# MTF065 Turbulent Flow: An Example of a Report

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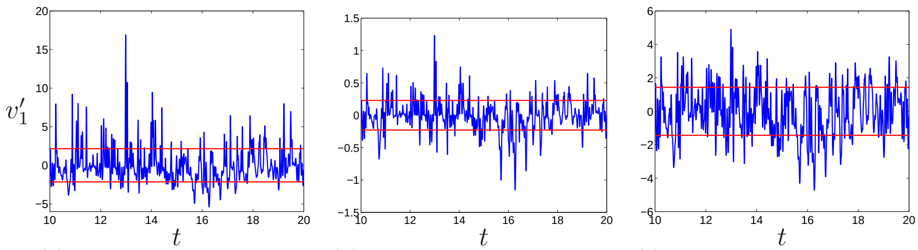



(a) $S = 1.51$ and $F = 7.68$ (b) $S = -0.27$ and $F = 5.81$ (c) $S = -0.09$ and $F = 2.77$

Figure 1: Time history of $v_1'$. Horisontal red lines show $\pm v_{1,rms}$.

# 1 Turbulent mean flow

There is no definition on turbulent flow, but it has a number of characteristic features (see Pope [1] and Tennekes & Lumley [2]) such as:

**I. Irregularity**. Turbulent flow is irregular, random and chaotic. The flow consists of a spectrum of different scales (eddy sizes). We do not have any exact definition of an *turbulent eddy*, but we suppose that it exists in a certain region in space for a certain time and that it is subsequently destroyed (by the cascade process or by dissipation, see below). It has a characteristic velocity and length (called a velocity and length scale). The region covered by a large eddy may well enclose also smaller eddies. The largest eddies are of the order of the flow geometry (i.e. boundary layer thickness, jet width, etc). At the other end of the spectra we have the smallest eddies which are dissipated by viscous forces (stresses) into thermal energy resulting in a temperature increase. Even though turbulence is chaotic it is deterministic and is described by the Navier-Stokes equations.

**turbulent eddy**

$S = 1.51$, $-0.27$ and $-0.09$. The flatness are $F = 7.68$, $5.81$ and $2.77$.

Consider the probability density functions of the fluctuations. The second moment corresponds to the variance of the fluctuations (or the square of the RMS, i.e.

$$\overline{v'^2} = \int_{-\infty}^{\infty} v'^2 f(v') dv'$$

$\overline{v'^2}$ is usually computed by integrating in time.

# 2 Turbulent mean flow

## 2.1 Time averaged Navier-Stokes

When the flow is turbulent it is preferable to decompose the instantaneous variables (for example the velocity components and the pressure) into a mean value and a fluctuating value, i.e.

$$\begin{aligned} v_i &= \bar{v}_i + v_i' \\ p &= \bar{p} + p' \end{aligned} \tag{1}$$

where the bar, $\bar{\cdot}$, denotes the time averaged value. One reason why we decompose the variables is that when we measure flow quantities we are usually interested in their mean values rather than their time histories. Another reason is that when we want to solve the Navier-Stokes equation numerically it would require a very fine grid to resolve all turbulent scales and it would also require a fine resolution in time (turbulent flow is always unsteady).

The continuity equation and the Navier-Stokes equation for incompressible flow with constant viscosity read

$$\begin{gathered}\frac{\partial v_i}{\partial x_i} = 0 \\ \rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}\end{gathered} \tag{2}$$

The gravitation term, $-\rho g_i$, has been omitted which means that the $p$ is the *hydrostatic* pressure. Inserting Eq. 1 at p. 2 into the continuity equation (2) and the Navier-Stokes equation we obtain the *time averaged* continuity equation and Navier-Stokes equation

$$\frac{\partial \bar{v}_i}{\partial x_i} = 0 \tag{3}$$

$$\rho\frac{\partial \bar{v}_i}{\partial t} + \rho\frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j} - \rho\overline{v_i' v_j'}\right) \tag{4}$$

This equation is the time-averaged Navier-Stokes equation and it is often called the *Reynolds equation*. A new term $\rho\overline{v_i' v_j'}$ appears on the right side of Eq. 4 which is called the *Reynolds stress tensor*. The tensor is symmetric (for example $\overline{v_1' v_2'} = \overline{v_2' v_1'}$). It represents correlations between fluctuating velocities. It is an additional stress term due to turbulence (fluctuating velocities) and it is unknown. We need a model for $\overline{v_i' v_j'}$ to close the equation system in Eq. 4. This is called the *closure problem*: the number of unknowns (ten: three velocity components, pressure, six stresses) is larger than the number of equations (four: the continuity equation and three components of the Navier-Stokes equations).

