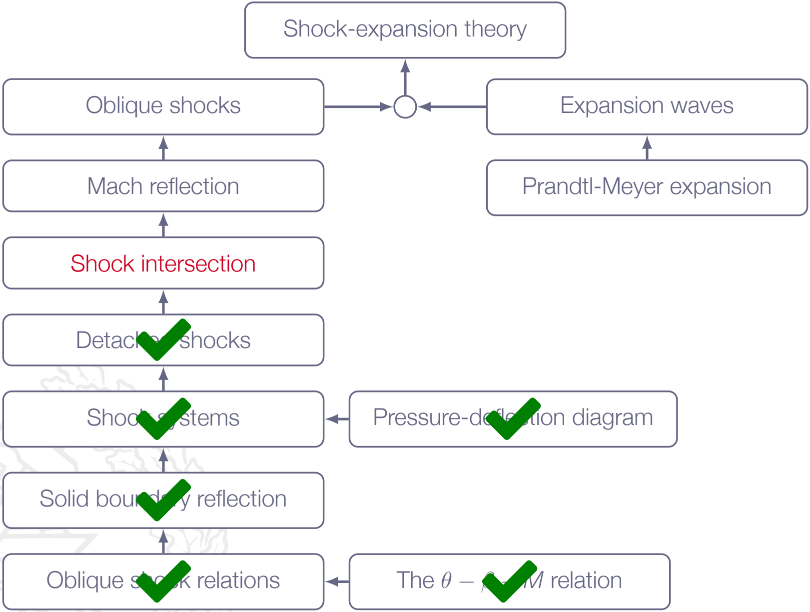
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

  e oblique shocks in 2D*
  f shock reflection at solid walls*
  g contact discontinuities
  h Prandtl-Meyer expansion fans in 2D
  i detached blunt body shocks, nozzle flows

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

what is the opposite of a shock?

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.10
Intersection of Shocks of the Same Family

Mach Waves

Oblique shock, angle β, flow deflection θ:

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

where

$$M_{n_1} = M_1 \sin(\beta)$$

and

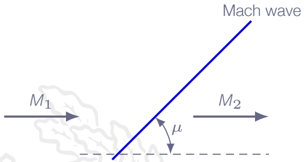
$$M_{n_2} = M_2 \sin(\beta - \theta)$$

Now, let $M_{n_1} \rightarrow 1$ and $M_{n_2} \rightarrow 1 \Rightarrow$ infinitely weak shock!

Such very weak shocks are called Mach waves

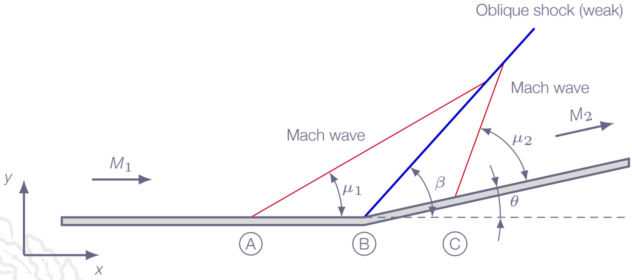
Mach Waves

$$M_{n_1} = 1 \Rightarrow M_1 \sin(\beta) = 1 \Rightarrow \beta = \arcsin(1/M_1)$$



- $M_2 \approx M_1$
- $\theta \approx 0$
- $\mu = \arcsin(1/M_1)$

Mach Waves



Oblique shock (weak)
Mach wave
M_2
μ_2
Mach wave
M_1
μ_1
β
θ
y
x
A
B
C

Mach Waves

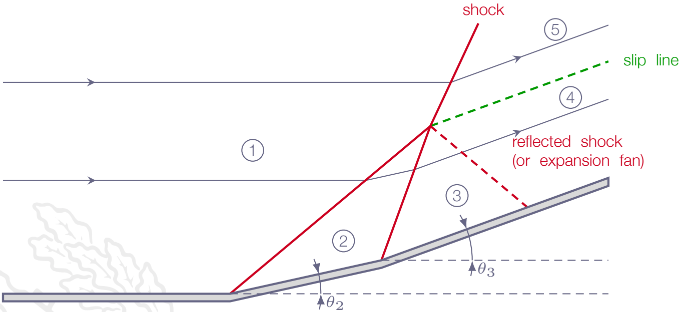
- Mach wave at A: $\sin(\mu_1) = 1/M_1$

- Mach wave at C: $\sin(\mu_2) = 1/M_2$

- Oblique shock at B: $M_{n_1} = M_1 \sin(\beta) \Rightarrow \sin(\beta) = M_{n_1}/M_1$
  - Existence of shock requires $M_{n_1} > 1 \Rightarrow \beta > \mu_1$
  - Mach wave intercepts shock!

