Non-orthogonal Finite Volume Method for Heat Conduction

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1 Equation

The 2D equation reads

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+S=0$$

or in vector notation

$$\nabla\cdot(k\nabla T)+S=0. \tag{1}$$

2 Discretization

Integrate Eq. 1 over a control volume

$$\int_V\left[\nabla\cdot(k\nabla T)+S\right]dV=0.$$

Rewrite the divergence term using Gauss' law

$$\int_A(k\nabla T)\cdot\mathbf{dA}+\int_V SdV=0.$$

where $\mathbf{dA}$ is the (vector) area bounding the volume V. Using $\mathbf{dA}=\mathbf{n}|\mathbf{dA}|=\mathbf{n}dA$, where $\mathbf{n}$ is the unit normal vector of $\mathbf{dA}$ we get

$$\underbrace{\int_A(k\nabla T)\cdot\mathbf{n}dA}_{1}+\underbrace{\int_V SdV}_{2}=0. \tag{2}$$

Now we concentrate on the area integral (term 1 = T_1). Write out the dot product so that

$$T_1 = \int_A (k\nabla T)\cdot \mathbf{n} dA = \int_A k\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right)dA$$

Going from the continuous level (integral) to the discrete level (sum) we can write

$$T_1 = \sum_{i=e,w,n,s}\left\{k\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right)A\right\}_i$$

Term T_1 is a (negative) heat flux, i.e.

$$T_1 = \sum_{i=e,w,n,s} -\dot{Q}_i \tag{3}$$

Let us look at the heat flux at the east face e

$$-\dot{Q}_e = \left\{\underbrace{k\frac{\partial T}{\partial x}An_x}_{I} + \underbrace{k\frac{\partial T}{\partial y}An_y}_{II}\right\}_e \tag{4}$$

In term I we need to express $\partial T/\partial x$. In order to do that we use Gauss' law over a control volume centered at face e (dashed control volume in Fig. 1). To compute $\partial T/\partial x$ we use the x-component of the normal vector $\mathbf{n} = (n_x, n_y)$, i.e.

$$\int_V \frac{\partial T}{\partial x}dV = \int_A Tn_x dA$$

Assuming that $\partial T/\partial x$ is constant in the volume V we obtain

$$\frac{\partial T}{\partial x} = \frac{1}{V}\int_A Tn_x dA$$

For the east we we can write (see Fig. 1)

$$\left(\frac{\partial T}{\partial x}\right)_e = \frac{1}{V_e}\sum_{i=E,ne,P,se}(Tn_x^e A)_i =$$

$$\frac{1}{V_e}\left\{(TAn_x^e)_E + (TAn_x^e)_{ne} + (TAn_x^e)_P + (TAn_x^e)_{se}\right\}$$

The high index e (in n_x^e for example) denotes that the normal vector is related to the control volume centered at e. The dashed control volume in Fig. 1 is

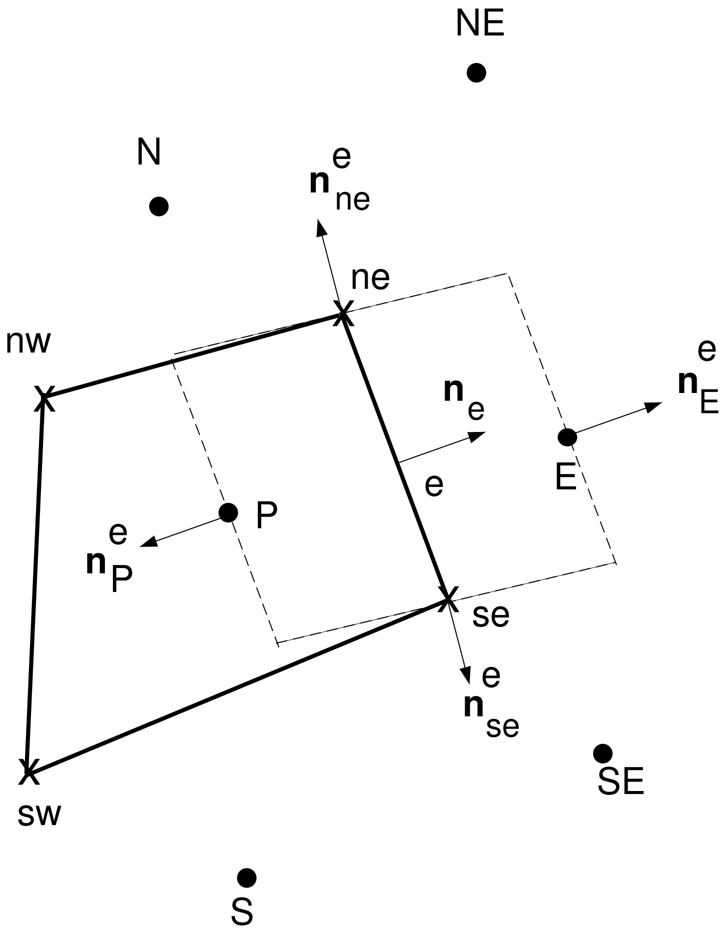

Figure 1: *Control volume. Temperature T and konductivity k are stored at nodes (P, E, ...). Coordinates x, y are given at the corners (ne, nw, ...). Solid lines shows the ordinary control volume. Dashed line shows the control volume centred at face e which is used when computing $(\partial T/\partial x)_e$ and $(\partial T/\partial y)_e$.*

chosen so that $(n_x^e A)_E = -(n_x^e A)_P = (n_x A)_e$, and $(n_x^e A)_{ne} = -(n_x^e A)_{se}$, which gives

$$\left(\frac{\partial T}{\partial x}\right)_e = \frac{1}{V_e}\left\{(An_x)_e(T_E - T_P) + (An_x^e)_{ne}(T_{ne} - T_{se})\right\} \tag{5}$$

