THESIS FOR THE DEGREE OF MASTER OF SCIENCE

Formulation and Implementation of a $\overline{v\theta} - g$ Model to Improve the Wall Normal Turbulent Heat Flux Predictions in a Fully Developed Channel Flow

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by

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Abstract

The performance of three different $k - \omega$ models and a $k - \varepsilon$ model is studied in a fully developed 1D channel flow. The results are compared with the existing DNS data.

All $\overline{v^2} - f$ models of today are based on a HRN $k - \varepsilon$ model. There are two main disadvantages with $k - \varepsilon - \overline{v^2} - f$ model. One is that it is numerically unstable for grids with well resolved wall regions (low y^+ values near the wall). The second disadvantage is the uncertainty in specifying ε near the wall. In the present work, $\overline{v^2} - f$ model based on HRN $k - \omega$ model is developed which has numerically appealing boundary condition for ω near the wall. The model is implemented to predict the properties in a fully developed 1D channel flow. A new set of model constants were determined by tuning the results to match DNS data. However further testing is needed since the new model is intended for computing complex engineering flows.

The prediction of heat transport in turbulent flows is of great importance in many industrial applications. A new turbulent heat flux model named $\overline{v\theta} - g$ has been proposed to improve the prediction of wall-normal turbulent heat flux. The main advantage of $\overline{v\theta} - g$ model over the traditional model is its ability to account for the elliptic nature of pressure strain term through g-equation. The performance of the model is assessed in a fully developed 1D channel flow and the results are shown to be in good agreement with the DNS data.

Acknowledgments

I would like to express my sincere gratitude to my supervisor, Prof. Lars Davidson for all his support and advice. I specially thank him for the weekly discussions we used to have, no matter how busy his shedule was. I can say for sure that past four and a half months have been the most productive days of my learning at Chalmers.

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Finally, I would like to thank everyone at the Department, for creating such a stimulating and work friendly atmosphere. Being a part of Thermo is real fun!

Contents

Chapter 1

Introduction

Two equation eddy viscosity models have served as the foundation for much of turbulence research for past two decades. The main reason for their popularity is that they are complete, *ie* they can predict properties of a given turbulent flow with no prior knowledge of turbulent structures.

All two equation eddy viscosity models use turbulent kinetic energy as one of the solved turbulent quantities. Along with the transport equation for k, another transport equation is solved for a second turbulent quantity. The only difference in all two equation models is the choice of this second quantity we solve for. In the present work, the performances of four different two-equation models was analyzed in a fully developed 1D channel flow.

1.1 Why $k - \omega - \overline{v^2} - f$ Model ?

Most of the two equation eddy viscosity models use damping functions to account for the near wall effect. The $\overline{v^2} - f$ model came up with a desire to eliminate the need to damp turbulence models in order to predict the near-wall phenomena.

The $\overline{v^2} - f$ model has been developed from a simplified second moment closure approach, taking into account the near wall turbulence, while keeping the eddy viscosity assumption. Hence, $\overline{v^2} - f$ models are valid through the whole flow domain, automatically becoming a near wall model close to the wall. For many years, $\overline{v^2} - f$ models have proved to be very accurate by consistently predicting more accurate results than standard two equation eddy viscosity models.

All $\overline{v^2} - f$ models of today are based on a High Reynolds Number k-ε model. One of the main problems with $k - \varepsilon - \overline{v^2} - f$ model is the boundary condition for ε near the wall which reads,

$$\varepsilon_{wall} = \nu \frac{2k}{y^2}$$

For a given problem ν is constant and for a given mesh y is constant. But, k keeps on changing and hence ε. The f - equation is highly diffusive in nature and is very sensitive to the changes in ε. This strong variable coupling near the wall causes numerical difficulties.

On contrary, boundary condition for ω reads,

$$\omega_{wall} = \frac{6\nu}{\beta y^2}$$

which is always a constant for a given mesh. Hence this boundary condition is numerically appealing.

Another drawback with $k-\varepsilon-\overline{v^2}-f$ is that it is numerically unstable for grids with well resolved wall regions [11]. An attempt to overcome these problems lead to the formulation of $k - \omega - \overline{v^2} - f$ model.

1.2 Why $\overline{v\theta} - g$ Model ?

The traditional model which is used to solve the wall-normal heat flux is of the form,

$$\frac{D\overline{v\theta}}{Dt} = P_{\theta i} + D_{\theta i} - \varepsilon_{\theta i} - \pi_{\theta i}$$

The terms on right hand side are Production, Diffusion, Dissipation and Pressure Scramble term respectively. The main disadvantage of this model is that it doesn't account for the elliptic nature of the pressure-strain term. This drawback is overcome by $\overline{v\theta} - g$ model by allowing the pressure field have its elliptic nature, through g-equation. Detailed formulation of the model is presented in chapter 6.

Chapter 2

Governing Equations

All fluid motions (laminar or turbulent) are governed by a set of dynamical equations namely the *Continuity equation* and the *Momentum equation*,

$$\begin{aligned} \left[\frac{\partial \tilde{\rho}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\rho}}{\partial x_j}\right] + \tilde{\rho}\frac{\partial \tilde{u}_j}{\partial x_j} &= 0 && (2.1) \\ \tilde{\rho}\left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}\right] &= -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial \tilde{T}_{ij}^{(v)}}{\partial x_j} && (2.2) \end{aligned}$$

$\tilde{u}_i(\vec{x}, t)$ represents the i-th component of the fluid velocity at a point in space $\vec{x}$ and time t.

$\tilde{p}(\vec{x}, t)$ is the static pressure.

$\tilde{T}_{ij}(\vec{x}, t)$ are the viscous stresses.

$\tilde{\rho}$ is the fluid density.

The tilde symbol indicates that an instantaneous quantity is being considered.

For many flows of interest, the fluid behaves as a Newtonian fluid in which the viscous stresses are related to the incompressible fluid motion using a property of fluid, viscosity.

$$\tilde{T}_{ij}^{(v)} = 2\mu\left(\tilde{s}_{ij} - \frac{1}{3}\tilde{s}_{kk}\delta_{ij}\right) \tag{2.3}$$

$\tilde{s}_{ij}$ is the instantaneous strain rate tensor given by,

$$\tilde{s}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) \tag{2.4}$$

For incompressible flows, the derivative of density following the fluid material is zero and hence 2.1 & 2.2 are simplified to,

$$\frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0 \tag{2.5}$$

$$\left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}\right] = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \tag{2.6}$$

When the flow is turbulent it is convenient to analyze the flow in two parts, a mean component and a fluctuating component,

$$\begin{aligned} \tilde{u}_i &= U_i + u_i \\ \tilde{p} &= P + p \\ \tilde{T}_{ij}^{(v)} &= T_{ij}^{(v)} + \tau_{ij}^{(v)} \end{aligned}$$

This technique of decomposing is referred to as *Reynolds Decomposition*. Inserting these decomposition in to the instantaneous equations and time averaging results in *Reynolds Averaged Navier Stokes*(RANS) equations.

$$\frac{\partial U_j}{\partial x_j} = 0 \tag{2.7}$$

$$\left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right] = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\{\overline{u_i u_j}\} \tag{2.8}$$

The last term in 2.8 represents the correlation between fluctuating velocities and is called as *Reynolds stress tensor*. All the effects of turbulent fluid motion on the mean flow is lumped in to this single term by the process of averaging. This will enable great savings in terms of computational requirements. On the other hand, the process of averaging generated six new unknown variables. Now, in total we have ten unknowns (3-velocity, 1-pressure, 6-stresses) and only four equations (1-continuity, 3 components of Navier Stokes equation). Hence we are six equations too few. This is referred to as the *Closure problem*. Based on the way we close the Reynolds stress tensor, there are two main categories,

(a) Eddy Viscosity Models.

(b) Reynolds stress Models.

Eddy Viscosity Models

The Reynolds stress tensor resulting from time averaging of Navier Stokes is closed by replacing it with an eddy viscosity multiplied by velocity gradients. This is referred to as the *Boussinesq assumption*.

$$\rho\overline{u_i u_j} = -\mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{2.9}$$

In order to make the above equation valid upon contraction, we rewrite it as,

$$\rho\overline{u_i u_j} = -\mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{2}{3}\rho\delta_{ij}k \tag{2.10}$$

The eddy viscosity is treated as a scalar quantity and is determined using a turbulent velocity scale v and a length scale l, based on the dimensional analysis.

$$\nu_t = vl \tag{2.11}$$

There are different types of EVM's based on the way we close the eddy viscosity. Algebraic or zero equation EVM's normally use a geometric relation to compute the eddy viscosity. In one equation EVM's we solve for one turbulence quantity and a second turbulent quantity is obtained from algebraic expression. These two quantities are used to describe the eddy viscosity. In two equation EVM's the two turbulent quantities are solved to describe the eddy viscosity.

Reynolds Stress Models

In RSM's we solve an equation for the Reynolds stress and one length scale determining equation. Since we solve for Reynolds stress, we don't need any model to close it. However RSM's are computationally much more demanding when compared to EVM's.

Chapter 3

Numerical Method & The Flow Domain

3.1 The Solver

Computations were carried out employing an incompressible finite volume code, CALC-BFC by Davidson and Farhanieh [4]. The code employs a general non-orthogonal boundary fitted co-ordinate system, on a non-staggered grid. TDMA (Tri-Diagonal Matrix Algorithm) is used as the matrix solver. Hybrid central/upwind differencing, the QUICK scheme and the Van Leer scheme are available.

3.2 The Differencing Schemes

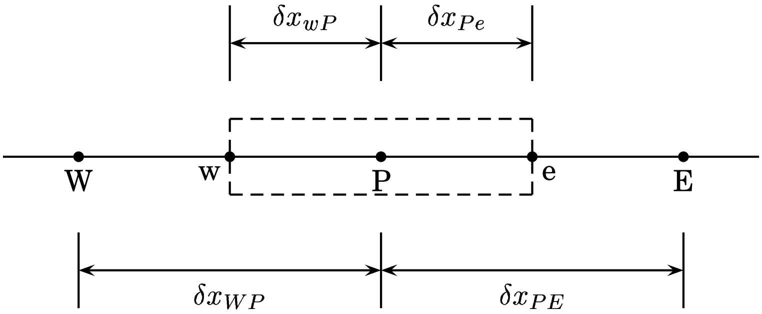
To understand the differencing schemes used, we first start by looking at one-dimensional steady state convection-diffusion equation as in Versteegh [12] which reads,

$$\frac{d}{dx}(\rho u A \phi) = \frac{d}{dx}\left(\Gamma A \frac{d\phi}{dx}\right) \tag{3.1}$$

The continuity equation is given by,

$$\frac{d}{dx}(\rho u A) = 0 \tag{3.2}$$

Define a one-dimensional control volume as shown in the figure,



Integrating the transport equations over the control volume gives,

$$\int_w^e \frac{d}{dx}(\rho u A\phi)dx = \int_w^e \frac{d}{dx}\left(\Gamma A\frac{d\phi}{dx}\right)dx \tag{3.3}$$

$$\int_w^e \frac{d}{dx}(\rho u A)dx = 0 \tag{3.4}$$

Therefore,

$$(\rho u A\phi)_e - (\rho u A\phi)_w = \left(\Gamma A\frac{d\phi}{dx}\right)_e - \left(\Gamma A\frac{d\phi}{dx}\right)_w \tag{3.5}$$

$$(\rho u A)_e - (\rho u A)_w = 0 \tag{3.6}$$

Employing central differencing to represent the diffusion terms on the right hand side of 3.5 results in,

$$(\rho u A\phi)_e - (\rho u A\phi)_w = (\Gamma A)_e\left(\frac{\phi_E - \phi_P}{dx_{PE}}\right) - (\Gamma A)_w\left(\frac{\phi_P - \phi_W}{dx_{WP}}\right) \tag{3.7}$$

Defining two variables F and D, the convective mass flux per unit area (ρu) and the diffusion per unit area (Γ/dx) respectively,

$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) \tag{3.8}$$

$$F_e - F_w = 0 \tag{3.9}$$

The Central Differencing Scheme

In 3.8 the cell face values of property ϕ for the convective terms considering a uniform grid is given by,

$$\phi_e = \frac{\phi_P + \phi_E}{2} \tag{3.10}$$

$$\phi_w = \frac{\phi_P + \phi_W}{2} \tag{3.11}$$

Therefore,

$$\frac{F_e}{2}(\phi_P + \phi_E) - \frac{F_w}{2}(\phi_P + \phi_W) = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) \quad (3.12)$$

$$\left[\left(D_w + \frac{F_w}{2}\right) + \left(D_e - \frac{F_e}{2}\right) + (F_e - F_w)\right]\phi_P = \left(D_w - \frac{F_w}{2}\right)\phi_W + \left(D_e - \frac{F_e}{2}\right)\phi_E \quad (3.13)$$

This is of the form,

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \quad (3.14)$$

The coefficients are,

$a_W = D_w + F_w/2$	$a_E = D_e - F_e/2$	$a_P = a_W + a_E + (F_e - F_w)$

Since we employ Finite Volume technique, the central differencing scheme is *conservative*. But, it is not *Bounded*. This means that it allows negative coefficients. This is justified by looking at a_E coefficient. For this coefficient to be positive, the ratio $F_e/D_e = P_e$ must always be less than 2 which usually is not the case. Also, the central differencing scheme doesn't reflect the way information is transported.

The Upwind Differencing Scheme

When the flow is in positive x-direction, $u_w > 0, u_e > 0$ $(F_w > 0,\ F_e > 0)$ then, $\phi_w = \phi_W$ and $\phi_e = \phi_P$. Therefore 3.8 becomes,

$$[(D_w + F_w) + D_e + (F_e - F_w)]\,\phi_P = (D_w + F_w)\phi_W + D_e\phi_E \quad (3.15)$$

When the flow is in negative x-direction, $u_w < 0, u_e < 0$ $(F_w < 0, F_e < 0)$ then, $\phi_w = \phi_P$ and $\phi_e = \phi_E$. Therefore 3.8 becomes,

$$[D_w + (D_e - F_e) + (F_e - F_w)]\,\phi_P = D_w\phi_W + (D_e - F_e)\phi_E \quad (3.16)$$

Hence the coefficients are given by,

$F_w > 0,\ F_e > 0$	$a_W = D_w + F_w$	$a_E = D_e$
$F_w < 0,\ F_e < 0$	$a_W = D_w$	$a_E = D_e - F_e$
In general	$a_W = D_w + \max(F_w, 0)$	$a_E = D_e + max(0, -F_e)$

Since we employ Finite Volume technique, the upwind scheme is *Conservative*. Upwind scheme is *Bounded* since all coefficients are positive. Also, upwind scheme accounts for the direction of the flow. Hence it is *Transportive*. The only drawback of upwind is that it is *First order accurate*.