**Reynolds equations**

**closure problem**

### 2.1.1 Boundary-layer approximation

For steady ($\partial/\partial t = 0$), two-dimensional ($\bar{v}_3 = \partial/\partial x_3 = 0$) boundary-layer type of flow (i.e. boundary layers along a flat plate, channel flow, pipe flow, jet and wake flow, etc.) where

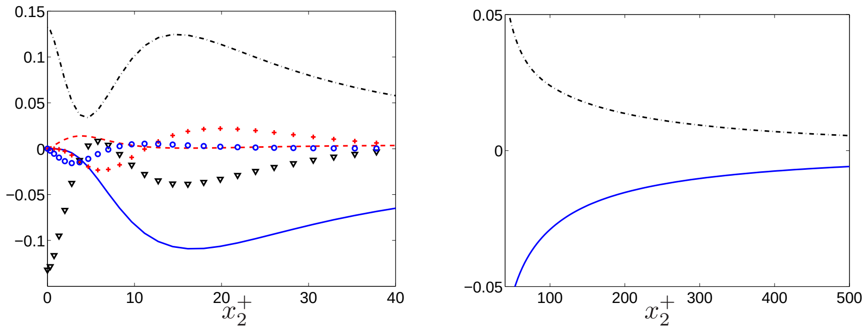
$$\bar{v}_2 \ll \bar{v}_1, \ \frac{\partial}{\partial x_1} \ll \frac{\partial}{\partial x_2}, \tag{5}$$

Eq. 4 reads

$$\rho\frac{\partial \bar{v}_1 \bar{v}_1}{\partial x_1} + \rho\frac{\partial \bar{v}_2 \bar{v}_1}{\partial x_2} = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\underbrace{\left[\mu\frac{\partial \bar{v}_1}{\partial x_2} - \rho\overline{v_1' v_2'}\right]}_{\tau_{tot}} \tag{6}$$

$x_1$ and $x_2$ denote the streamwise and wall-normal coordinate, respectively.

If you want to learn more how to derive transport equations of turbulent quantities, see [5] which can be downloaded here
`http://www.tfd.chalmers.se/~lada/allpapers.html`




(a) Zoom near the wall (b) Outer region

Figure 2: Channel flow at $Re_\tau = 2000$. Terms in the $\overline{v_1' v_2'}$ equation scaled by $u_\tau^4/\nu$. DNS data [3, 4]. —— : $P_{12}$; - - - : $-\varepsilon_{12}$; $\nabla$ : $-\partial \overline{v'p'}/\partial x_2$; - · - : $\Pi_{12}$; $+$: $-\partial(\overline{v_1' v_2'^2})/\partial x_2$; $\circ$: $\nu \partial^2 \overline{v_1' v_2'}/\partial x_2^2$.

| $i$ | $n$ | $j$ | $k$ | $\varepsilon_{inm}\varepsilon_{mjk}$ | $\delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj}$ |
|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | $\varepsilon_{12m}\varepsilon_{m12} = \varepsilon_{123}\varepsilon_{312} = 1 \cdot 1 = 1$ | $1 - 0 = 1$ |
| 2 | 1 | 1 | 2 | $\varepsilon_{21m}\varepsilon_{m12} = \varepsilon_{213}\varepsilon_{312} = -1 \cdot 1 = -1$ | $0 - 1 = -1$ |
| 1 | 2 | 2 | 1 | $\varepsilon_{12m}\varepsilon_{m21} = \varepsilon_{123}\varepsilon_{321} = 1 \cdot -1 = -1$ | $0 - 1 = -1$ |
| | | | | | |
| 1 | 3 | 1 | 3 | $\varepsilon_{13m}\varepsilon_{m13} = \varepsilon_{132}\varepsilon_{213} = -1 \cdot -1 = 1$ | $1 - 0 = 1$ |
| 3 | 1 | 1 | 3 | $\varepsilon_{31m}\varepsilon_{m13} = \varepsilon_{312}\varepsilon_{213} = 1 \cdot -1 = -1$ | $0 - 1 = -1$ |
| 1 | 3 | 3 | 1 | $\varepsilon_{13m}\varepsilon_{m31} = \varepsilon_{132}\varepsilon_{231} = -1 \cdot 1 = -1$ | $0 - 1 = -1$ |
| | | | | | |
| 2 | 3 | 2 | 3 | $\varepsilon_{23m}\varepsilon_{m23} = \varepsilon_{231}\varepsilon_{123} = 1 \cdot 1 = 1$ | $1 - 0 = 1$ |
| 3 | 2 | 2 | 3 | $\varepsilon_{32m}\varepsilon_{m23} = \varepsilon_{321}\varepsilon_{123} = -1 \cdot 1 = -1$ | $0 - 1 = -1$ |
| 2 | 3 | 3 | 2 | $\varepsilon_{23m}\varepsilon_{m32} = \varepsilon_{231}\varepsilon_{132} = 1 \cdot -1 = -1$ | $0 - 1 = -1$ |

Table 1: The components of the $\varepsilon - \delta$ identity which are non-zero.

# A $\varepsilon - \delta$ identity

The $\varepsilon - \delta$ identity reads

$$\varepsilon_{inm}\varepsilon_{mjk} = \varepsilon_{min}\varepsilon_{mjk} = \varepsilon_{nmi}\varepsilon_{mjk} = \delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj}$$

In Table 1 the components of the $\varepsilon - \delta$ identity are given.

# References

[1] S.B. Pope. *Turbulent Flow*. Cambridge University Press, Cambridge, UK, 2001.

[2] H. Tennekes and J.L. Lumley. *A First Course in Turbulence*. The MIT Press, Cambridge, Massachusetts, 1972.

[3] S. Hoyas and J. Jiménez. Scaling of the velocity fluctuations in turbulent channels up to $Re_\tau = 2003$. *Physics of Fluids A*, 18(011702), 2006.

[4] S. Hoyas and J. Jiménez. http://torroja.dmt.upm.es/ftp/channels/data/statistics/. 2006.

[5] L. Davidson. Transport equations in incompressible URANS and LES. Report 2006/01, Div. of Fluid Dynamics, Dept. of Applied Mechanics, Chalmers University of Technology, Göteborg, Sweden, 2006.