We derive an analogous expression for $(\partial T/\partial y)_e$ as

$$\left(\frac{\partial T}{\partial y}\right)_e = \frac{1}{V_e}\left\{(An_y)_e(T_E - T_P) + (An_y^e)_{ne}(T_{ne} - T_{se})\right\} \tag{6}$$

and insert Eqs. 5, 6 into Eq. 4 we get

$$-\dot{Q}_e = \frac{k_e A_e}{V_e}\left\{(n_{xe}^2 + n_{ye}^2)(T_E - T_P)A_e + \left[n_{xe}(An_x^e)_{ne} + n_{ye}(An_y^e)_{ne}\right](T_{ne} - T_{se})\right\}_e \tag{7}$$

Since $\mathbf{n}$ is a unit vector $n_{xe}^2 + n_{ye}^2 = 1$, it can be written

$$-\dot{Q}_e = \frac{k_e A_e}{V_e}\left\{(T_E - T_P)A_e + \left[n_{xe}(An_x^e)_{ne} + n_{ye}(An_y^e)_{ne}\right](T_{ne} - T_{se})\right\} \tag{8}$$

Having derived $\dot{Q}_e$, we derive $\dot{Q}_w, \dot{Q}_n$ and $\dot{Q}_s$ in the same way and insert the heat fluxes into Eq. 3 and include the source term (term II in Eq. 2), we arrive at the discretized equation

$$\begin{aligned}
a_P T_P &= a_E T_E + a_W T_W + a_N T_N + a_S T_S + b \\
a_E &= \frac{k_e A_e^2}{V_e} \\
a_W &= \frac{k_w A_w^2}{V_w} \\
a_N &= \frac{k_n A_n^2}{V_n} \\
a_S &= \frac{k_s A_s^2}{V_s} \\
b &= S V_P + \frac{k_e A_e}{V_e}\left[n_{xe}(An_x^e)_{ne} + n_{ye}(An_y^e)_{ne}\right](T_{ne} - T_{se}) \\
&+ \frac{k_n A_n}{V_n}\left[n_{xn}(An_x^n)_{nw} + n_{yn}(An_y^n)_{nw}\right](T_{nw} - T_{ne}) \\
&+ \frac{k_w A_w}{V_w}\left[n_{xw}(An_x^w)_{sw} + n_{yw}(An_y^w)_{sw}\right](T_{sw} - T_{nw}) \\
&+ \frac{k_s A_s}{V_s}\left[n_{xs}(An_x^s)_{se} + n_{ys}(An_y^s)_{se}\right](T_{se} - T_{sw})
\end{aligned}$$

2.1 The temperature at corners

The temperature is stored at nodes (P, E, ...), but we need also the temperature at corners (ne, se, ...). The easiest way is compute it as 1/4-th of its node-neighbours. The temperature T_{ne}, for example, is computed as

$$T_{ne} = \frac{1}{4}\left(T_E + T_{NE} + T_N + T_P\right)$$

2.2 Geometrical quantities

It is useful to first compute the unit vectors $\mathbf{s}$ along the control volume. For the east face, for example, we get

$$\begin{aligned} s_{ex} &= \frac{x_{ne} - x_{se}}{d_e} \\ s_{ey} &= \frac{y_{ne} - y_{se}}{d_e} \\ d_e &= \sqrt{(x_{ne} - x_{se})^2 + (y_{ne} - y_{se})^2} \end{aligned}$$

(note that the area of the east face A_e is equal to d_e since $\Delta z = 1$). The relation between the normal vector $\mathbf{n}$, $\mathbf{s}$ and the unit vector in the z-direction

$$\begin{aligned} \mathbf{s} \cdot \mathbf{n} &= 0 \\ \mathbf{s} \times \hat{z} &= \mathbf{n}, \end{aligned}$$

gives us the normal vector for the east face as

$$\begin{aligned} n_{ex} &= s_{ey} \\ n_{ey} &= -s_{ex}. \end{aligned}$$

When computing the $\mathbf{s}_{ne}$-vector, we can assume that it is parallel to the vector from node P to node E. Compute the coordinates of the nodes as 1/4-th of the surrounding grid points. For node P, for example, we obtain

$$\begin{aligned} x_P &= \frac{1}{4}\left(x_{ne} + x_{nw} + x_{sw} + x_{se}\right) \\ y_P &= \frac{1}{4}\left(y_{ne} + y_{nw} + y_{sw} + y_{se}\right) \end{aligned}$$

The volume V_e of the dashed control volume in Fig. 1 can be computed as

$$V_e = \frac{1}{2}\left(V_P + V_E\right)$$

where the volume of the control volume P, for example, is computed as the vector product

$$
\begin{aligned}
V_P &= \frac{1}{2}|\mathbf{a} \times \mathbf{b}| \\
a_x &= x_{ne} - x_{sw} \\
a_y &= y_{ne} - y_{sw} \\
b_x &= x_{nw} - x_{se} \\
b_y &= y_{nw} - y_{se}.
\end{aligned}
$$

This vector product has the form

$$
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix} = (0, 0, a_x b_y - a_y b_x)
$$