The Hybrid Scheme

The hybrid scheme is a mere combination of central differencing and upwind differencing schemes. When Peclet number (P_e) which is defined as the ratio of convection to diffusion is less than 2 ($P_e < 2$) central differencing scheme which is second order accurate is used. If $P_e \geq 2$ then, first order upwind scheme which accounts for the transportiveness is employed.
The coefficients are given by,

$a_W = \max\left[F_w, (D_w + F_w/2), 0\right]$	$a_E = \max\left[-F_e, (D_e - F_e/2), 0\right]$

3.3 The TDMA Solver

The Tri-Diagonal Matrix Algorithm is actually a *direct* solution technique for linear algebraic equations, which is applied in an iterative fashion. CALC-BFC employs a segregated TDMA solver which is discussed below for a 2D case with variable ϕ.
The 2D discretised equation is given by,

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + S_U \tag{3.17}$$

Rewriting the above equation in the form,

$$a_i\phi_i = b_i\phi_{i+1} + c_i\phi_{i-1} + d_i \tag{3.18}$$

where,

$$a_i = a_P, b_i = a_E, c_i = a_W, d_i = a_N\phi_N + a_S\phi_S + S_U \tag{3.19}$$

The variable ϕ in 3.18 is determined as,

$$\phi_i = P_i\phi_{i+1} + Q_i \tag{3.20}$$

3.18 is written in matrix form as,

$$\begin{bmatrix} a_2 & -b_2 & 0 & \cdots & & \\ -c_3 & a_3 & -b_3 & o & \cdots & \\ 0 & -c_4 & a_4 & -b_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} d_2 + c_2\phi_1 \\ d_3 \\ d_4 \\ \vdots \end{bmatrix}$$

Divide first row by a_2,

$$\begin{bmatrix} 1 & -P_2 & 0 & \cdots & & \\ -c_3 & a_3 & -b_3 & o & \cdots & \\ 0 & -c_4 & a_4 & -b_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} Q_2 \\ d_3 \\ d_4 \\ \vdots \end{bmatrix}$$

where,

$$P_2 = \frac{b_2}{a_2}, Q_2 = \frac{d_2 + c_2\phi_1}{a_2} \tag{3.21}$$

To eliminate c in the second row, multiply first row by c_3, add it to second row and finally divide the entire second row by $a_3 - c_3P_2$ to obtain,

$$\begin{bmatrix} 1 & -P_2 & 0 & \cdots & & \\ 0 & 1 & -P_3 & o & \cdots & \\ 0 & -c_4 & a_4 & -b_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} Q_2 \\ Q_3 \\ d_4 \\ \vdots \end{bmatrix}$$

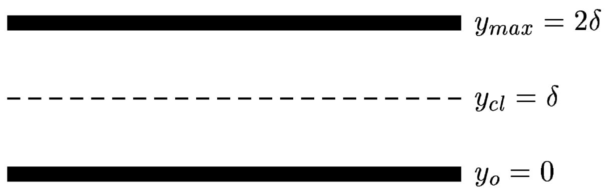
where,

$$P_3 = \frac{b_3}{a_3 - c_3P_2}, Q_3 = \frac{d_3 + c_3Q_2}{a_3 - c_3P_2} \tag{3.22}$$

3.22 becomes recursive and hence 3.20 is solved from $i = 4$ to $i_{max} - 1$ with P_i and Q_i of the form,

$$P_i = \frac{b_i}{a_i - c_iP_{i-1}}, Q_i = \frac{d_i + c_iQ_{i-1}}{a_i - c_iP_{i-1}} \tag{3.23}$$

3.4 The Flow Domain



The computational domain is a fully developed 1D-Channel flow, with variations only in wall-normal direction. Hence, we don't have any convection terms. Therefore, for simplicity, Hybrid scheme has been used for all the flow variables we solve.

Reynolds number based on channel half width and the friction velocity is given by,

$$R_{e_\tau} = \frac{u_*\delta}{\nu} \tag{3.24}$$

Wall shear stress,

$$\tau_w = \mu \left(\frac{\partial U}{\partial y}\right)_{wall} = \rho u_*^2 \tag{3.25}$$

Therefore, friction velocity,

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \tag{3.26}$$

From global momentum balance with δ equal to half the channel width,

$$-\delta\frac{\partial P}{\partial x} = \tau_w = \mu\left(\frac{\partial U}{\partial y}\right)_{wall} \tag{3.27}$$

From 3.26,

$$u_*^2 = \frac{\tau_w}{\rho}$$

Therefore,

$$\rho u_*^2 = -\delta\frac{\partial P}{\partial x} \tag{3.28}$$

Letting δ = 1 (y_{max} = 2) and $\rho = u_* = 1$,

$$-\frac{\partial P}{\partial x} = 1. \tag{3.29}$$

Since the flow is fully developed, there are no gradients in streamwise direction,

$$\frac{\partial \phi}{\partial x} = 0 \tag{3.30}$$

'ϕ' is any variable that we solve for.

The Continuity Equation is given by,

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{3.31}$$

Since $\partial/\partial x = 0$, we get,

$$\frac{\partial V}{\partial y} = 0 \tag{3.32}$$

This implies $V = Constant$. Since $V = 0$ at the wall, it is zero everywhere.

Chapter 4

Two Equation Models

4.1 Abe-Kondoh-Nagano k-ε Model

Since the dissipation rate of turbulent kinetic energy ε appears naturally in the exact equation for turbulent kinetic energy, it is the obvious choice for the second turbulent quantity in the formation of ν_t. The AKN Model [1] is used. The modeled k and ε equations read,

$$
\begin{aligned}
\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \varepsilon && (4.1) \\
\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1}\frac{\epsilon}{k}P_k - C_{\varepsilon 2} f_2 \frac{\varepsilon^2}{k} && (4.2) \\
f_2 &= \left[1 - \exp\left(-\frac{y^*}{3.1}\right)\right]^2 \left[1 - 0.3\,\exp\left\{-\left(\frac{R_t}{6.5}\right)^2\right\}\right] \\
y^* &= \frac{\epsilon^{1/4} y}{\nu^{3/4}}
\end{aligned}
$$

The values of constants are,

$C_\mu = 0.09$	$C_{\varepsilon 1} = 1.5$	$C_{\varepsilon 2} = 1.9$	$\sigma_k = 1.4$	$\sigma_\varepsilon = 1.4$

The production term is given by,

$$P_k = 2\nu_t S_{ij} S_{ij} \tag{4.3}$$

The turbulent length scale is given by,

$$l = \frac{k^{3/2}}{\varepsilon} \tag{4.4}$$

The turbulent viscosity is computed as,

$$\nu_t = C_\mu f_\mu v l = C_\mu f_\mu k^{1/2}\frac{k^{3/2}}{\varepsilon} = C_\mu f_\mu \frac{k^2}{\varepsilon} \tag{4.5}$$

$$f_\mu = \left[1 - \exp\left(-\frac{y^*}{14}\right)\right]^2 \left[1 + \frac{5}{R_t^{3/4}} \exp\left\{-\left(\frac{R_t}{200}\right)^2\right\}\right]$$

Time scale is obtained using velocity scale and the length scale,

$$\tau = \frac{l}{v} = \frac{l}{k^{1/2}} = \frac{k^{3/2}}{\varepsilon k^{1/2}} = \frac{k}{\varepsilon} \tag{4.6}$$

Near the walls τ goes to zero causing a singularity in ε equation. Durbin [6] suggested a lower bound on the time scale using Kolmogorov variables,

$$\tau \geq 6\sqrt{\frac{\nu}{\varepsilon}} \tag{4.7}$$

Boundary Condition for ε Equation

(a) Boundary condition for ε is derived by looking at how k equation behaves near the wall. The exact k equation reads,

$$\frac{\partial}{\partial x_j}(\rho U_j k) = -\rho\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{u_j p} + \frac{1}{2}\overline{\rho u_j u_i u_i} - \mu\frac{\partial k}{\partial x_j}\right) - \mu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} \tag{4.8}$$

The largest terms close to the wall are the dissipation term and the viscous diffusion term. Hence it simplifies to ,

$$0 = \mu\frac{\partial^2 k}{\partial y^2} - \rho\varepsilon \tag{4.9}$$

Therefore,

$$\varepsilon_{wall} = \nu\frac{\partial^2 k}{\partial y^2} \tag{4.10}$$

(b) An alternative boundary condition may be derived by looking at the dissipation term in the exact dissipation equation which reads,

$$\varepsilon = \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} \tag{4.11}$$

Using Taylors expansion the fluctuating velocities can be expressed as follows,

$$\begin{aligned} u &= a_o + a_1 y + a_2 y^2 + \cdots \\ v &= b_o + b_1 y + b_2 y^2 + \cdots \\ w &= c_o + c_1 y + c_2 y^2 + \cdots \end{aligned} \tag{4.12}$$

The no-slip condition implies,

$$a_o = b_o = c_o = 0 \tag{4.13}$$

Since $(\partial u/\partial x)_{y=0} = (\partial w/\partial z)_{y=0} = 0$, from continuity we have, $(\partial v/\partial y)_{y=0} = 0$. Hence, $b_1 = 0$. Squaring and averaging 4.12 we get,

$$\begin{aligned}
\overline{u^2} &= \overline{a_1^2}y^2 \cdots = \mathcal{O}(y^2) \\
\overline{v^2} &= \overline{b_2^2}y^4 \cdots = \mathcal{O}(y^4) \\
\overline{w^2} &= \overline{c_1^2}y^2 \cdots = \mathcal{O}(y^2)
\end{aligned} \tag{4.14}$$

$$k = \frac{1}{2}\overline{(a_1^2 + c_1^2)}y^2 \cdots = \mathcal{O}(y^2) \tag{4.15}$$

$$\frac{\partial U}{\partial y} = \overline{a_1} \cdots = \mathcal{O}(y^0) \tag{4.16}$$

$$\varepsilon = \nu\overline{\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial w}{\partial y}\right)^2} = \nu\overline{(a_1^2 + c_1^2)} \cdots = \mathcal{O}(y^0) \tag{4.17}$$

Therefore,

$$\varepsilon = \nu\tfrac{2k}{y^2} \qquad \text{as} \qquad y \to 0.$$

Deficiencies in k-ε Model

The k-ε Model presented above has two major deficiencies,

1. Using DNS data it has been shown that ε is finite and non-zero at the wall. Also, it is not a constant since DNS data appear to indicate a Reynolds number dependency. Hence it is very difficult to assign a specific value near the wall boundaries.
2. It has strong variable coupling at wall boundaries which causes numerical difficulties.

4.2 The Standard k-ω Model

The original model suggested by Wilcox [13] is commonly referred to as the standard k-ω model. Since no damping functions are used, it is referred to as High Reynolds Number (HRN) model.

The modeled k and ω equations read,

$$\frac{\partial k}{\partial t} + U_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\nu + \nu_t\sigma^*)\frac{\partial k}{\partial x_j}\right] + P_k - \beta^*\omega k \tag{4.18}$$

$$\frac{\partial \omega}{\partial t} + U_j\frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\nu + \sigma\nu_t)\frac{\partial \omega}{\partial x_j}\right] + \xi\frac{\omega}{k}P_k - \beta\omega^2 \tag{4.19}$$

The values of constants are,

$\beta = 3/40$	$\beta^* = 9/100$	$\xi = 5/9$	$\sigma = 1/2$	$\sigma^* = 1/2$

The production term is given by,

$$P_k = 2\nu_t S_{ij} S_{ij} \tag{4.20}$$

The turbulent length scale is given by,

$$l = \frac{k^{1/2}}{\omega} \tag{4.21}$$

The turbulent viscosity is computed as,

$$\nu_t = vl = k^{1/2} \frac{k^{1/2}}{\omega} = \frac{k}{\omega} \tag{4.22}$$

Time scale is obtained using velocity scale and the length scale,

$$\tau = \frac{l}{v} = \frac{l}{k^{1/2}} = \frac{k^{1/2}}{\omega k^{1/2}} = \frac{1}{\omega} \tag{4.23}$$

Boundary Condition for ω Equation

Boundary condition for ω is derived by looking at how ω equation behaves near the wall. The largest terms close to the wall are the dissipation term and the viscous diffusion term which read,

$$0 = \nu \frac{\partial^2 \omega}{\partial y^2} - \beta \omega^2 \tag{4.24}$$

The solution to this equation is,

$$\omega_{wall} = \frac{6\nu}{\beta y^2} \tag{4.25}$$

The only flow variable in the expression is ν which is also a constant. Hence, the value of first interior node depends only on the mesh size which is numerically appealing.

4.3 The Low Reynolds Number k-ω Model

This model was suggested by Wilcox [14] which employs damping functions to capture the effect of wall boundaries. It is commonly referred to as the Wilcox Low Reynolds Number (LRN) k-ω model.

The modeled k and ω equations read,

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\nu + \nu_t \sigma^*)\frac{\partial k}{\partial x_j}\right] + P_k - \beta^* \omega k \tag{4.26}$$

$$\beta^* = \frac{9}{100}\frac{5/18 + (R_t/8)^4}{1 + (R_t/8)^4} \tag{4.27}$$

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\nu + \sigma \nu_t)\frac{\partial \omega}{\partial x_j}\right] + \alpha \frac{\omega}{k} P_k - \beta \omega^2 \tag{4.28}$$

$$\alpha = \frac{1}{\alpha^*}\frac{5}{9}\frac{0.1 + R_t/6}{1 + R_t/6} \tag{4.29}$$

The values of constants are,

$\beta = 3/40$	$\sigma = 1/2$	$\sigma^* = 1/2$

The production term is given by,

$$P_k = 2\nu_t S_{ij} S_{ij} \tag{4.30}$$

The turbulent length scale is given by,

$$l = \frac{k^{1/2}}{\omega} \tag{4.31}$$

The turbulent viscosity is computed as,

$$\nu_t = \alpha^* v l = \alpha^* k^{1/2} \frac{k^{1/2}}{\omega} = \alpha^* \frac{k}{\omega} \tag{4.32}$$

where,

$$\alpha^* = \frac{0.025 + 10R_t/27}{1 + 10R_t/27} \tag{4.33}$$

The turbulent Reynolds number, R_t is given by,

$$R_t = \frac{k}{\omega \nu} \tag{4.34}$$

Time scale is obtained using velocity scale and the length scale,

$$\tau = \frac{l}{v} = \frac{l}{k^{1/2}} = \frac{k^{1/2}}{\omega k^{1/2}} = \frac{1}{\omega} \tag{4.35}$$

Advantages of Low Reynolds Number k-ω Model

The two main attractive features of Wilcox Low Reynolds Number model are,

1. It possesses a non-trivial solution for ω as k goes to zero. It is thus expected to capture flow characteristics which a Low Reynolds Number k-ε model fails to handle.

2. It uses damping functions that only depend on turbulent Reynolds number and hence it is convenient to apply this model to internal flows with complex geometries.

4.4 The Modified Low Reynolds Number k-ω Model

This model was suggested by Peng-Davidson-Holmberg [10]. The Low Reynolds Number model of Wilcox which was discussed in previous section doesn't reproduce correct asymptotic behavior at near wall boundaries. A *first order turbulent cross-diffusion term* is added to the existing second order diffusion term in the transport equation of specific dissipation ω. The model constants are re-established. New R_t dependent damping functions are devised to make the model asymptotically consistent as the wall is approached.

The modeled k and ω equations read,

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - C_k f_k \omega k \tag{4.36}$$

$$f_k = 1 - 0.722 \exp\left[-\left(\frac{R_t}{10}\right)^4\right] \tag{4.37}$$

$$\begin{aligned}\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + C_{\omega 1} f_\omega \frac{\omega}{k} P_k + \cdots \\ &\quad -C_{\omega 2}\omega^2 + C_\omega \frac{\nu_t}{k}\left(\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}\right)\end{aligned} \tag{4.38}$$

$$f_w = 1 + 4.3 \exp\left[-\left(\frac{R_t}{1.5}\right)^{\frac{1}{2}}\right] \tag{4.39}$$

The values of constants are,

$C_k = 0.09$	$C_\mu = 1.0$	$C_{\omega 1} = 0.42$	$C_{\omega 2} = 0.075$
$C_\omega = 0.75$	$\sigma_k = 0.8$	$\sigma_\omega = 1.35$	$-$

The production term is given by,

$$P_k = 2\nu_t S_{ij} S_{ij} \tag{4.40}$$

The turbulent length scale is given by,

$$l = \frac{k^{1/2}}{\omega} \tag{4.41}$$

The turbulent viscosity is computed as,

$$\nu_t = C_\mu f_\mu v l = C_\mu f_\mu k^{1/2}\frac{k^{1/2}}{\omega} = C_\mu f_\mu \frac{k}{\omega} \tag{4.42}$$

where,

$$f_\mu = 0.025 + \left\{1 - \exp\left[-\left(\frac{R_t}{10}\right)^{\frac{3}{4}}\right]\right\}\left\{0.975 + \frac{0.001}{R_t}\exp\left[-\left(\frac{R_t}{200}\right)^2\right]\right\} \tag{4.43}$$

The turbulent Reynolds number, R_t is given by,

$$R_t = \frac{k}{\omega\nu} \tag{4.44}$$

Time scale is obtained using velocity scale and the length scale,

$$\tau = \frac{l}{v} = \frac{l}{k^{1/2}} = \frac{k^{1/2}}{\omega k^{1/2}} = \frac{1}{\omega} \tag{4.45}$$

4.5 Performance of Two Equation Eddy Viscosity Models

Boundary Condition

At Wall

At the channel wall we have no-slip boundary condition for velocity components and the kinetic energy.

$$U = k = 0$$

ε and ω are fixed at the second node using the following wall functions.

$$\varepsilon_{wall} = \nu \frac{2k}{y^2}$$
$$\omega_{wall} = \frac{6\nu}{\beta y^2}$$

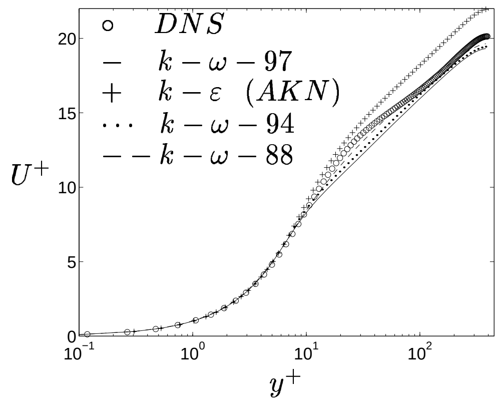
At Symmetry Line

At the center of the channel, we have Neumann boundary condition for all the properties.