- Also, $M_{n_2} = M_2 \sin(\beta - \theta) \Rightarrow \sin(\beta - \theta) = M_{n_2}/M_2$
  - For finite shock strength $M_{n_2} < 1 \Rightarrow (\beta - \theta) < \mu_2$
  - Again, Mach wave intercepts shock

Shock Intersection - Same Family



Shock Intersection - Same Family

Case 1: Streamline going through regions 1, 2, 3, and 4 (through two oblique (weak) shocks)

Case 2: Streamline going through regions 1 and 5 (through one oblique (weak) shock)

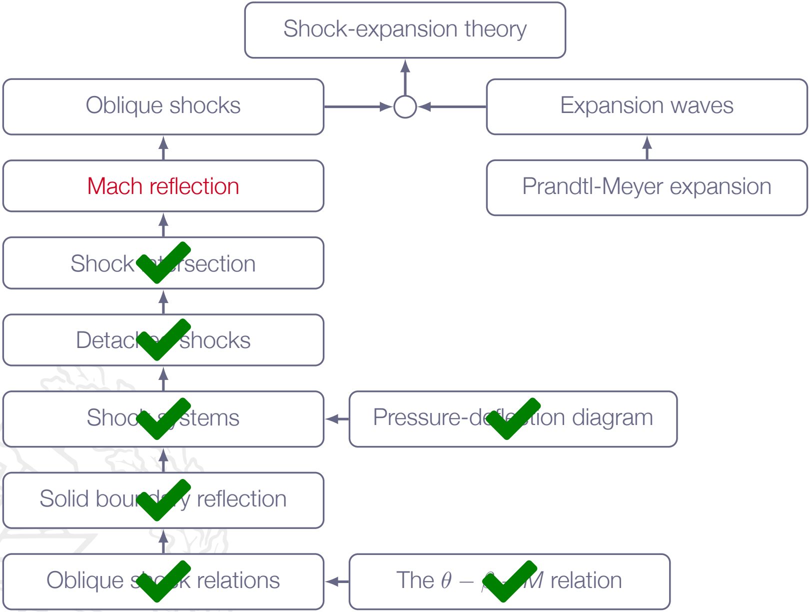
Problem: Find conditions 4 and 5 such that

a. $p_4 = p_5$
b. flow angle in 4 equals flow angle in 5

Solution may give either reflected shock or expansion fan, depending on actual conditions

A slip line usually appears, across which there is a discontinuity in all variables except p and flow angle

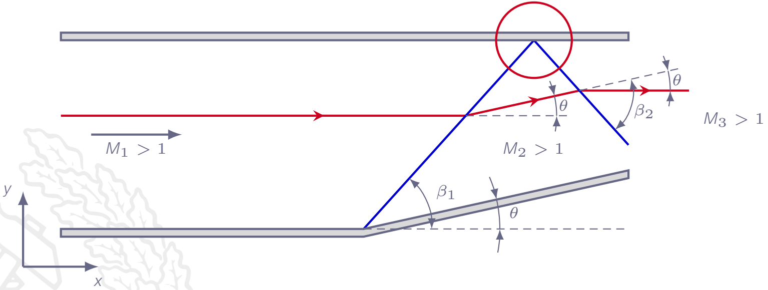
Roadmap - Oblique Shocks and Expansion Waves



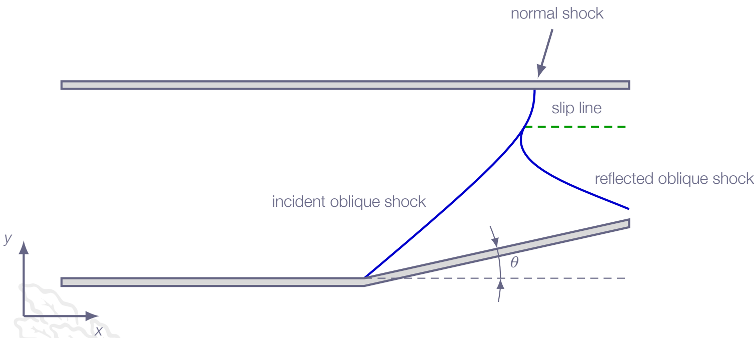
Chapter 4.11
Mach Reflection

Regular Shock Reflection

Regular reflection possible if both primary and reflected shocks are weak (see θ-β-M relation)



