

Fluid Mechanics - MTF053

Lecture Notes

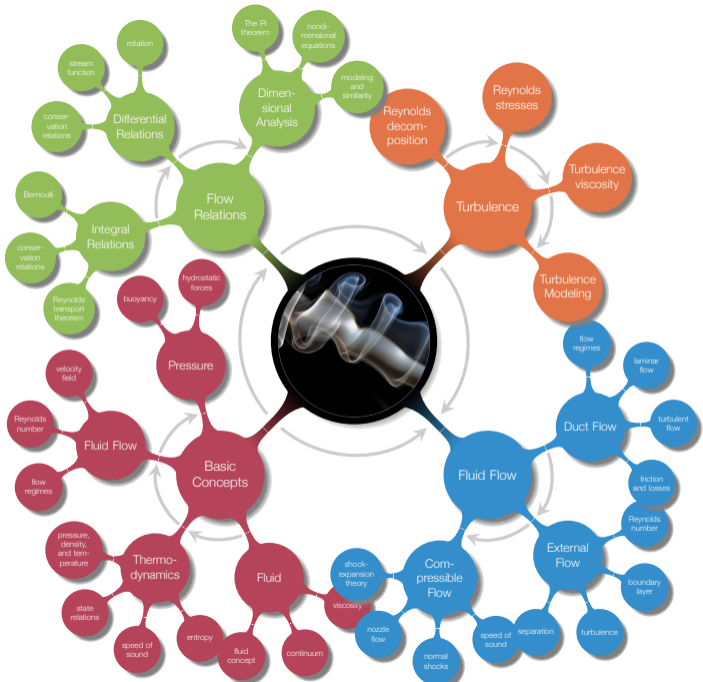
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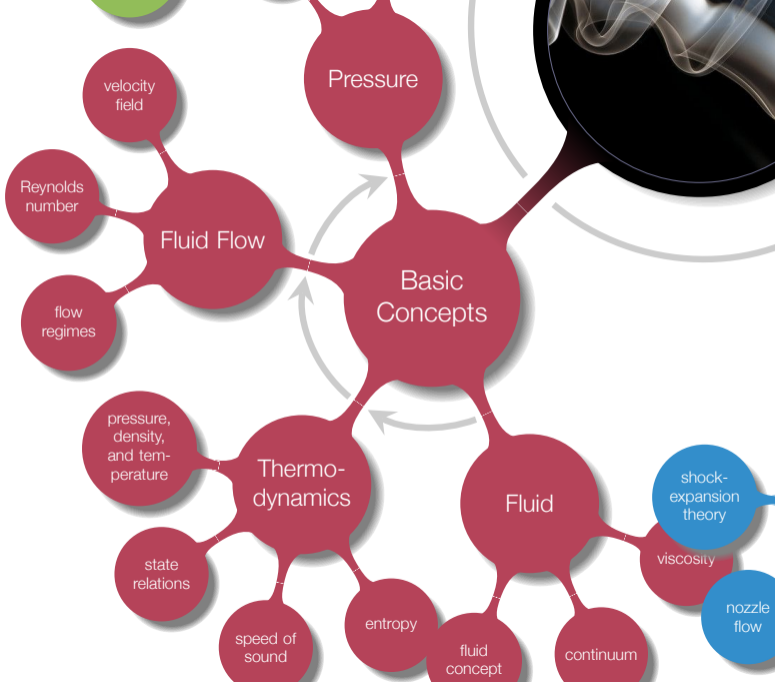
Overview