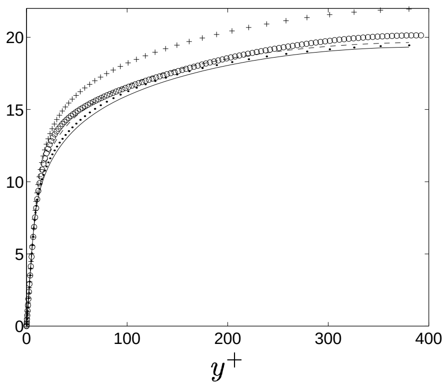
$$\frac{\partial U}{\partial y} = \frac{\partial k}{\partial y} = \frac{\partial \varepsilon}{\partial y} = \frac{\partial \omega}{\partial y} = 0$$

The results are compared with the existing DNS data with a Reynolds number based on the channel half width and the friction velocity, $R_e = u_* \delta / \nu = 395$. The number of grid points in the y-direction, starting from wall to half the channel width is 68. The value of y^+ closest to the wall is 0.098. A geometric stretching factor of 1.08 is used. The DNS turbulent viscosity is computed using the definition $\nu_t = -\overline{uv}/\frac{\partial U}{\partial y}$.

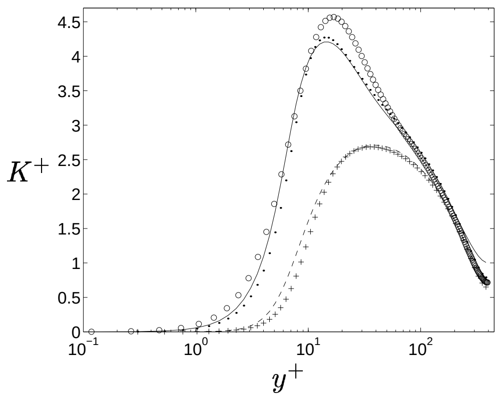
4.1(a) shows the velocity profile for different two equation models. It is seen that AKN k-ε model over-predicts the velocity after $y^+ \simeq 20$. This is due to under-prediction of turbulent viscosity after $y^+ \simeq 20$ which is shown in 4.1(g). ν_t in the far-field region is best predicted by AKN k-ε model. However, away from the wall, higher values of ν_t by all other models are balanced by low velocity gradients resulting in a very good estimate of $\overline{uv}$ for all the models. 4.1(c) shows that both low Reynolds number $k - \omega$ models better predicts th kinetic energy than AKN k-ε model.



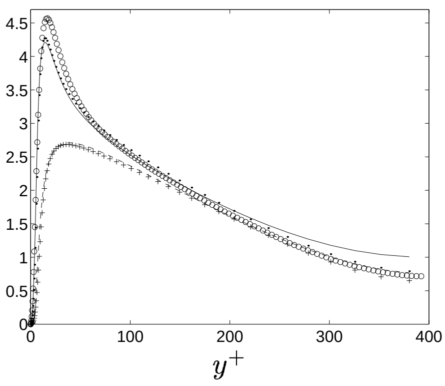
(a) U



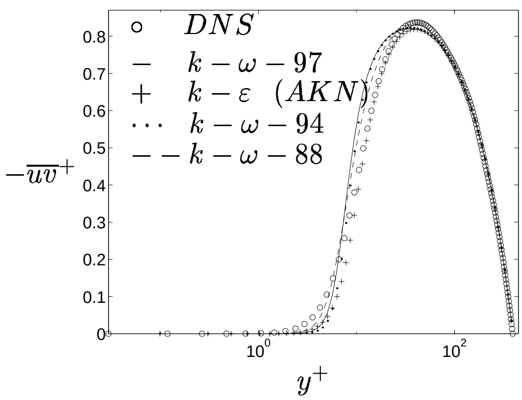
(b) U



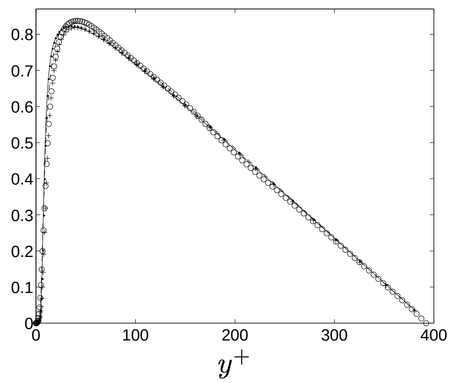
(c) K.



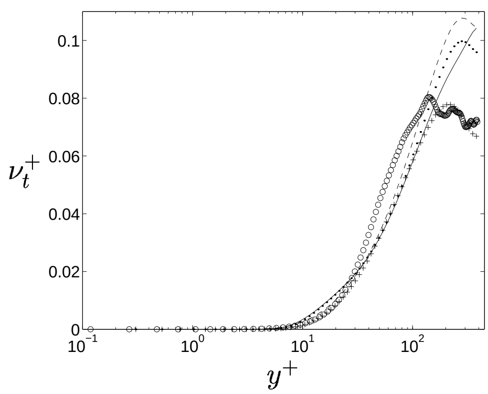
(d) K



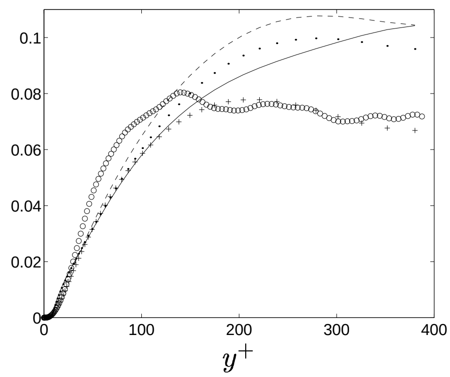
(e) $-\overline{uv}$



(f) $-\overline{uv}$



(g) ν_t



(h) ν_t

Figure 4.1: Turbulent quantities compared with the DNS data [8], $R_e = 395$, (a,b) Velocity contour (c,d) Kinetic Energy (e,f) Reynolds shear stress (g,h) Turbulent viscosity

Chapter 5

$\overline{v^2} - f$ Models

5.1 Formulation of $k - \omega - \overline{v^2} - f$ Model

In the present work, the standard k-ω model of Wilcox [13] is solved along with modified $\overline{v^2} - f$ model of Lien and Kalitzin [7].

The $k - \omega$ Model

The High Reynolds Number(HRN) Model of Wilcox [13] read,

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \beta^* \omega k \tag{5.1}$$

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \xi \frac{\omega}{k} P_k - \beta \omega^2 \tag{5.2}$$

The $\overline{v^2} - f$ Model

The modified $\overline{v^2} - f$ model of Lien and Kalitzin [7] based on a HRN $k - \varepsilon$ model read,

$$\frac{\partial \overline{v^2}}{\partial t} + U_j \frac{\partial \overline{v^2}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right] + kf - 6\overline{v^2}\frac{\varepsilon}{k} \tag{5.3}$$

$$L^2 \frac{\partial^2 f}{\partial {x_j}^2} - f = \frac{C_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3}\right) - C_2 \frac{P_k}{k} - \frac{1}{T}\left(6\frac{\overline{v^2}}{k} - \frac{2}{3}\right) \tag{5.4}$$

Using the definition $\varepsilon/k = \beta^* \omega$, the last term in 5.3 is re-written as $-6\overline{v^2}\beta^*\omega$. Hence the $\overline{v^2}$ equation modifies to,

$$\frac{\partial \overline{v^2}}{\partial t} + U_j \frac{\partial \overline{v^2}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right] + kf - 6\overline{v^2}\beta^*\omega \tag{5.5}$$

Since we don't have any ε terms in the f-equation, it remains unchanged.

The eddy-viscosity is computed as,

$$\nu_t = C_\mu \overline{v^2} T \tag{5.6}$$

The Time & Length Scales

The turbulent time scale and the length scales as suggested by Durbin [6] are given by,

$$T = \max\left(\frac{k}{\varepsilon}\ ,\ C_\zeta \sqrt{\frac{\nu}{\varepsilon}}\right) \tag{5.7}$$

$$L = C_l \max\left(\frac{k^{3/2}}{\varepsilon}\ ,\ C_\eta \frac{\nu^{3/4}}{\varepsilon^{1/4}}\right) \tag{5.8}$$

Again, using the definition $\varepsilon/k = \beta^*\omega$, the above time and length scales modifies to,

$$T = \max\left(\frac{1}{\beta^*\omega}\ ,\ C_\zeta \sqrt{\frac{\nu}{\beta^*\omega k}}\right) \tag{5.9}$$

$$L = C_l \max\left(\frac{k^{1/2}}{\beta^*\omega}\ ,\ C_\eta \frac{\nu^{3/4}}{(\beta^*\omega k)^{1/4}}\right) \tag{5.10}$$

The $k - \omega - \overline{v^2} - f$ Model

The final form of $k - \omega - \overline{v^2} - f$ model reads,

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \beta^*\omega k \tag{5.11}$$

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \xi\frac{\omega}{k} P_k - \beta\omega^2 \tag{5.12}$$

$$\frac{\partial \overline{v^2}}{\partial t} + U_j \frac{\partial \overline{v^2}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right] + kf - 6\overline{v^2}\beta^*\omega \tag{5.13}$$

$$L^2 \frac{\partial^2 f}{\partial {x_j}^2} - f = \frac{C_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3}\right) - C_2\frac{P_k}{k} - \frac{1}{T}\left(6\frac{\overline{v^2}}{k} - \frac{2}{3}\right) \tag{5.14}$$

The C_ζ & C_η Constants

To prevent singularity as $y \to 0$, Durbin [6] suggested a lower bound on the time and length scales by not letting them go below the Kolmogoroff scales in the near wall region. C_ζ and C_η are the constants that go along with the time and length scales. The values of C_ζ and C_η in $k - \varepsilon - \overline{v^2} - f$

model are 6 and 70 respectively. Using the same values in the above formulated $k - \omega - \overline{v^2} - f$ model revealed that they were too high and the shifting of scales from Kolmogoroff to the normal scale took place very far from the wall (high y^+). Hence the constants were retuned so that the shifting of scales took place at approximately same y^+ values as in $k - \varepsilon - \overline{v^2} - f$ model. The constants were found to be $C_\zeta = 0.5$ and $C_\eta = 17$.

The C_μ Constant

In the log-region, we know from experiments that convection and diffusive terms in k-equation are negligible. Hence the k-equation reduces to,

$$\begin{aligned} 0 &= P_k - \beta^* \omega k \\ -(\overline{uv}) \frac{\partial \overline{U}}{\partial y} &= \beta^* \omega k \\ -\left(\nu_t \frac{\partial \overline{U}}{\partial y}\right) \frac{\partial \overline{U}}{\partial y} &= \beta^* \omega k \\ -\left(\nu_t \frac{\partial \overline{U}}{\partial y}\right)^2 &= \nu_t \beta^* \omega k \\ (\overline{uv})^2 &= C_\mu \overline{v^2} T \beta^* \omega k \end{aligned}$$

We know by definition, $T = k/\epsilon = 1/\beta^*\omega$. Hence we get,

$$\begin{aligned} (\overline{uv})^2 &= C_\mu \overline{v^2} \frac{1}{\beta^* \omega} \beta^* \omega k \\ \frac{(\overline{uv})^2}{k\overline{v^2}} &= C_\mu \end{aligned}$$

Therefore,

$$C_\mu = \left(\frac{\overline{uv}}{k}\right)^2 \left(\frac{k}{\overline{v^2}}\right) \tag{5.15}$$

From experiments, we know that $\overline{uv}/k = 0.3$ in the log-region. Now, the value of $k/\overline{v^2}$ determines the C_μ constant.

	$\overline{v^2}/k$	C_μ
From DNS data of Moser (1999)	0.33	0.27
From Experiments (a consensus of near-wall turbulence data)	0.24	0.37
To get the best fit with DNS	0.28	0.32

Using the present DNS data [8], $R_{e_\tau} = 590$, and approximating the log-region to be between $y^+ \simeq 30$ to 250, the ratio $\left(\overline{v^2}/k\right)_{dns}$ is found to be 0.33 which results in $C_\mu = 0.27$. From experiments, $\overline{v^2}/k = 0.24$ which gives $C_\mu = 0.37$. But, $\overline{v^2}/k$ determined by tuning the results to match the DNS data was found to be 0.28 which gives $C_\mu = 0.32$.

The Prandtl Numbers

The Prandtl numbers are again determined by trying to get the best fit with DNS data. The Prandtl numbers for kinetic energy was chosen to be 2.5 and for ω to be 0.5. Since $\overline{v^2}$ is a part of k, we expect σ_{v2} to be same or at-least approximately around σ_k.

Choosing $\sigma_{v2} = 2.5$ gave low $\overline{v^2}$ in the centerline region of the channel. Hence the diffusion in $\overline{v^2}$-equation is increased by decreasing σ_{v2} to 0.5 which results in good prediction of $\overline{v^2}$ at the centerline when compared with DNS data.