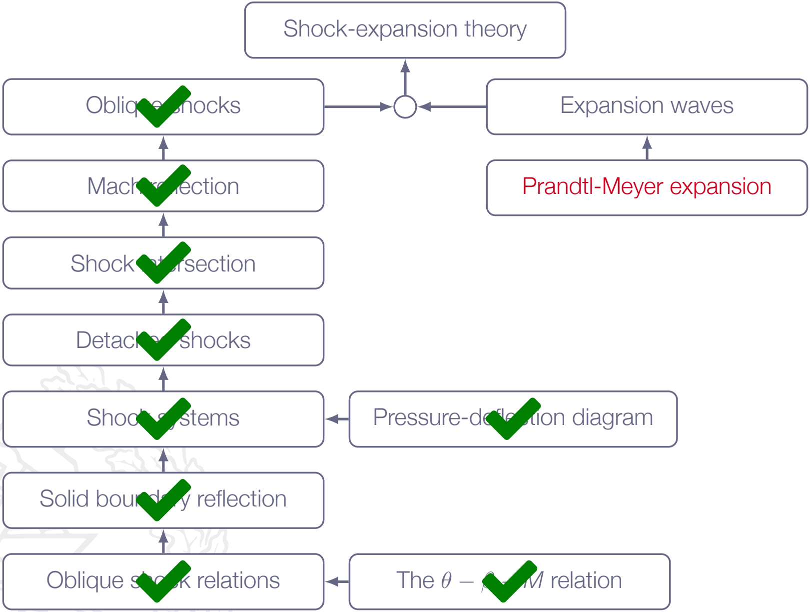
Mach Reflection



Mach reflection:

- appears when regular reflection is not possible
- more complex flow than for a regular reflection
- no analytic solution - numerical solution necessary

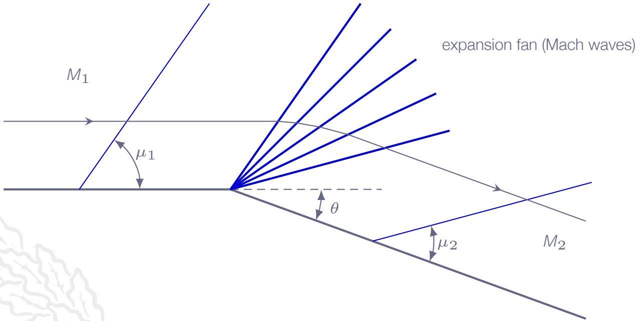
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.14 Prandtl-Meyer Expansion Waves

Prandtl-Meyer Expansion Waves

An expansion fan is a centered simple wave (also called Prandl-Meyer expansion)



- $M_2 > M_1$ (the flow accelerates through the expansion fan)
- $p_2 < p_1$, $\rho_2 < \rho_1$, $T_2 < T_1$

Prandtl-Meyer Expansion Waves

- Continuous expansion region
- Infinite number of weak Mach waves
- Streamlines through the expansion wave are smooth curved lines
- $ds = 0$ for each Mach wave $\Rightarrow$ the expansion process is ISENTROPIC!

Prandtl-Meyer Expansion Waves

- upstream of expansion $M_1 > 1$, $\sin(\mu_1) = 1/M_1$
- flow accelerates as it curves through the expansion fan
- downstream of expansion $M_2 > M_1$, $\sin(\mu_2) = 1/M_2$
- flow is isentropic $\Rightarrow s$, p_o, T_o, ρ_o, a_o, ... are constant along streamlines
- flow deflection: θ

Prandtl-Meyer Expansion Waves

It can be shown that $d\theta = \sqrt{M^2 - 1}\dfrac{dv}{v}$, where $v = |\mathbf{v}|$ (valid for all gases)

Integration gives

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dv}{v}$$

the term $\dfrac{dv}{v}$ needs to be expressed in terms of Mach number

$v = Ma \Rightarrow \ln v = \ln M + \ln a \Rightarrow$

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

Prandtl-Meyer Expansion Waves

Calorically perfect gas and adiabatic flow gives

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\left\{a = \sqrt{\gamma R T},\ a_o = \sqrt{\gamma R T_o}\right\} \Rightarrow \frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 \Rightarrow$$

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

or

$$a = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1/2}$$

Prandtl-Meyer Expansion Waves

Differentiation gives:

$$da = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-3/2} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

or

$$da = a \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

which gives

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a} = \frac{dM}{M} + \frac{-\frac{1}{2}(\gamma - 1)MdM}{1 + \frac{1}{2}(\gamma - 1)M^2} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2}\frac{dM}{M}$$