Chapter 1 - Introduction

Overview

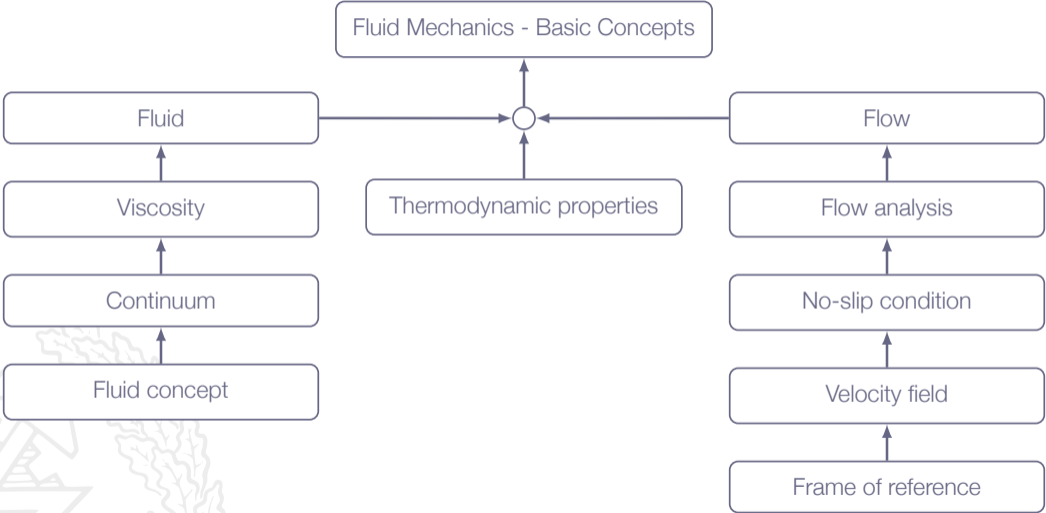


Learning Outcomes

- 1 **Explain** the difference between a fluid and a solid in terms of forces and deformation
- 2 **Understand** and be able to explain the viscosity concept
- 3 **Define** the Reynolds number
- 5 **Explain** the difference between Lagrangian and Eulerian frame of reference and know when to use which approach
- 7 **Explain** the concepts: streamline, pathline and streakline
- 8 **Understand** and be able to **explain** the concept shear stress
- 9 **Explain** how to do a force balance for fluid element (forces and pressure gradients)
- 10 **Understand and explain** buoyancy and cavitation
- 16 **Understand** and **explain** the concept Newtonian fluid

in this lecture we will find out what a fluid flow is

Roadmap - Introduction to Fluid Mechanics



Fluid Mechanics

*"Fluid mechanics is the branch of physics concerned with the **mechanics of fluids** (liquids, gases, and plasmas) and the **forces** on them. It has applications in a wide range of disciplines, including mechanical, civil, chemical and biomedical engineering, geophysics, oceanography, meteorology, astrophysics, and biology."*

Wikipedia



Fluids Flows in Your Daily Life



Fluids Flows in Your Daily Life



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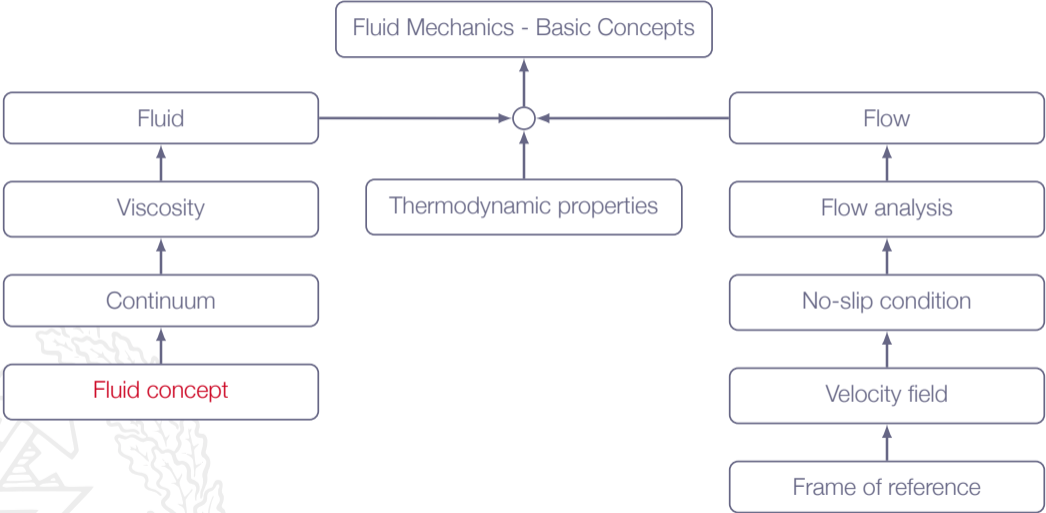


Fluid Flow Applications

- ▶ Analytical solutions limited to very specific simplified cases
- ▶ Complex geometries and flows leads to the need for experiments and Computational Fluid Dynamics (CFD)
- ▶ Understanding the basic principles is a key factor for a correct analysis



Roadmap - Introduction to Fluid Mechanics



The Concept of a Fluid

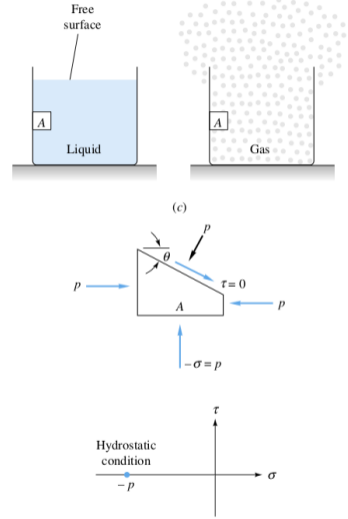
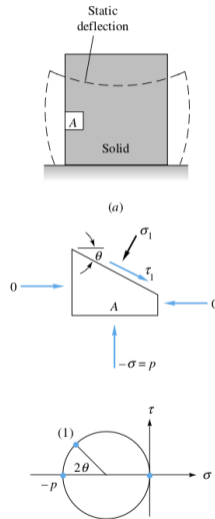
"In physics, a fluid is a substance that continually deforms (flows) under an applied shear stress, or external force. Fluids are a phase of matter and include liquids, gases and plasmas. They are substances with zero shear modulus, or, in simpler terms, substances which cannot resist any shear force applied to them."