Finally, the model constants used are given in the table below,

$\beta^* = 0.09$	$\beta = 0.075$	$C_l = 0.23$	$C_1 = 1.4$	$C_\eta = 17$
$C_2 = 0.3$	$\sigma_k = 2.5$	$\sigma_\omega = 0.5$	$\sigma_{v2} = 0.5$	$C_\zeta = 0.5$
$C_\mu = 0.32$	$\xi = 5/9$	–	–	–

For a fully developed 1D channel flow with variations only in wall-normal direction, the model simplifies to,

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \beta^*\omega k \tag{5.16}$$

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial y}\right] + \xi\frac{\omega}{k}P_k - \beta\omega^2 \tag{5.17}$$

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial y}\right] + kf - 6\overline{v^2}\beta^*\omega \tag{5.18}$$

$$L^2\frac{\partial^2 f}{\partial y^2} - f = \frac{C_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3}\right) - C_2\frac{P_k}{k} - \frac{1}{T}\left(6\frac{\overline{v^2}}{k} - \frac{2}{3}\right) \tag{5.19}$$

$$T = \max\left(\frac{1}{\beta^*\omega}\ ,\ C_\zeta\sqrt{\frac{\nu}{\beta^*\omega k}}\right) \tag{5.20}$$

$$L = C_l \max\left(\frac{k^{1/2}}{\beta^*\omega}\ ,\ C_\eta\frac{\nu^{3/4}}{(\beta^*\omega k)^{1/4}}\right) \tag{5.21}$$

5.2 Performance of $k - \omega - \overline{v^2} - f$ Model

Boundary Condition

At Wall

At the wall we have no-slip boundary condition for velocity components, kinetic energy, wall-normal stress and the f-equation.

$$U = k = \overline{v^2} = f = 0$$

ω is fixed at the second node using the following wall functions.

$$\omega_{wall} = \frac{6\nu}{\beta y^2}$$

At Symmetry Line

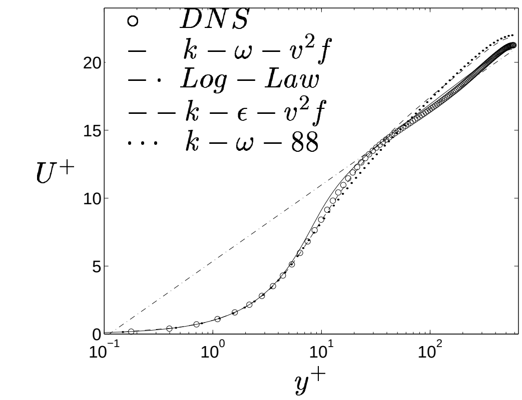
At the center of the channel, we have Neumann boundary condition for all the properties.

$$\frac{\partial U}{\partial y} = \frac{\partial k}{\partial y} = \frac{\partial \varepsilon}{\partial y} = \frac{\partial \omega}{\partial y} = \frac{\partial \overline{v^2}}{\partial y} = \frac{\partial f}{\partial y} = 0$$

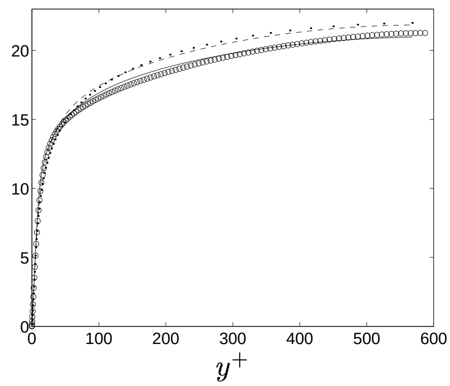
The results are compared with the existing DNS data with a Reynolds number based on the channel half width and the friction velocity, $R_{e_\tau} = u_*\delta/\nu = 590$. The number of grid points in the y-direction, starting from wall to half the channel width is 82. The value of y^+ closest to the wall is 0.05. A geometric stretching factor of 1.08 is used. The DNS viscosity is computed using the definition $\nu_t = -\overline{uv}/\frac{\partial U}{\partial y}$.

5.1(a) shows that the velocity in the log-region is well predicted by $k - \omega - \overline{v^2} - f$ model when compared to $k - \varepsilon - \overline{v^2} - f$ which overpredicts after $y^+ \simeq 50$. This is because the turbulent viscosity, ν_t is better predicted by $k - \omega - \overline{v^2} - f$ than other two models as shown in 5.1(g). From 5.1(j) it is obvious that the normal stress $\overline{v^2}$ which plays a very important role in improvement of near wall behavior is equally well predicted by $k - \omega - \overline{v^2} - f$ model when compared to $k - \varepsilon - \overline{v^2} - f$. The only drawback of the model is poor estimation of kinetic energy.

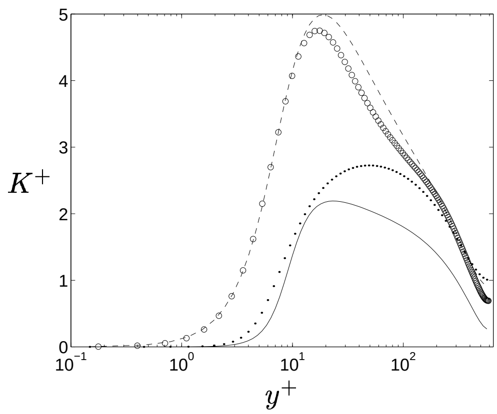
To check the model consistency, the performance of the model is analyzed at a higher Reynolds number, $R_{e_\tau} = 8000$. This corresponds to a Reynolds number based on centerline velocity, $R_{e_c} = 220,000$. The results are plotted for both $k - \omega - \overline{v^2} - f$ and $k - \varepsilon - \overline{v^2} - f$ model. It can again be seen from 5.2(a) that the velocity is well predicted in the log-region by $k - \omega - \overline{v^2} - f$ model.



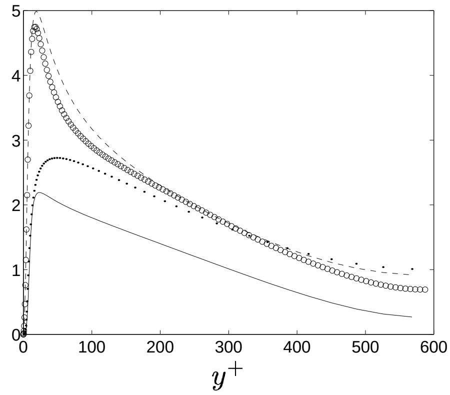
(a) U



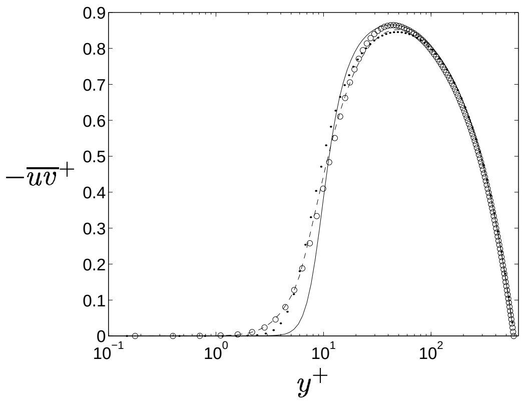
(b) U



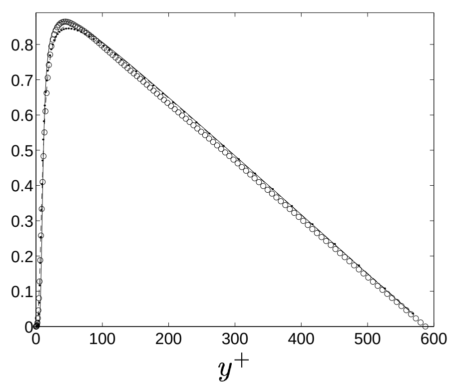
(c) K



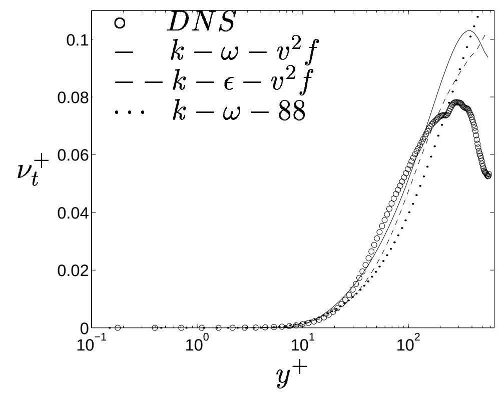
(d) K



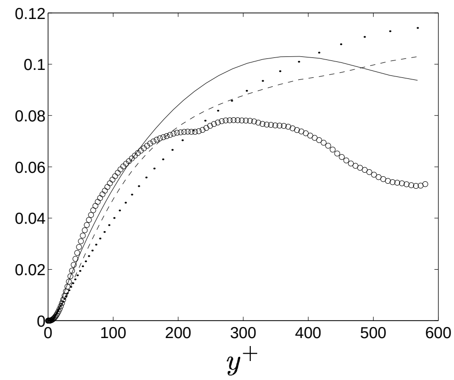
(e) $-\overline{uv}$



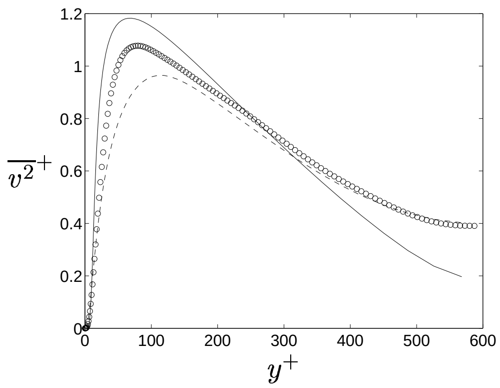
(f) $-\overline{uv}$



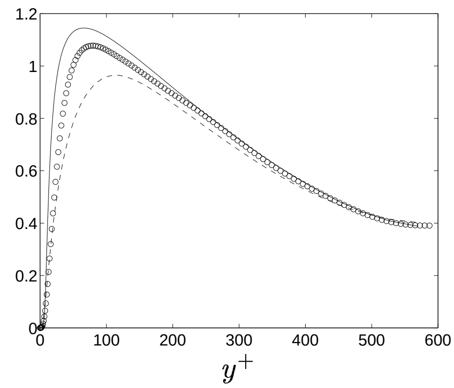
(g) ν_t



(h) ν_t

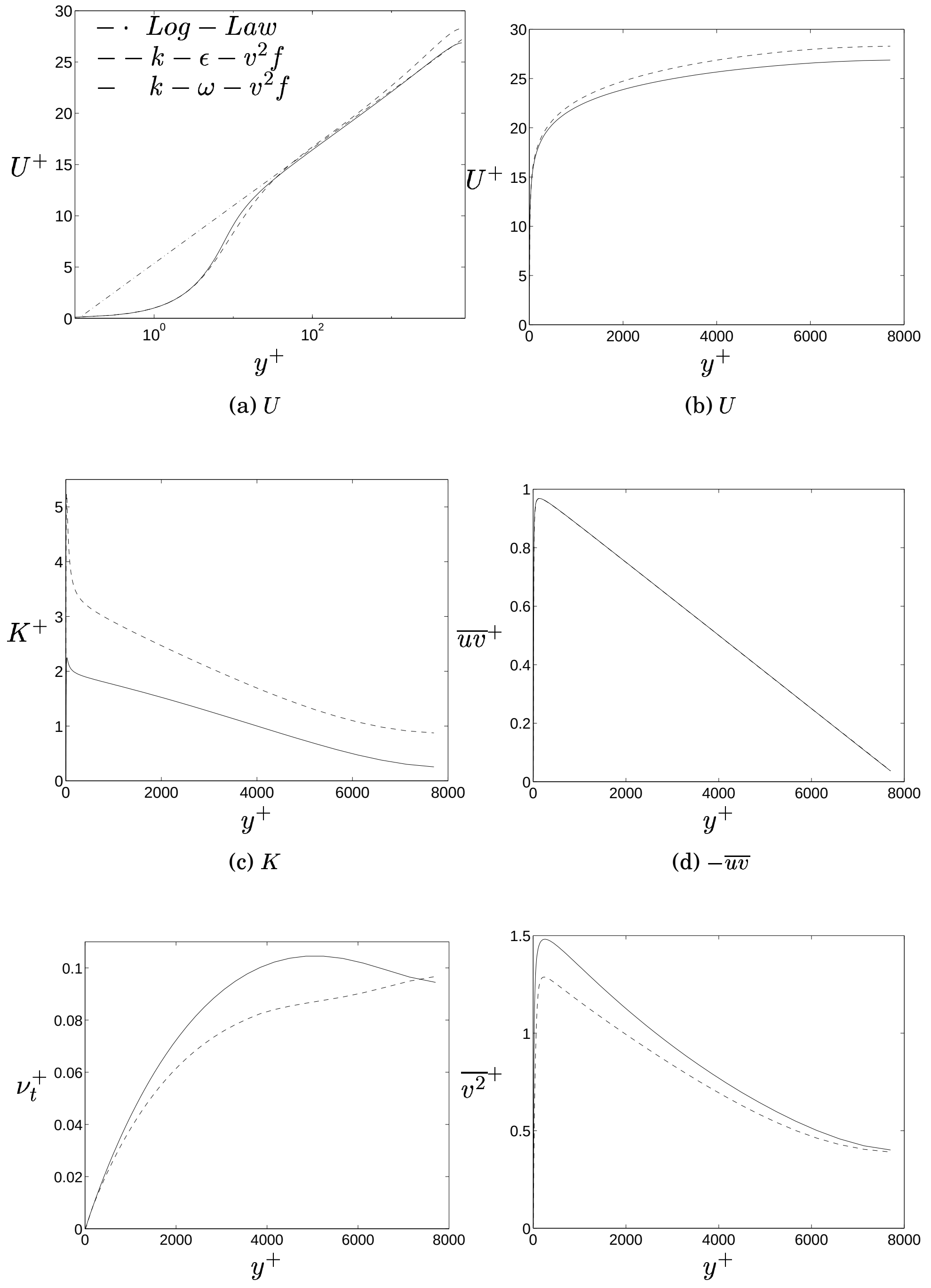


(i) $\overline{v^2}$ with $\sigma_{v2} = 2.5$



(j) $\overline{v^2}$ with $\sigma_{v2} = 0.5$

Figure 5.1: Turbulent quantities compared with the DNS data [8], $R_{e_\tau} = 590$, (a,b) Velocity contour (c,d) Kinetic Energy (e,f) Reynolds shear stress (g,h) Turbulent viscosity (i,j) Normal Reynolds stress.



(a) U (b) U

(c) K (d) $-\overline{uv}$

(e) ν_t (f) $\overline{v^2}$

Figure 5.2: Turbulent quantities at $R_{e_\tau} = 8000$, (a,b) Velocity contour (c) Kinetic Energy (d) Reynolds shear stress (e) Turbulent viscosity (f) Normal Reynolds stress.

Chapter 6

$\overline{v\theta} - g$ Model

In turbulent flows, the prediction of heat transfer, especially near the walls is of great importance for many industrial applications. In the present work, a new model named $\overline{v\theta} - g$ has been formulated to predict the wall-normal turbulent heat flux. The development of the model is discussed in the subsequent sections.

6.1 Turbulent Heat Flux Transport Equation

The transport equation for the turbulent heat flux is given by,

$$\begin{aligned}
\frac{D\overline{u_i\theta}}{Dt} \quad = \quad & \underbrace{-\overline{u_i u_k}\frac{\partial T}{\partial x_k}}_{P^{th}_{\theta i}} \underbrace{-\overline{\theta u_k}\frac{\partial U_i}{\partial x_k}}_{P^{m}_{\theta i}} \underbrace{-g_i\beta\overline{\theta^2}}_{G_{\theta i}} \\
& + \underbrace{\overline{\frac{\partial \theta}{\partial x_i}\frac{p}{\rho}}}_{\phi_{\theta i}} - \underbrace{(a+\nu)\overline{\frac{\partial \theta}{\partial x_k}\frac{\partial u_i}{\partial x_k}}}_{\varepsilon_{\theta i}} \\
& + \frac{\partial}{\partial x_k}\left[\underbrace{\left(a\overline{\frac{\partial \theta}{\partial x_k}u_i} + \nu\overline{\theta\frac{\partial u_i}{\partial x_k}}\right)}_{D^v_{\theta i}} \underbrace{-\overline{\theta u_i u_k}}_{D^t_{\theta i}} - \underbrace{\frac{\overline{\theta p}}{\rho}\delta_{ik}}_{D^p_{\theta i}}\right] \qquad (6.1)
\end{aligned}$$

6.2 Term by Term Modeling

Viscous Diffusion

The viscous diffusion term in exact transport equation of heat flux reads,

$$D^v_{\theta i} = \frac{\partial}{\partial x_k}\left(a\overline{\frac{\partial \theta}{\partial x_k}u_i} + \nu\overline{\theta\frac{\partial u_i}{\partial x_k}}\right) \qquad (6.2)$$

Peeters and Henkes [9] suggested a model by splitting the viscous diffusion in to the model $d^v_{\theta i}$ and residual terms.

$$D^v_{\theta i} = \underbrace{\frac{1}{2}(a+\nu)\frac{\partial^2 \overline{\theta u_i}}{\partial x_k^2}}_{d^v_{\theta i}} - \frac{1}{2}(a-\nu)\overline{\theta\frac{\partial^2 u_i}{\partial x_k^2}} + \frac{1}{2}(a-\nu)\overline{u_i\frac{\partial^2 \theta}{\partial x_k^2}} \tag{6.3}$$

The last two terms are the residuals and are subsequently neglected.