Prandtl-Meyer Expansion Waves

Thus,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M} = \nu(M_2) - \nu(M_1)$$

where

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M}$$

is the so-called Prandtl-Meyer function

Prandtl-Meyer Expansion Waves

Performing the integration gives:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1} \sqrt{M^2-1}$$

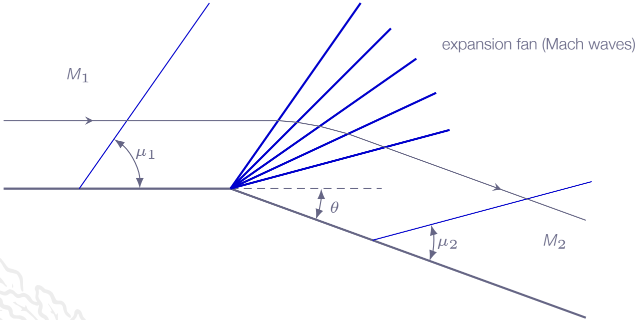
We can now calculate the deflection angle $\Delta\theta$ as:

$$\Delta\theta = \nu(M_2) - \nu(M_1)$$

$\nu(M)$ is tabulated in Table A.5 for a range of Mach numbers ($\gamma = 1.4$)

Prandtl-Meyer Expansion Waves

Example:



- $\theta_1 = 0$, $M_1 > 1$ is given
- θ_2 is given
- problem: find M_2 such that $\theta_2 = \nu(M_2) - \nu(M_1)$
- $\nu(M)$ for $\gamma = 1.4$ can be found in Table A.5

Prandtl-Meyer Expansion Waves

Since flow is isentropic, the usual isentropic relations apply:

(p_o and T_o are constant)

Calorically perfect gas:

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

Prandtl-Meyer Expansion Waves

since $p_{o_1} = p_{o_2}$ and $T_{o_1} = T_{o_2}$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right) \Big/ \left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

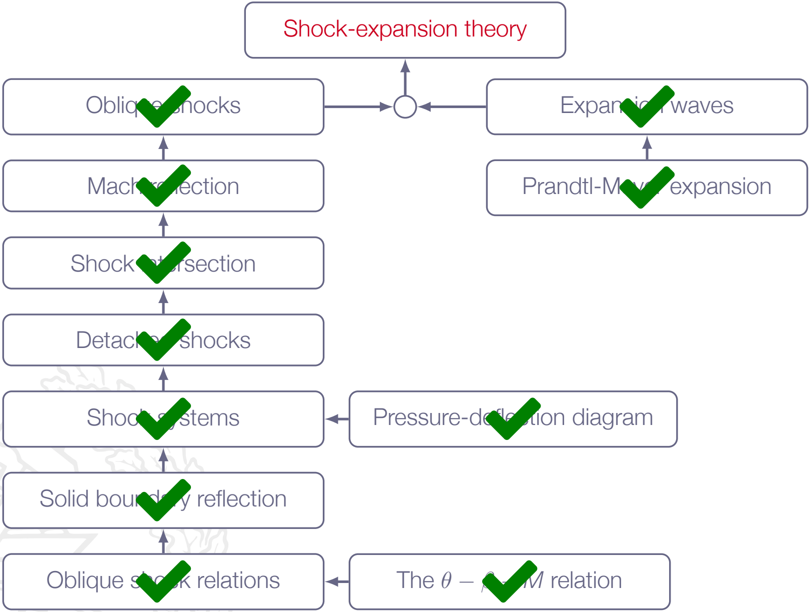
$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right) \Big/ \left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

Prandtl-Meyer Expansion Waves

Alternative solution:

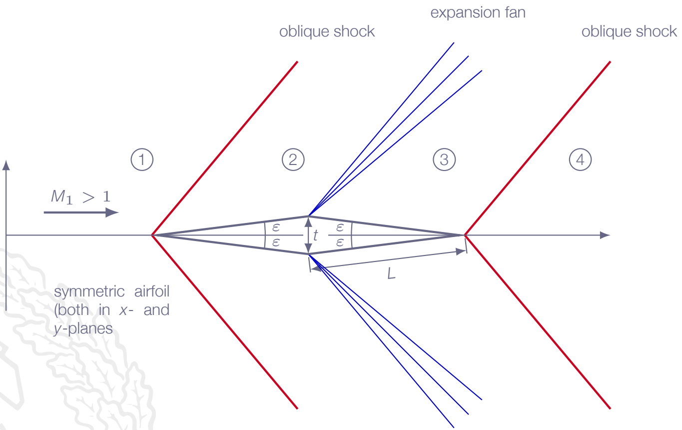
1. determine M_2 from $\theta_2 = \nu(M_2) - \nu(M_1)$
2. compute p_{o_1} and T_{o_1} from p_1, T_1, and M_1 (or use Table A.1)
3. set $p_{o_2} = p_{o_1}$ and $T_{o_2} = T_{o_1}$
4. compute p_2 and T_2 from p_{o_2}, T_{o_2}, and M_2 (or use Table A.1)

