MTF256 Turbulent Flow: Formulas

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The continuity equation and the Navier-Stokes equation for incompressible flow with constant viscosity read

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation and Navier-Stokes equation for incompressible flow with constant viscosity read

$$\begin{aligned}\frac{\partial \bar{v}_i}{\partial x_i} &= 0\\ \rho\frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j} - \rho\overline{v_i'v_j'}\right)\end{aligned} \tag{1}$$

The exact k equation reads

$$\frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

The exact $\bar{K}$ equation reads

$$\begin{aligned}\frac{\partial \bar{v}_j \bar{K}}{\partial x_j} = {} & \nu\frac{\partial^2 \bar{K}}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \bar{v}_i \bar{p}}{\partial x_i} - \nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j}\\ & - \frac{\partial \bar{v}_i \overline{v_i'v_j'}}{\partial x_j} + \overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}.\end{aligned}$$

The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}\frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) = {} & -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k}\\ & -\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right)\\ & + \frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}\end{aligned}$$

In the Boussinesq assumption an eddy (i.e. a *turbulent*) viscosity is introduced to model the unknown Reynolds stresses in Eq. 1. The stresses are modelled as

$$\overline{v_i'v_j'} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}k = -2\nu_t\bar{s}_{ij} + \frac{2}{3}\delta_{ij}k$$

Trick 1:

$$A_i\frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j\frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i\frac{\partial A_i}{\partial x_j} = \frac{1}{2}\frac{\partial A_i A_i}{\partial x_j}$$