Turbulent Diffusion

The following model can be derived from exact transport equation of $\overline{\theta u_i u_j}$

$$D^t_{\theta i} = \frac{\partial}{\partial x_k}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{u_k u_l}\frac{\partial \overline{\theta u_i}}{\partial x_l} + \overline{u_i u_l}\frac{\partial \overline{\theta u_k}}{\partial x_l}\right)\right] \tag{6.4}$$

The above equation is invariant with coordinate rotation. The constant C_θ is taken to be 0.11 [5].

Pressure Diffusion

It has been shown that it is much easier to model the pressure diffusion $D^p_{\theta i}$ along with the pressure-temperature gradient correlation $\phi_{\theta i}$ than $D^p_{\theta i}$ alone. Hence, $D^p_{\theta i}$ is re-written as $\pi_{\theta i} = -\overline{(\theta/\rho)(\partial p/\partial x_i)}$ which is nothing but a temperature-pressure gradient correlation and is called as the Pressure Scrambling term.

Pressure Scrambling

The model for pressure scrambling $\pi_{\theta i}$ is given by [9],

$$\begin{aligned}\pi_{\theta i} \quad = \quad & \underbrace{-C_{1\theta}\frac{\varepsilon}{k}\overline{\theta u_i}}_{\phi_{\theta i,1}} \underbrace{-C_{2\theta}P^m_{\theta i}}_{\phi_{\theta i,2}} \underbrace{-C_{3\theta}G_{\theta i}}_{\phi_{\theta i,3}} \\ & - \left[\underbrace{C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k}}_{\phi^w_{\theta i,1}} + \underbrace{C_{2\theta w}\pi_{\theta k,2}}_{\phi^w_{\theta i,2}} + \underbrace{C_{3\theta w}\pi_{\theta k,3}}_{\phi^w_{\theta i,3}}\right] f_w n_k n_i\end{aligned} \tag{6.5}$$

where,

$$f_w = min\left(\sum_{walls}\frac{k^{3/2}}{C_w \varepsilon y_n}, 1.4\right) \tag{6.6}$$

The constants were selected and tuned by Peeters and Henkes [9] which read,

$C_{1\theta} = 3.75$	$C_{2\theta} = C_{3\theta} = 0.5$	$C_{1\theta w} = 0.75$	$C_{2\theta w} = C_{3\theta w} = 0$	$C_w = 2.53$

Dissipation

The following model was suggested by Peeters and Henkes [9].

$$-\varepsilon_{\theta i} = \frac{-(a+\nu)(\overline{\theta u_i} + \overline{\theta u_k} n_k n_i)}{y_n^2} \tag{6.7}$$

y_n is the normal distance to the nearest wall.

6.3 Temperature Equation

The temperature equation is given by,

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{ a \frac{\partial \theta}{\partial x_i} - \overline{u_i \theta} \right\} \tag{6.8}$$

where, a is Thermal diffusivity

$$a = \nu / P_r$$

Eddy Viscosity Closure Model for the Heat Flux

$$-\overline{u_i \theta} = \frac{\nu_t}{\sigma_\theta} \frac{\partial \theta}{\partial x_i} \tag{6.9}$$

6.4 Formulation of $\overline{v\theta} - g$ Model

The exact transport equation for the wall-normal heat flux component reads,

$$\begin{aligned} \frac{D\overline{v\theta}}{Dt} = & \underbrace{-\overline{vu_k}\frac{\partial T}{\partial x_k}}_{P_{\theta 2}^{th}} \underbrace{-\overline{\theta u_k}\frac{\partial V}{\partial x_k}}_{P_{\theta 2}^{m}} \underbrace{-g_2 \beta \overline{\theta^2}}_{G_{\theta 2}} \\ & + \underbrace{\overline{\frac{\partial p}{\partial y}\frac{\theta}{\rho}} - \frac{\partial}{\partial y}\left(\frac{\overline{p\theta}}{\rho}\right)}_{\phi_{\theta 2}} \underbrace{- (a+\nu)\overline{\frac{\partial \theta}{\partial x_k}\frac{\partial v}{\partial x_k}}}_{\varepsilon_{\theta 2}} \\ & + \frac{\partial}{\partial x_k}\left[\underbrace{\left(a\overline{\frac{\partial \theta}{\partial x_k} v} + \nu \overline{\theta \frac{\partial v}{\partial x_k}} \right)}_{D_{\theta 2}^{v}} \underbrace{-\overline{\theta v u_k}}_{D_{\theta 2}^{t}} \underbrace{-\frac{\overline{\theta p}}{\rho}\delta_{2k}}_{D_{\theta 2}^{p}} \right] \end{aligned} \tag{6.10}$$

Since we don't have any Buoyancy effect, the term, $G_{\theta 2}$ is zero. Modeling each term individually as discussed in the previous section results in the modeled $\overline{v\theta}$ equation which is given by,

$$
\begin{aligned}
\frac{D\overline{v\theta}}{Dt} \quad = \quad & -\overline{vu_k}\frac{\partial T}{\partial x_k} - \overline{\theta u_k}\frac{\partial V}{\partial x_k} \\
& -\frac{(a+\nu)(\overline{\theta v} + \overline{\theta u_k} n_k n_2)}{y_n^2} \\
& +\frac{1}{2}(a+\nu)\frac{\partial^2 \overline{\theta v}}{\partial x_k^2} - \frac{\partial}{\partial x_k}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{u_k u_l}\frac{\partial \overline{\theta v}}{\partial x_l} + \overline{vu_l}\frac{\partial \overline{\theta u_k}}{\partial x_l}\right)\right] \\
& \underbrace{-C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v}}_{\phi_{\theta 2,1}} \underbrace{-C_{2\theta}P_{\theta 2}^m}_{\phi_{\theta 2,2}} \\
& -\left[\underbrace{C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k}}_{\phi_{\theta 2,1}^w} + \underbrace{C_{2\theta w}\pi_{\theta k,2}}_{\phi_{\theta 2,2}^w} + \underbrace{C_{3\theta w}\pi_{\theta k,3}}_{\phi_{\theta 2,3}^w}\right] f_w n_k n_2
\end{aligned}
\tag{6.11}
$$

where,

$$
f_w = min\left(\sum_{walls}\frac{k^{3/2}}{C_w \varepsilon y_n}, 1.4\right) \tag{6.12}
$$

Since $C_{2\theta w}$ and $C_{3\theta w}$ are chosen to be zero, the above equation reduces to,

$$
\begin{aligned}
\frac{D\overline{v\theta}}{Dt} \quad = \quad & -\overline{vu_k}\frac{\partial T}{\partial x_k} - \overline{\theta u_k}\frac{\partial V}{\partial x_k} \\
& -\frac{(a+\nu)(\overline{\theta v} + \overline{\theta u_k} n_k n_2)}{y_n^2} \\
& +\frac{1}{2}(a+\nu)\frac{\partial^2 \overline{\theta v}}{\partial x_k^2} - \frac{\partial}{\partial x_k}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{u_k u_l}\frac{\partial \overline{\theta v}}{\partial x_l} + \overline{vu_l}\frac{\partial \overline{\theta u_k}}{\partial x_l}\right)\right] \\
& \underbrace{-C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v}}_{\phi_{\theta 2,1}} \underbrace{-C_{2\theta}P_{\theta 2}^m}_{\phi_{\theta 2,2}} - \underbrace{C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k}}_{\phi_{\theta 2,1}^w} f_w n_k n_2
\end{aligned}
\tag{6.13}
$$

Let us define a new variable g as,

$$
g = -C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v} - C_{2\theta}P_{\theta 2}^m - C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k} f_w n_k n_2 \tag{6.14}
$$

Therefore, the above equation becomes,

$$
\begin{aligned}
\frac{D\overline{v\theta}}{Dt} &= -\overline{vu_k}\frac{\partial T}{\partial x_k} - \overline{\theta u_k}\frac{\partial V}{\partial x_k} \\
&\quad -\frac{(a+\nu)(\overline{\theta v} + \overline{\theta u_k} n_k n_2)}{y_n^2} \\
&\quad +\frac{1}{2}(a+\nu)\frac{\partial^2 \overline{\theta v}}{\partial x_k^2} - \frac{\partial}{\partial x_k}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{u_k u_l}\frac{\partial \overline{\theta v}}{\partial x_l} + \overline{vu_l}\frac{\partial \overline{\theta u_k}}{\partial x_l}\right)\right] \\
&\quad +g
\end{aligned}
\tag{6.15}
$$

g is nothing but the pressure scramble term in $\overline{v\theta}$-equation. The elliptic nature of pressure scramble term is accounted by adding a differential operator on left hand side of g as follows,

$$
L^2 \frac{\partial^2 g}{\partial {x_j}^2} - g = C_{1\theta}\overline{\theta v}\frac{\varepsilon}{k} + C_{2\theta}P^m_{\theta 2} + C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k} f_w n_k n_2 \tag{6.16}
$$

The turbulent length scale as suggested by Durbin [6] is given by,

$$
L = C_l \max\left(\frac{k^{3/2}}{\varepsilon}\ ,\ C_\eta \frac{\nu^{3/4}}{\varepsilon^{1/4}}\right) \tag{6.17}
$$

The above g-equation acts so as to let the pressure scramble term (g) go from its wall value to the value of its sink term over the scale L. In the far-field $\partial^2 g/\partial {x_j}^2 \simeq 0$ and thus g takes its source value. In this way, the reduction in pressure scramble term as the wall is approached is modeled.

The constants read,

$C_{1\theta} = 3.75$	$C_{2\theta} = C_{3\theta} = 0.5$	$C_{1\theta w} = 0.75$	$C_{2\theta w} = C_{3\theta w} = 0$
$C_w = 2.53$	$C_l = 0.23$	$C_\eta = 70$	$C_\theta = 0.11$

P_r for air is 0.72.

Finally, the $\overline{v\theta} - g$ model reads,

$$
\begin{aligned}
\frac{D\overline{v\theta}}{Dt} &= -\overline{vu_k}\frac{\partial T}{\partial x_k} - \overline{\theta u_k}\frac{\partial V}{\partial x_k} \\
&\quad -\frac{(a+\nu)(\overline{\theta v} + \overline{\theta u_k} n_k n_2)}{y_n^2} \\
&\quad +\frac{1}{2}(a+\nu)\frac{\partial^2 \overline{\theta v}}{\partial x_k^2} - \frac{\partial}{\partial x_k}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{u_k u_l}\frac{\partial \overline{\theta v}}{\partial x_l} + \overline{vu_l}\frac{\partial \overline{\theta u_k}}{\partial x_l}\right)\right] \\
&\quad +g \qquad (6.18)
\end{aligned}
$$

$$
L^2 \frac{\partial^2 g}{\partial {x_j}^2} - g = C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v} + C_{2\theta} P^m_{\theta 2} + C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k} f_w n_k n_2 \qquad (6.19)
$$

For a fully developed 1D channel flow with variations only in wall-normal direction, the $\overline{v\theta} - g$ model reads,

$$
\begin{aligned}
\frac{D\overline{v\theta}}{Dt} &= -\overline{v^2}\frac{\partial T}{\partial y} - 2\frac{(a+\nu)\overline{\theta v}}{y_n^2} + \frac{1}{2}(a+\nu)\frac{\partial^2 \overline{\theta v}}{\partial y^2} \\
&\quad -2\frac{\partial}{\partial y}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{v^2}\frac{\partial \overline{\theta v}}{\partial y}\right)\right] + g \qquad (6.20)
\end{aligned}
$$

$$
L^2 \frac{\partial^2 g}{\partial y^2} - g = C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v} + C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta v} f_w \qquad (6.21)
$$

6.5 Performance of $\overline{v\theta} - g$ Model

Boundary Condition

Since there was a lot of uncertianity in specifying the boundary condition for temperature at mid span, the whole channel is solved.

At Both Walls

At the wall we have no-slip boundary condition for velocity components, kinetic energy, wall-normal stress, wall-normal heat flux, f-equation and the g-equation.

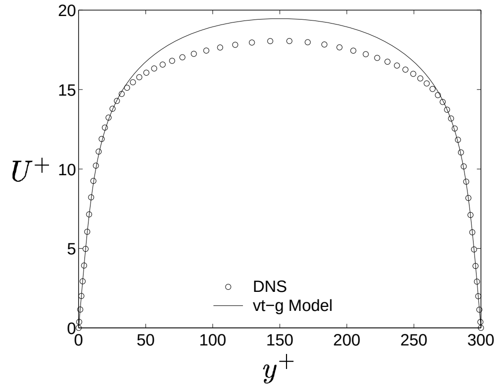
$$U = k = \overline{v^2} = \overline{v\theta} = f = g = 0$$

ε is fixed at the second node using the following wall functions.

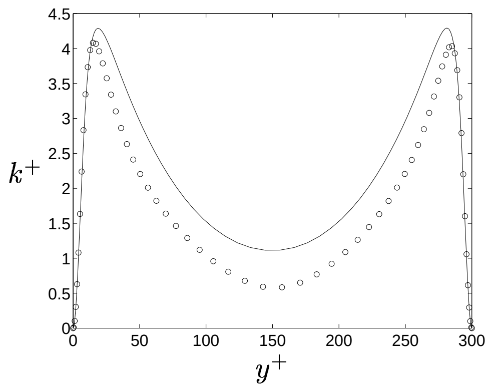
$$\varepsilon_{wall} = \nu \frac{2k}{y^2}$$

The results are compared with the existing DNS data with a Reynolds number based on the channel half width and the friction velocity, $R_{e_\tau} = u_* \delta / \nu = 150$. The number of grid points in the y-direction, starting from one wall to the other is 136. The value of y^+ closest to the wall is 0.034. A geometric stretching factor of 1.08 is used.

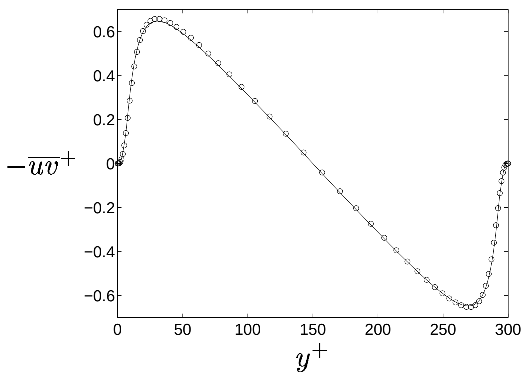
6.1(g) shows that the wall-normal heat flux predicted by $\overline{v\theta} - g$ model is better than just $\overline{v\theta}$ model. However, 6.1(i) shows that temperature calculated using $\overline{v\theta}$ model and the eddy-viscosity model for heat-flux is in better agreement with the DNS than $\overline{v\theta} - g$ model. Since the computations are done at very low Reynolds number($R_{e_\tau} = 150$), further testing of the model at higher Reynolds number was necessary. Performance of the model is analyzed at a higher Reynolds number of $R_{e_\tau} = 10,000$ and the results are compared with the log-law. 6.2(a) shows that the velocity profile is in good agreement with the log-law. However, 6.2(j) shows that the temperature prediction by none of the models is comparable to log-law.