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.15
Shock Expansion Theory

Diamond-Wedge Airfoil



Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil

- symmetric airfoil
- zero incidence flow (freestream aligned with flow axis)

gives:

- symmetric flow field
- zero lift force on airfoil

Diamond-Wedge Airfoil

Drag force:

$$D = -\oiint_{\partial\Omega} p(\mathbf{n} \cdot \mathbf{e}_x) dS$$

$\partial\Omega$	airfoil surface
p	surface pressure
$\mathbf{n}$	outward facing unit normal vector
$\mathbf{e}_x$	unit vector in x-direction

Diamond-Wedge Airfoil

Since conditions 2 and 3 are constant in their respective regions, we obtain:

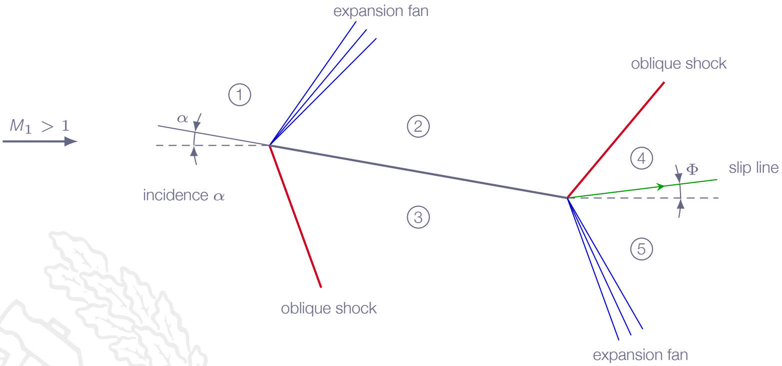
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

Flat-Plate Airfoil

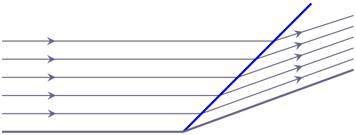
It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the missmatch of flow angles out

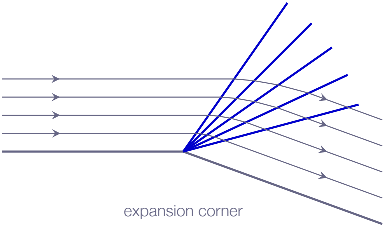
Flat-Plate Airfoil

- Flow states 4 and 5 must satisfy:
  - $p_4 = p_5$
  - flow direction 4 equals flow direction 5 (Φ)
- Shock between 2 and 4 as well as expansion fan between 3 and 5 will unjust themselves to comply with the requirements
- For calculation of lift and drag only states 2 and 3 are needed
- States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves



compression corner

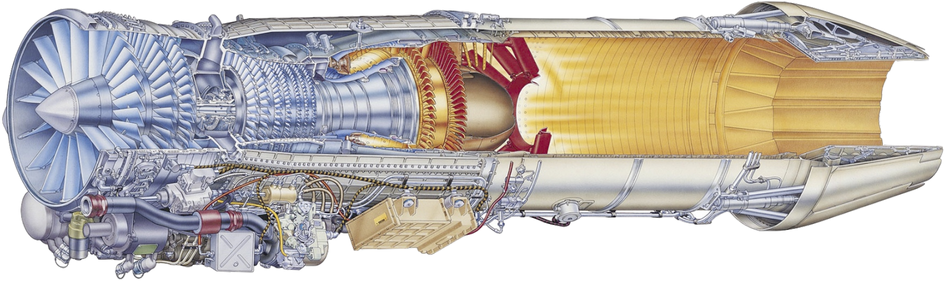


expansion corner

- M decrease
- $\|\mathbf{v}\|$ decrease
- p increase
- ρ increase
- T increase

- M increase
- $\|\mathbf{v}\|$ increase
- p decrease
- ρ decrease
- T decrease

Oblique Shocks and Expansion Waves



Roadmap - Oblique Shocks and Expansion Waves