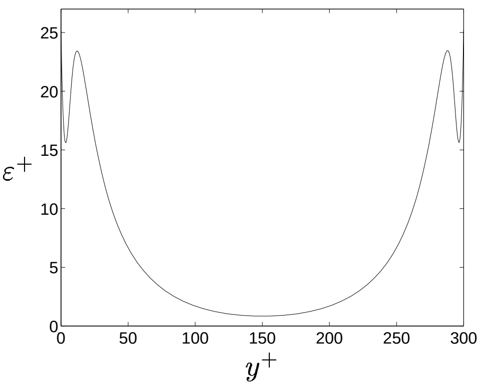
(a) U



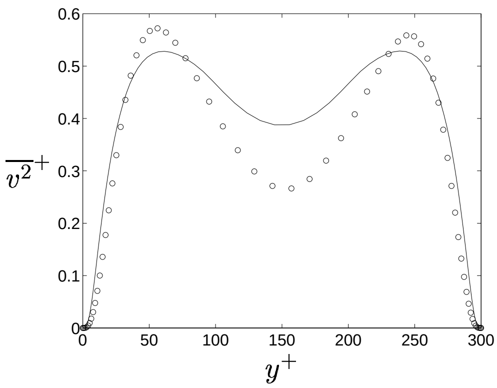
(b) K



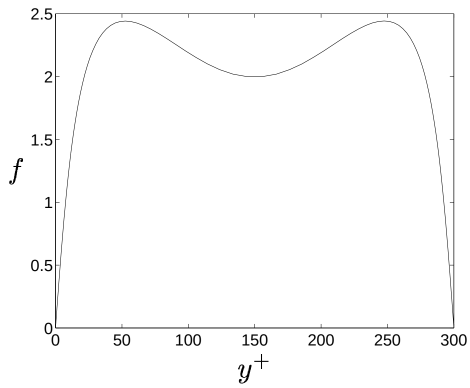
(c) $-\overline{uv}$



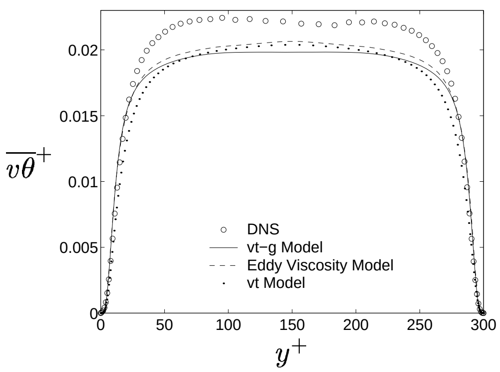
(d) ε



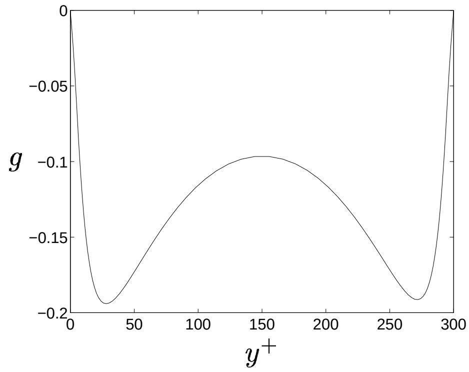
(e) $\overline{v^2}$



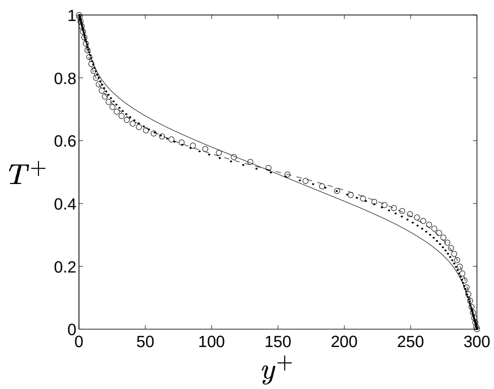
(f) f



(g) $\overline{v\theta}$

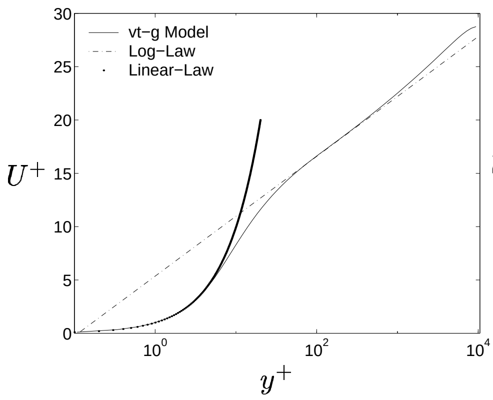


(h) g

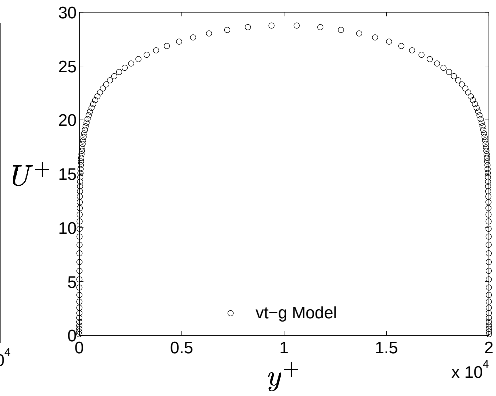


(i) T

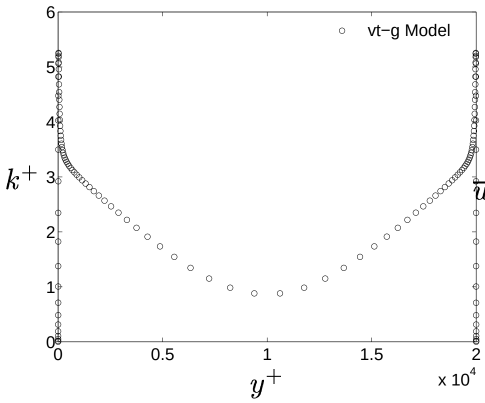
Figure 6.1: Turbulent quantities compared with the DNS data, $R_{e_\tau} = 150$ (a) Velocity contour (b) Kinetic Energy (c) Reynolds shear stress (d) Dissipation (e) Normal Reynolds stress (f) f-equation (g) Wall-normal heat flux (h) g-equation (i) Temperature



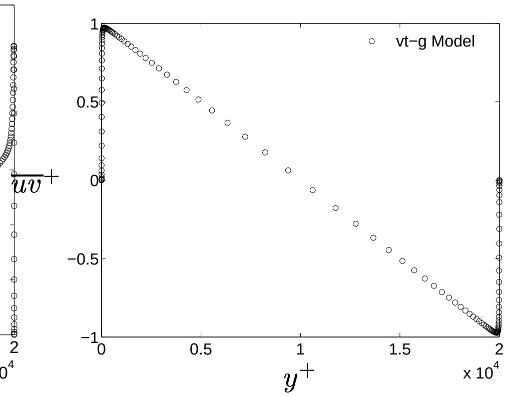
(a) U



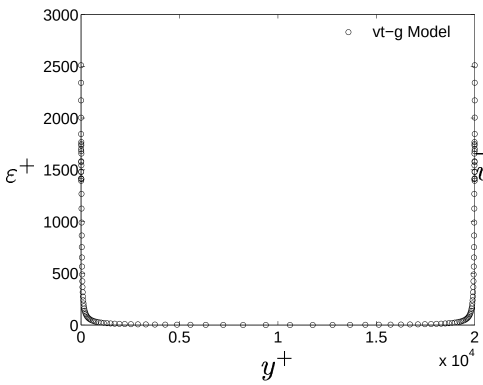
(b) U



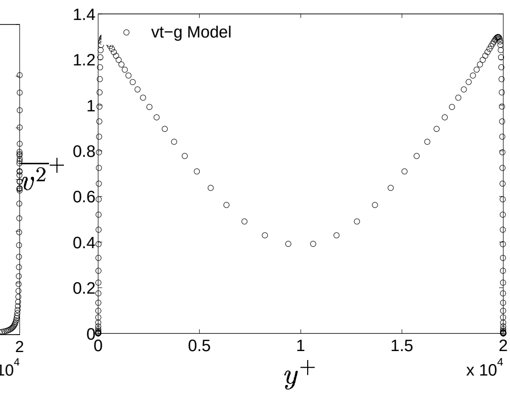
(c) K



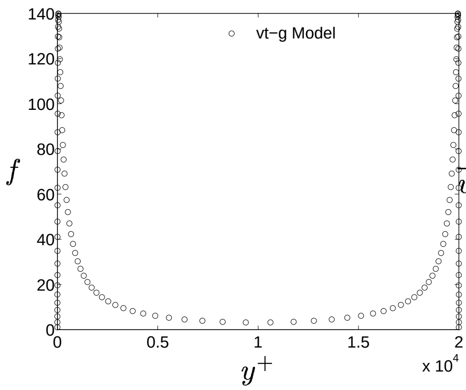
(d) $-\overline{uv}$



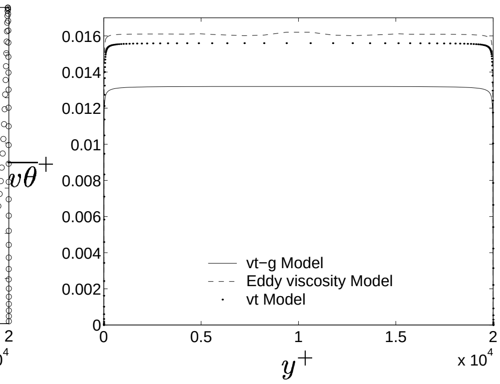
(e) ε



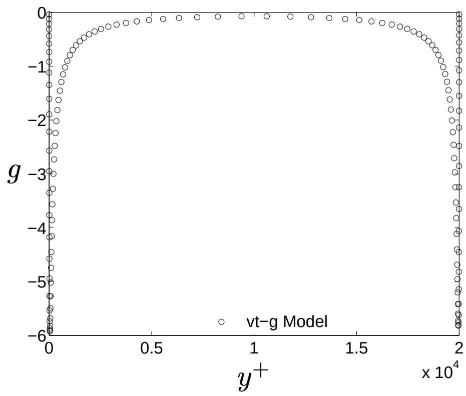
(f) $\overline{v^2}$



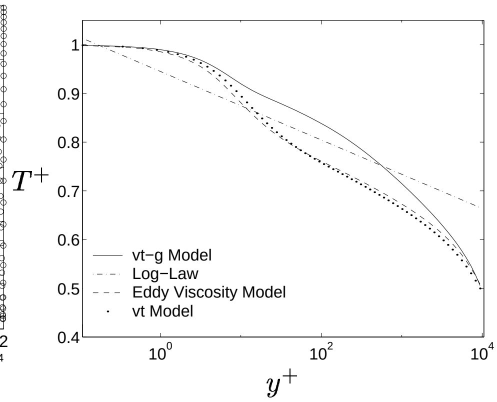
(g) f



(h) $\overline{v\theta}$

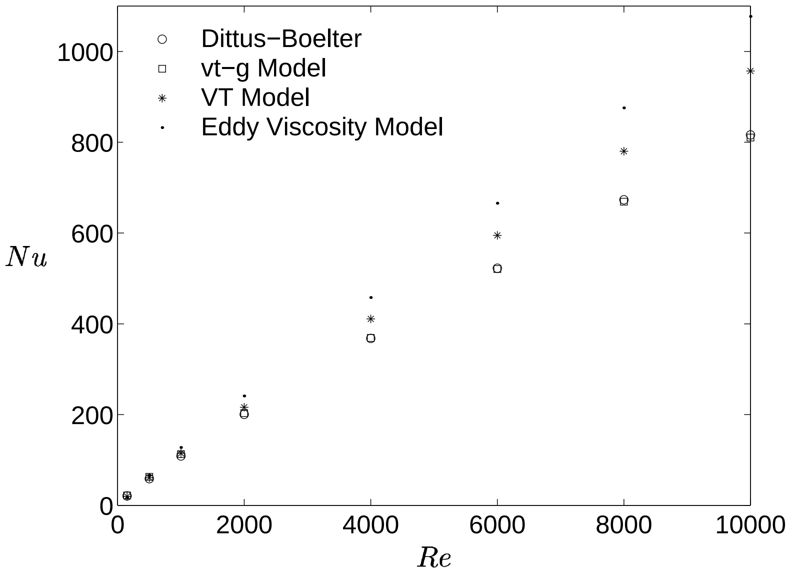


(i) g



(j) T

Figure 6.2: Turbulent quantities compared with the DNS data, $R_{e_\tau} = 10,000$ (a,b) Velocity contour (c) Kinetic Energy (d) Reynolds shear stress (e) Dissipation (f) Normal Reynolds stress (g) f-equation (h) Wall-normal heat flux (i) g-equation (j) Temperature



(a) $Nu \quad vs \quad Re$

Figure 6.3: Comparision of Nusselt Number at different Reynolds numbers.

Since the performance of $\overline{v\theta}-g$ model remained inconclusive with all the previous results, the heat transfer predictions which is usually visualized using Nusselt number was calculated at different Reynolds numbers and compared to the Nusselt number calculated using empirical Dittus-Boelter formula which reads,

$$Nu_s = 0.023\, Re^{0.8}\, Pr^{0.4} \tag{6.22}$$

The Nusselt number with a prescribed heat transfer at the wall, q_w is defined as,

$$Nu_s = \frac{2q_w PrH}{\mu c_p(T_w - T_{mean})} \tag{6.23}$$

where H is the channel height, Pr is the Prandtl number, T_w is the wall temperature and T_{mean} is the mean temperature.

From 6.3(a) it is obvious that the heat transfer prediction is better for $\overline{v\theta} - g$ model than traditional $\overline{v\theta}$ model. However, the difference is less prominent at lower Reynolds number, which is why the results were inconclusive at $R_e = 150$.

Chapter 7

Future Work

(a) The $k-\omega-\overline{v^2}-f$ model developed is intended for computing complex engineering flows. Hence, further testing of the model is needed.

(b) It has been found that $k-\varepsilon-\overline{v^2}-f$ model is very sensitive to y^+ values in the near wall region. Hence one must try to use wall functions and fix all the parameters we solve for. Doing so may also result in faster convergence.

(c) If we have accurate wall-normal heat flux, the wall-parallel heat flux can be estimated by relating the stresses to heat fluxes as discussed below,

$$\frac{\overline{u^2}}{\overline{v^2}} = \frac{\overline{u\theta}}{\overline{v\theta}} \tag{7.1}$$

Therefore,

$$\overline{u\theta} = \frac{\overline{u^2}}{\overline{v^2}}\overline{v\theta} \tag{7.2}$$

$\overline{v^2}$ and $\overline{v\theta}$ are known. $\overline{u^2}$ is estimated as follows,

We know that,

$$\overline{u^2} + \overline{v^2} + \overline{w^2} = 2k \tag{7.3}$$

Also, we know that $\overline{u^2}$ is the largest, $\overline{v^2}$ is the smallest and $\overline{w^2}$ lies somewhere in between $\overline{u^2}$ and $\overline{v^2}$. Hence we can take,

$$\overline{w^2} = \frac{1}{2}\left(\overline{u^2} + \overline{v^2}\right) \tag{7.4}$$

Hence 7.3 becomes,

$$\frac{3}{2}\left(\overline{u^2} + \overline{v^2}\right) = 2k \tag{7.5}$$

$$\overline{u^2} = \frac{4k}{3} - \overline{v^2} \tag{7.6}$$

Substituting in 7.2 we get,

$$\overline{u\theta} = \frac{4k/3 - \overline{v^2}}{\overline{v^2}} \overline{v\theta} \tag{7.7}$$

In the case of fully developed 1D channel flow, the wall-parallel heat flux is not of importance. Hence the above made hypothesis needs to be verified in a flow case where wall-parallel heat flux is of importance.

Bibliography

[1] K. Abe, T. Kondoh, and Y. Nagano. A new turbulence model for predicting fluid flow and heat transfer in separating and reattaching flows - 1. Flow field calculations. *Int. J. Heat Mass Transfer*, 37:139–151, 1994.

[2] J. Bredberg. *Turbulence Modelling for Internal Cooling of Gas-Turbine Blades*. PhD thesis, Dept. of Thermo and Fluid Dynamics, Chalmers University of Technology, Göteborg, Sweden, 2002.

[3] L. Davidson. An introduction to turbulence models. Technical Report 97/2, Dept. of Thermo and Fluid Dynamics, Chalmers University of Technology, Gothenburg, 1997.

[4] L. Davidson and B. Farhanieh. CALC-BFC: A finite-volume code employing collocated variable arrangement and cartesian velocity components for computation of fluid flow and heat transfer in complex three-dimensional geometries. Rept. 95/11, Dept. of Thermo and Fluid Dynamics, Chalmers University of Technology, Gothenburg, 1995.

[5] H.S. Dol, K. Hanjalić, and S. Kenjeres. A comparative assessment of the second-moment differential and algebraic models in turbulent natural convection. *Int. J. Heat Fluid Flow*, 18:4–14, 1997.

[6] P.A. Durbin. Near-wall turbulence closure modeling without damping functions. *Theoretical and Computational Fluid Dynamics*, 3:1–13, 1991.

[7] F-S Lien and G. Kalitzin. Computations of transonic flow with the v2f turbulence model. *Int. J. Heat Fluid Flow*, 22(1):53–61, 2001.

[8] R.D. Moser, J.D. Kim, and N.N. Mansour. Direct numerical simulation of turbulent channel flow up to $Re_\tau = 590$. *Physics of Fluids A*, 11:943–945, 1999.

[9] T.W.J. Peeters and R.A.W.M. Henkes. The reynolds-stress model of turbulence applied to the natural-convection boundary layer along a heated vertical plate. *Int. J. Heat Mass Transfer*, 35:403–420, 1992.

[10] S-H Peng, L. Davidson, and S. Holmberg. A modified low-Reynolds-number $k - \omega$ model for recirculating flows. *ASME: Journal of Fluids Engineering*, 119:867–875, 1997.

[11] A. Sveningsson. private communication. Dept. of Thermo and Fluid Dynamics, Chalmers University of Technology, Gothenburg, Sweden, 2003.

[12] H.K. Versteegh and W. Malalasekera. *An Introduction to Computational Fluid Dynamics - The Finite Volume Method*. Longman Scientific & Technical, Harlow, England, 1995.

[13] D.C. Wilcox. Reassessment of the scale-determining equation. *AIAA Journal*, 26:1299–1310, 1988.

[14] D.C. Wilcox. Comparison of two-equation turbulence models for boundary layers wi th pressure-gradient. *AIAA Journal*, 31:1414–1421, 1993.

Appendix A

Log - Laws

Velocity

The log-law for Velocity as given in Davidson [3] reads,

$$\frac{U_p}{u_*} = \frac{1}{\kappa} \ln \left(\frac{E u_* y}{\nu} \right) \tag{A.1}$$

U_p = Velocity component parallel to the wall.
u_* = Friction velocity.
y = Normal distance from the nearest wall.
E , κ = Constants determined from experiments. ($E = 9$ & $\kappa = 0.41$)

In my case, $u_* = 1$. But, generally u_* is not known and we iterate for u_* using A.1. A few iterations usually suffice.

The log-law given by A.1 is valid only in the log-region which is between $30 \leq y^+ \leq 100$. However the upper limit on y^+ depends on the Reynolds number we solve for.

Temperature

The log-law for Temperature as given in Bredberg [2] reads,

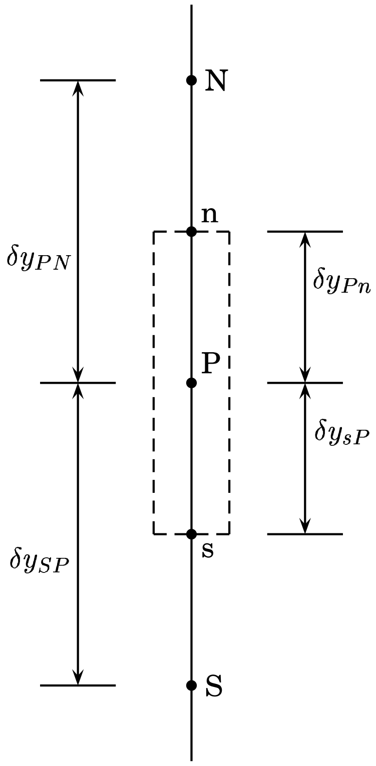
$$\frac{(T_w - T_p)\, \rho C_p u_*}{q_w} = \sigma + \frac{\sigma_t}{\kappa} \ln \left(\frac{y^+}{13.2} \right) \tag{A.2}$$

T_w = Wall temperature.
ρ = Density of air = 1.25 kg/m^3.
C_p = Specific heat of air at constant pressure = 0.715 $J/g/k$.
σ = Prandtl number for air = 0.72
σ_t = Turbulent Prandtl number = 0.89
κ = Kappa = 0.41

Appendix B

1D Discretisation of the Governing Equations

Define a one-dimensional control volume as shown in the figure.



1. **Momentum Equation**

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[(\nu + \nu_t)\frac{\partial U_i}{\partial x_j}\right] \tag{B.1}$$

since the flow is fully developed, we have,

$$V = \frac{\partial \phi}{\partial x} = 0$$

where 'ϕ' is any variable that we solve for.
We set the pressure gradient to be 1.

$$-\frac{\partial P}{\partial x} = 1$$

Therefore B.1 for 1D case reduces to,

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[(\nu + \nu_t)\frac{\partial U}{\partial y}\right]$$

$$0 = \int_s^n 1dy + \int_s^n \frac{\partial}{\partial y}\left[(\nu + \nu_t)\frac{\partial U}{\partial y}\right] dy$$

$$0 = \triangle y + (\nu + \nu_t)_n \left(\frac{U_N - U_P}{\delta y_{NP}}\right) - (\nu + \nu_t)_s \left(\frac{U_P - U_S}{\delta y_{PS}}\right)$$

$$\left\{\frac{(\nu + \nu_t)_n}{\delta y_{NP}} + \frac{(\nu + \nu_t)_s}{\delta y_{PS}}\right\} U_P = \left\{\frac{(\nu + \nu_t)_n}{\delta y_{NP}}\right\} U_N + \left\{\frac{(\nu + \nu_t)_s}{\delta y_{PS}}\right\} U_S + \triangle y$$

This is of the form,

$$a_P U_P = a_N U_N + a_S U_S + S_u$$

The coefficients are,

$S_u = \triangle y$	$S_p = 0$
$a_N = (\nu + \nu_t)_n / \delta y_{NP}$	$a_S = (\nu + \nu_t)_s / \delta y_{PS}$
$a_P = a_N + a_S - S_p$	

2. **Kinetic Energy Equation**

$$U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \varepsilon$$

For a fully developed 1D channel flow,

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \varepsilon$$

$$0 = \int_s^n \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] dy + \int_s^n P_k dy - \int_s^n \varepsilon dy$$

$$0 = \left(\nu + \frac{\nu_t}{\sigma_k}\right)_n \left(\frac{k_N - k_P}{\delta y_{NP}}\right) - \left(\nu + \frac{\nu_t}{\sigma_k}\right)_s \left(\frac{k_P - k_S}{\delta y_{PS}}\right) + P_k \triangle y - \varepsilon \triangle y$$

$$\left\{ \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_n}{\delta y_{NP}} + \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_s}{\delta y_{PS}} + \frac{\varepsilon \triangle y}{k_{old}} \right\} k_P =$$

$$\left\{ \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_n}{\delta y_{NP}} \right\} k_N + \left\{ \frac{\left(\nu + \frac{\nu_t}{\sigma_k}\right)_s}{\delta y_{PS}} \right\} k_S + P_k \triangle y$$

This is of the form,

$$a_P k_P = a_N k_N + a_S k_S + S_u$$

The coefficients are,

$S_u = P_k \triangle y$	$S_p = -\varepsilon \triangle y / k_{old}$
$a_N = \left(\nu + \frac{\nu_t}{\sigma_k}\right)_n / \delta y_{NP}$	$a_S = \left(\nu + \frac{\nu_t}{\sigma_k}\right)_s / \delta y_{PS}$
$a_P = a_N + a_S - S_p$	

3. ε **Equation**

$$U_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{\varepsilon}{k}(C_{\varepsilon 1} P_k - C_{\varepsilon 2} f_2 \varepsilon)$$

For a fully developed 1D channel flow,

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] + \frac{\varepsilon}{k}(C_{\varepsilon 1} P_k - C_{\varepsilon 2} f_2 \varepsilon)$$

$$0 = \int_s^n \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] dy + \int_s^n \frac{\varepsilon}{k}(C_{\varepsilon 1} P_k - C_{\varepsilon 2} f_2 \varepsilon) dy$$

$$0 = \left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_n \left(\frac{\varepsilon_N - \varepsilon_P}{\delta y_{NP}}\right) - \left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_s \left(\frac{\varepsilon_P - \varepsilon_S}{\delta y_{PS}}\right) + \frac{C_{\varepsilon 1} P_k \varepsilon}{k} \triangle y - \frac{C_{\varepsilon 2} f_2 \varepsilon^2}{k} \triangle y$$

$$\left\{ \frac{\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_n}{\delta y_{NP}} + \frac{\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_s}{\delta y_{PS}} + \frac{C_{\varepsilon 2} f_2 \varepsilon_{old} \triangle y}{k} \right\} \varepsilon_P =$$

$$\left\{ \frac{\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_n}{\delta y_{NP}} \right\} \varepsilon_N + \left\{ \frac{\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_s}{\delta y_{PS}} \right\} \varepsilon_S + \left\{ \frac{C_{\varepsilon 1} \varepsilon P_k \triangle y}{k} \right\}$$

This is of the form,

$$a_P \varepsilon_P = a_N \varepsilon_N + a_S \varepsilon_S + S_u$$

The coefficients are,

$S_u = C_{\varepsilon 1}\varepsilon P_k \triangle y/k$	$S_p = -C_{\varepsilon 2} f_2 \varepsilon_{old} \triangle y/k$
$a_N = \left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_n / \delta y_{NP}$	$a_S = \left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)_s / \delta y_{PS}$
$a_P = a_N + a_S - S_p$	

4. ω **Equation (Peng - 1997)**

$$U_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + C_{\omega 1} f_\omega \frac{\omega}{k} P_k - C_{\omega 2}\omega^2 + C_\omega \frac{\nu_t}{k}\left(\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}\right)$$

For a fully developed 1D channel flow,

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial y}\right] + C_{\omega 1} f_\omega \frac{\omega}{k} P_k - C_{\omega 2}\omega^2 + C_\omega \frac{\nu_t}{k}\left(\frac{\partial k}{\partial y}\frac{\partial \omega}{\partial y}\right)$$

$$\begin{aligned} 0 \;=\; & \int_s^n \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial y}\right] dy + \int_s^n C_{\omega 1} f_\omega \frac{\omega}{k} P_k dy \\ & - \int_s^n C_{\omega 2}\omega^2 dy + \int_s^n C_\omega \frac{\nu_t}{k}\left(\frac{\partial k}{\partial y}\frac{\partial \omega}{\partial y}\right) dy \end{aligned}$$

$$\left\{\frac{\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)_n}{\delta y_{NP}} + \frac{\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)_s}{\delta y_{PS}} + C_{\omega 2}\omega \triangle y\right\}\omega_P =$$

$$\left\{\frac{\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)_n}{\delta y_{NP}}\right\}\omega_N + \left\{\frac{\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)_s}{\delta y_{PS}}\right\}\omega_S$$

$$+\left\{C_{\omega 1} f_\omega \frac{\omega}{k} P_k \triangle y + C_\omega \frac{\nu_t}{k}\left(\frac{\partial k}{\partial y}\frac{\partial \omega}{\partial y}\right)\triangle y\right\}$$

This is of the form,

$$a_P \omega_P = a_N \omega_N + a_S \omega_S + S_u$$

The coefficients are,

$S_u = C_{\omega 1} f_\omega \frac{\omega}{k} P_k \triangle y + C_\omega \frac{\nu_t}{k}\left(\frac{\partial k}{\partial y}\frac{\partial \omega}{\partial y}\right)\triangle y$	$S_p = -C_{\omega 2}\omega \triangle y$
$a_N = \left(\nu + \frac{\nu_t}{\sigma_\omega}\right)_n / \delta y_{NP}$	$a_S = \left(\nu + \frac{\nu_t}{\sigma_\omega}\right)_s / \delta y_{PS}$
$a_P = a_N + a_S - S_p$	

5. $\overline{v^2}$ **Equation**

$$U_j \frac{\partial \overline{v^2}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right] + kf - 6\overline{v^2}\frac{\varepsilon}{k}$$

For a fully developed 1D channel flow,

$$0 = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial y}\right] + kf - 6\overline{v^2}\frac{\varepsilon}{k}$$

$$0 = \int_s^n \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_{v2}}\right)\frac{\partial \overline{v^2}}{\partial y}\right] dy + \int_s^n kf\,dy - \int_s^n 6\overline{v^2}\frac{\varepsilon}{k}dy$$

$$0 = \left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_n \left(\frac{\overline{v^2}_N - \overline{v^2}_P}{\delta y_{NP}}\right) - \left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_s \left(\frac{\overline{v^2}_P - \overline{v^2}_S}{\delta y_{PS}}\right) + kf\triangle y - 6\overline{v^2}\frac{\varepsilon}{k}\triangle y$$

$$\left\{\frac{\left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_n}{\delta y_{NP}} + \frac{\left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_s}{\delta y_{PS}} + 6\frac{\varepsilon}{k}\triangle y\right\}\overline{v^2}_P =$$
$$\left\{\frac{\left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_n}{\delta y_{NP}}\right\}\overline{v^2}_N + \left\{\frac{\left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_s}{\delta y_{PS}}\right\}\overline{v^2}_S + kf\triangle y$$

This is of the form,

$$a_P\overline{v^2}_P = a_N\overline{v^2}_N + a_S\overline{v^2}_S + S_u$$

The coefficients are,

$S_u = kf\triangle y$	$S_p = -6\varepsilon\triangle y/k$
$a_N = \left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_n/\delta y_{NP}$	$a_S = \left(\nu + \frac{\nu_t}{\sigma_{v^2}}\right)_s/\delta y_{PS}$
$a_P = a_N + a_S - S_p$	

6. f **Equation**

$$L^2\frac{\partial^2 f}{\partial {x_j}^2} - f = \frac{C_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3}\right) - C_2\frac{P_k}{k} - \frac{1}{T}\left(6\frac{\overline{v^2}}{k} - \frac{2}{3}\right)$$

$$-\frac{\partial^2 f}{\partial {x_j}^2} + \frac{f}{L^2} = -\frac{1}{L^2T}\left[(C_1 - 6)\frac{\overline{v^2}}{k} - \frac{2}{3}(C_1 - 1)\right] + \frac{C_2}{L^2}\frac{P_k}{k}$$

$$-\frac{\partial^2 f}{\partial {x_j}^2} + \frac{f}{L^2} + \frac{1}{L^2T}\left[C_1\frac{\overline{v^2}}{k} + \frac{2}{3}\right] = \frac{1}{L^2T}\left(6\frac{\overline{v^2}}{k} + \frac{2}{3}C_1\right) + \frac{C_2}{L^2}\frac{P_k}{k}$$

For a fully developed 1D channel flow,

$$-\int_s^n \frac{\partial^2 f}{\partial y^2}dy + \int_s^n \frac{f}{L^2}dy + \int_s^n \frac{1}{L^2T}\left(C_1\frac{\overline{v^2}}{k} + \frac{2}{3}\right)dy =$$
$$\int_s^n \frac{1}{L^2T}\left(6\frac{\overline{v^2}}{k} + \frac{2}{3}C_1\right)dy + \int_s^n \frac{C_2}{L^2}\frac{P_k}{k}dy$$

$$\left(\frac{f_P - f_S}{\delta y_{PS}}\right) - \left(\frac{f_N - f_P}{\delta y_{NP}}\right) + \frac{f}{L^2}\triangle y + \frac{1}{L^2T}\left(C_1\frac{\overline{v^2}}{k} + \frac{2}{3}\right)\triangle y =$$
$$\frac{1}{L^2T}\left(6\frac{\overline{v^2}}{k} + \frac{2}{3}C_1\right)\triangle y + \frac{C_2}{L^2}\frac{P_k}{k}\triangle y$$

$$\left\{\frac{1}{\delta y_{NP}} + \frac{1}{\delta y_{PS}} + \frac{1}{L^2}\triangle y + \frac{1}{L^2Tf}\left(C_1\frac{\overline{v^2}}{k} + \frac{2}{3}\right)\triangle y\right\} f_P =$$
$$\left\{\frac{1}{\delta y_{NP}}\right\} f_N + \left\{\frac{1}{\delta y_{PS}}\right\} f_S + \left\{\frac{1}{L^2T}\left(6\frac{\overline{v^2}}{k} + \frac{2}{3}C_1\right)\triangle y + \frac{C_2}{L^2}\frac{P_k}{k}\triangle y\right\}$$

This is of the form,

$$a_P f_P = a_N f_N + a_S f_S + S_u$$

The coefficients are,

$S_u = \frac{1}{L^2T}\left(6\frac{\overline{v^2}}{k} + \frac{2}{3}C_1\right)\triangle y + \frac{C_2}{L^2}\frac{P_k}{k}\triangle y$	$S_p = -\frac{1}{L^2}\triangle y - \frac{1}{L^2Tf}\left(C_1\frac{\overline{v^2}}{k} + \frac{2}{3}\right)\triangle y$
$a_N = 1/\delta y_{NP}$	$a_S = 1/\delta y_{PS}$
$a_P = a_N + a_S - S_p$	

7. $\overline{v\theta}$ Equation

(a) Viscous diffusion

$$D^v_{\theta i} = \frac{1}{2}(a + \nu)\frac{\partial^2 \overline{\theta u_i}}{\partial x_k^2}$$

For a fully developed 1D channel flow with $i = 2$,

$$D^v_{\theta 2} = \frac{1}{2}(a + \nu)\frac{\partial^2 \overline{\theta v}}{\partial y^2}$$

$$D^v_{\theta 2} = \int_s^n \frac{1}{2}(a + \nu)\frac{\partial^2 \overline{\theta v}}{\partial y^2}dy$$

$$D^v_{\theta 2} = \frac{1}{2}(a + \nu)\left[\left(\frac{\partial \overline{\theta v}}{\partial y}\right)_n - \left(\frac{\partial \overline{\theta v}}{\partial y}\right)_s\right]$$

$$D^v_{\theta 2} = \frac{1}{2}(a + \nu)\left[\left(\frac{\overline{\theta v}_N - \overline{\theta v}_P}{\delta y_{NP}}\right) - \left(\frac{\overline{\theta v}_P - \overline{\theta v}_S}{\delta y_{PS}}\right)\right]$$

(b) **Turbulent diffusion**

$$D^t_{\theta i} = \frac{\partial}{\partial x_k}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{u_k u_l}\frac{\partial \overline{\theta u_i}}{\partial x_l} + \overline{u_i u_l}\frac{\partial \overline{\theta u_k}}{\partial x_l}\right)\right]$$

For a fully developed 1D channel flow with $i = 2$,

$$D^t_{\theta 2} = \frac{\partial}{\partial y}\left[C_\theta \frac{k}{\varepsilon}\left(\overline{v^2}\frac{\partial \overline{\theta v}}{\partial y} + \overline{v^2}\frac{\partial \overline{\theta v}}{\partial y}\right)\right]$$

$$D^t_{\theta 2} = 0.22\int_s^n \frac{\partial}{\partial y}\left[\frac{k}{\varepsilon}\left(\overline{v^2}\frac{\partial \overline{\theta v}}{\partial y}\right)\right] dy$$

$$D^v_{\theta 2} = 0.22\left[\frac{k_n}{\varepsilon_n}\overline{v^2_n}\left(\frac{\overline{\theta v}_N - \overline{\theta v}_P}{\delta y_{NP}}\right) - \frac{k_s}{\varepsilon_s}\overline{v^2_s}\left(\frac{\overline{\theta v}_P - \overline{\theta v}_S}{\delta y_{PS}}\right)\right]$$

(c) **Production**

$$P_{\theta i} = -\overline{u_i u_k}\frac{\partial T}{\partial x_k} - \overline{\theta u_k}\frac{\partial U_i}{\partial x_k}$$

For a fully developed 1D channel flow with $i = 2$,

$$P_{\theta 2} = -\overline{v^2}\frac{\partial T}{\partial y}$$

$$P_{\theta 2} = -\int_s^n \overline{v^2}\frac{\partial T}{\partial y} dy$$

$$P_{\theta 2} = -\overline{v^2}\frac{\partial T}{\partial y}\triangle y$$

(d) **Dissipation**

$$-\varepsilon_{\theta i} = \frac{-(a+\nu)(\overline{\theta u_i} + \overline{\theta u_k} n_k n_i)}{y_n^2}$$

For a fully developed 1D channel flow with $i = 2$,

$$-\varepsilon_{\theta 2} = \frac{-(a+\nu)(\overline{\theta v} + \overline{\theta v})}{y_n^2}$$

$$-\varepsilon_{\theta 2} = \int_s^n \frac{-2(a+\nu)\overline{\theta v}}{y_n^2} dy$$

$$-\varepsilon_{\theta 2} = \frac{-2(a+\nu)\overline{\theta v}}{y_n^2}\triangle y$$

(e) **Pressure scrambling**

$$\pi_{\theta i} = \underbrace{-C_{1\theta}\frac{\varepsilon}{k}\overline{\theta u_i}}_{\phi_{\theta i,1}} \underbrace{-C_{2\theta}P^m_{\theta i}}_{\phi_{\theta i,2}} \underbrace{-C_{3\theta}G_{\theta i}}_{\phi_{\theta i,3}} - \left[\underbrace{C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k}}_{\phi^w_{\theta i,1}} + \underbrace{C_{2\theta w}\pi_{\theta k,2}}_{\phi^w_{\theta i,2}} + \underbrace{C_{3\theta w}\pi_{\theta k,3}}_{\phi^w_{\theta i,3}}\right] f_w n_k n_i$$

For a fully developed 1D channel flow with $i = 2$,

$$\pi_{\theta 2} = -C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v} - C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta v}$$

$$\pi_{\theta 2} = -\int_s^n C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v}dy - \int_s^n C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta v}dy$$

$$\pi_{\theta 2} = -C_{1\theta}\frac{\varepsilon}{k}\overline{\theta v}\triangle y - C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta v}\triangle y$$

Hence the transport equation for the turbulent heat flux becomes,

$$\frac{D\overline{u_i\theta}}{Dt} = P^{th}_{\theta 2} + P^m_{\theta 2} - \varepsilon_{\theta i} + D^v_{\theta i} + D^t_{\theta i}$$

In boundary layer approximation, $V \ll U$ and $\partial/\partial x \ll \partial/\partial y$. Hence, $P^m_{\theta 2} = 0$. For a fully developed 1D channel flow the wall-normal heat flux equation reads,

$$\begin{aligned} 0 \;=\;& -\overline{v^2}\frac{\partial T}{\partial y}\triangle y - \frac{2(a+\nu)\overline{\theta v}}{y_n^2}\triangle y \\ & +\frac{1}{2}(a+\nu)\left[\left(\frac{\overline{\theta v}_N - \overline{\theta v}_P}{\delta y_{NP}}\right) - \left(\frac{\overline{\theta v}_P - \overline{\theta v}_S}{\delta y_{PS}}\right)\right] \\ & +0.22\left[\frac{k_n}{\varepsilon_n}\overline{v_n^2}\left(\frac{\overline{\theta v}_N - \overline{\theta v}_P}{\delta y_{NP}}\right) - \frac{k_s}{\varepsilon_s}\overline{v_s^2}\left(\frac{\overline{\theta v}_P - \overline{\theta v}_S}{\delta y_{PS}}\right)\right] \end{aligned}$$

$$\left\{\frac{\frac{1}{2}(a+\nu) + 0.22\frac{k_n}{\varepsilon_n}\overline{v_n^2}}{\delta y_{NP}} + \frac{\frac{1}{2}(a+\nu) + 0.22\frac{k_s}{\varepsilon_s}\overline{v_s^2}}{\delta y_{PS}} + \frac{2(a+\nu)}{y_n^2}\triangle y\right\}\overline{\theta v}_P =$$

$$\left\{\frac{\frac{1}{2}(a+\nu) + 0.22\frac{k_n}{\varepsilon_n}\overline{v_n^2}}{\delta y_{NP}}\right\}\overline{\theta v}_N + \left\{\frac{\frac{1}{2}(a+\nu) + 0.22\frac{k_s}{\varepsilon_s}\overline{v_s^2}}{\delta y_{PS}}\right\}\overline{\theta v}_S \underbrace{-\overline{v^2}\frac{\partial T}{\partial y}\triangle y}_{prod}$$

The above equation is of the form,

$$a_P\overline{\theta v}_P = a_N\overline{\theta v}_N + a_S\overline{\theta v}_S + S_u$$

The coefficients are,

$S_u = -\overline{v^2}\frac{\partial T}{\partial y}\triangle y$	$S_p = -\left\{2(a+\nu)/y_n^2\triangle y\right\}/\overline{\theta v}$
$a_N = \frac{1}{2}(a+\nu) + 0.22\frac{k_n}{\varepsilon_n}\overline{v_n^2}/\delta y_{NP}$	$a_S = \frac{1}{2}(a+\nu) + 0.22\frac{k_s}{\varepsilon_s}\overline{v_s^2}/\delta y_{PS}$
$a_P = a_N + a_S - S_p$	

8. **g Equation**

$$L^2\frac{\partial^2 g}{\partial {x_j}^2} - g = C_{1\theta}\frac{\varepsilon}{k}\overline{\theta u_i} + C_{2\theta}P^m_{\theta i} + C_{1\theta w}\frac{\varepsilon}{k}\overline{\theta u_k}f_w n_k n_i$$

$$-\frac{\partial^2 g}{\partial {x_j}^2} + \frac{g}{L^2} = -C_{1\theta}\frac{\varepsilon}{kL^2}\overline{\theta u_i} - \frac{C_{2\theta}}{L^2}P^m_{\theta i} - C_{1\theta w}\frac{\varepsilon}{kL^2}\overline{\theta u_k}f_w n_k n_i$$

For a fully developed 1D channel flow,

$$-\frac{\partial^2 g}{\partial y^2} + \frac{g}{L^2} = -C_{1\theta}\frac{\varepsilon}{kL^2}\overline{\theta v} - C_{1\theta w}\frac{\varepsilon}{kL^2}\overline{\theta v}$$

$$-\int_s^n \frac{\partial^2 g}{\partial y^2}dy = -\int_s^n C_{1\theta}\frac{\varepsilon}{kL^2}(\overline{\theta v})dy - \int_s^n C_{1\theta w}\frac{\varepsilon}{kL^2}(\overline{\theta v})dy - \int_s^n \frac{g}{L^2}dy$$

$$\left(\frac{g_P - g_S}{\delta y_{PS}}\right) - \left(\frac{g_N - g_P}{\delta y_{NP}}\right) = -C_{1\theta}\frac{\varepsilon}{kL^2}(\overline{\theta v})\triangle y - C_{1\theta w}\frac{\varepsilon}{kL^2}(\overline{\theta v})\triangle y - \frac{g}{L^2}\triangle y$$

$$\begin{aligned}\left\{\frac{1}{\delta y_{NP}} + \frac{1}{\delta y_{PS}} + \frac{1}{L^2}\triangle y\right\}g_P \quad &= \quad \left\{\frac{1}{\delta y_{NP}}\right\}g_N + \left\{\frac{1}{\delta y_{PS}}\right\}g_S \\ &\quad + \left\{-C_{1\theta}\frac{\varepsilon}{kL^2}(\overline{\theta v})\triangle y - C_{1\theta w}\frac{\varepsilon}{kL^2}(\overline{\theta v})\triangle y\right\}\end{aligned}$$

This is of the form,

$$a_P g_P = a_N g_N + a_S g_S + S_u$$

The coefficients are,

$S_u = -C_{1\theta}\frac{\varepsilon}{kL^2}(\overline{\theta v})\triangle y - C_{1\theta w}\frac{\varepsilon}{kL^2}(\overline{\theta v})\triangle y$	$S_p = -\frac{1}{L^2}\triangle y$
$a_N = 1/\delta y_{NP}$	$a_S = 1/\delta y_{PS}$
$a_P = a_N + a_S - S_p$	

9. **Temperature Equation**

$$\frac{\partial\theta}{\partial t} + u_i\frac{\partial\theta}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{a\frac{\partial\theta}{\partial x_i} - \overline{u_i\theta}\right\}$$

For a fully developed 1D channel flow,

$$0 = \frac{\partial}{\partial y}\left\{a\frac{\partial\theta}{\partial y} - \overline{v\theta}\right\}$$

$$0 = \int_s^n \frac{\partial}{\partial y}\left\{a\frac{\partial \theta}{\partial y} - \overline{v\theta}\right\} dy$$

$$0 = a\left[\left(\frac{\partial \theta}{\partial y}\right)_n - \left(\frac{\partial \theta}{\partial y}\right)_s\right] - [(\overline{v\theta})_n - (\overline{v\theta})_s]$$

$$0 = a\left[\left(\frac{\theta_N - \theta_P}{\delta y_{NP}}\right) - \left(\frac{\theta_P - \theta_S}{\delta y_{PS}}\right)\right] - [(\overline{v\theta})_n - (\overline{v\theta})_s]$$

$$\left\{\frac{a}{\delta y_{NP}} + \frac{a}{\delta y_{PS}} + \frac{\overline{v\theta_n}}{\theta}\right\}\theta_P = \left\{\frac{a}{\delta y_{NP}}\right\}\theta_N + \left\{\frac{a}{\delta y_{PS}}\right\}\theta_S + (\overline{v\theta})_s$$

This is of the form,

$$a_P\theta_P = a_N\theta_N + a_S\theta_S + S_u$$

The coefficients are,

$S_u = (\overline{v\theta})_s$	$S_p = -(\overline{v\theta})_n/\theta$
$a_N = a/\delta y_{NP}$	$a_S = a/\delta y_{PS}$
$a_P = a_N + a_S - S_p$	

10. **Temperature Equation using Eddy Viscosity Model**

$$\frac{\partial \theta}{\partial t} + u_i\frac{\partial \theta}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{a\frac{\partial \theta}{\partial x_i} + \frac{\nu_t}{\sigma_\theta}\frac{\partial \theta}{\partial x_i}\right\}$$

For a fully developed 1D channel flow,

$$0 = \frac{\partial}{\partial y}\left\{a\frac{\partial \theta}{\partial y} + \frac{\nu_t}{\sigma_\theta}\frac{\partial \theta}{\partial y}\right\}$$

$$0 = \int_s^n \frac{\partial}{\partial y}\left\{\left(a + \frac{\nu_t}{\sigma_\theta}\right)\frac{\partial \theta}{\partial y}\right\} dy$$

$$0 = \left[\left(a + \frac{\nu_t}{\sigma_\theta}\right)_n\left(\frac{\partial \theta}{\partial y}\right)_n - \left(a + \frac{\nu_t}{\sigma_\theta}\right)_s\left(\frac{\partial \theta}{\partial y}\right)_s\right]$$

$$0 = \left[\left(a + \frac{\nu_t}{\sigma_\theta}\right)_n\left(\frac{\theta_N - \theta_P}{\delta y_{NP}}\right) - \left(a + \frac{\nu_t}{\sigma_\theta}\right)_s\left(\frac{\theta_P - \theta_S}{\delta y_{PS}}\right)\right]$$

$$\left\{\frac{\left(a + \frac{\nu_t}{\sigma_\theta}\right)_n}{\delta y_{NP}} + \frac{\left(a + \frac{\nu_t}{\sigma_\theta}\right)_s}{\delta y_{PS}}\right\}\theta_P = \left\{\frac{\left(a + \frac{\nu_t}{\sigma_\theta}\right)_n}{\delta y_{NP}}\right\}\theta_N + \left\{\frac{\left(a + \frac{\nu_t}{\sigma_\theta}\right)_s}{\delta y_{PS}}\right\}\theta_S$$

This is of the form,

$$a_P\theta_P = a_N\theta_N + a_S\theta_S + S_u$$

The coefficients are,

$S_u = 0$	$S_p = 0$
$a_N = \left(a + \frac{\nu_t}{\sigma_\theta}\right)_n / \delta y_{NP}$	$a_S = \left(a + \frac{\nu_t}{\sigma_\theta}\right)_s / \delta y_{PS}$
$a_P = a_N + a_S - S_p$	