Wikipedia

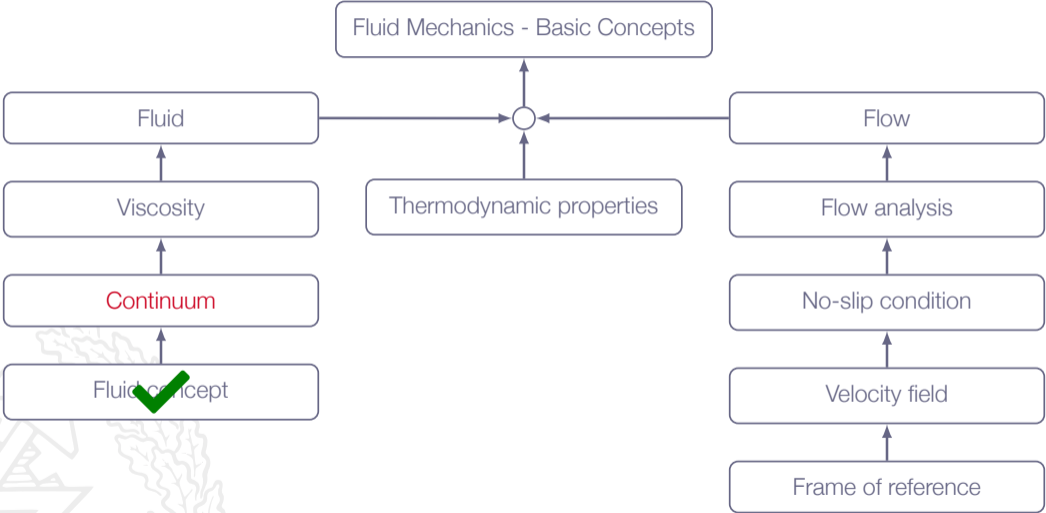


The Concept of a Fluid

"A solid can resist a shear stress by a static deflection; a fluid cannot"



Roadmap - Introduction to Fluid Mechanics

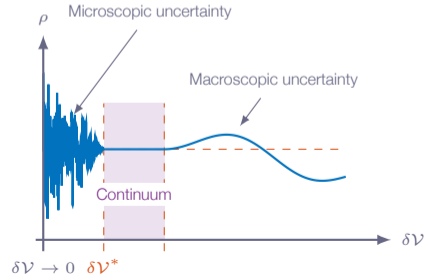


The Fluid as a Continuum

- ▶ Fluid **density** is essentially a **point function**
- ▶ fluid **properties** can be thought of as **varying continually in space**
- ▶ Volume large enough such that the **number of molecules** within the volume is **constant**
- ▶ Volume small enough **not** to **introduce macroscopic fluctuations**

$$\rho = \lim_{\delta V \rightarrow \delta V^*} \frac{\delta m}{\delta V}$$

standard air: $\delta V^* \approx 10^{-9} \text{mm}^3 \Rightarrow \sim 3 \times 10^7$ molecules

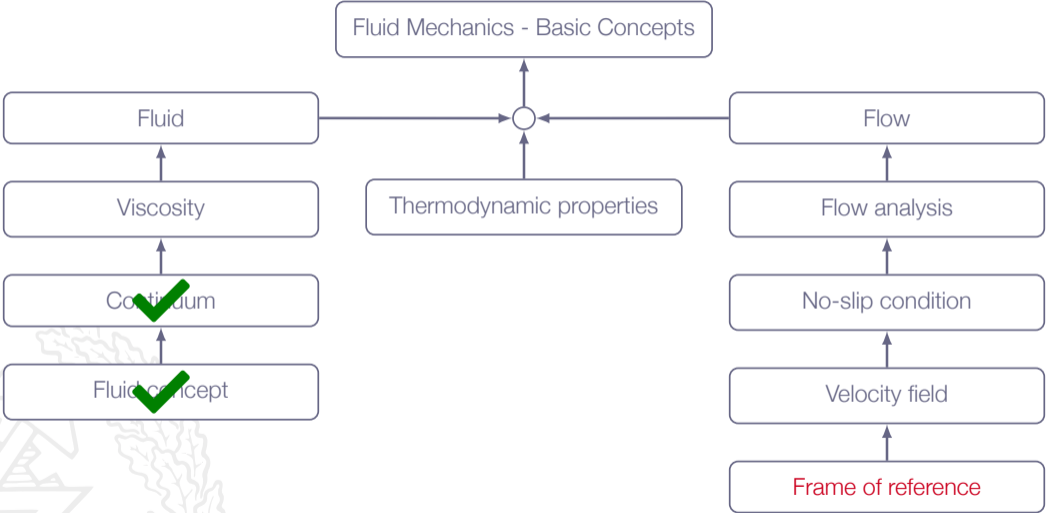


The Fluid as a Continuum

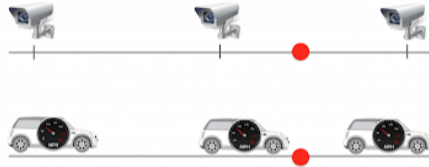
- ▶ Flow properties varies smoothly
- ▶ Differential calculus can be used



Roadmap - Introduction to Fluid Mechanics



Frame of Reference



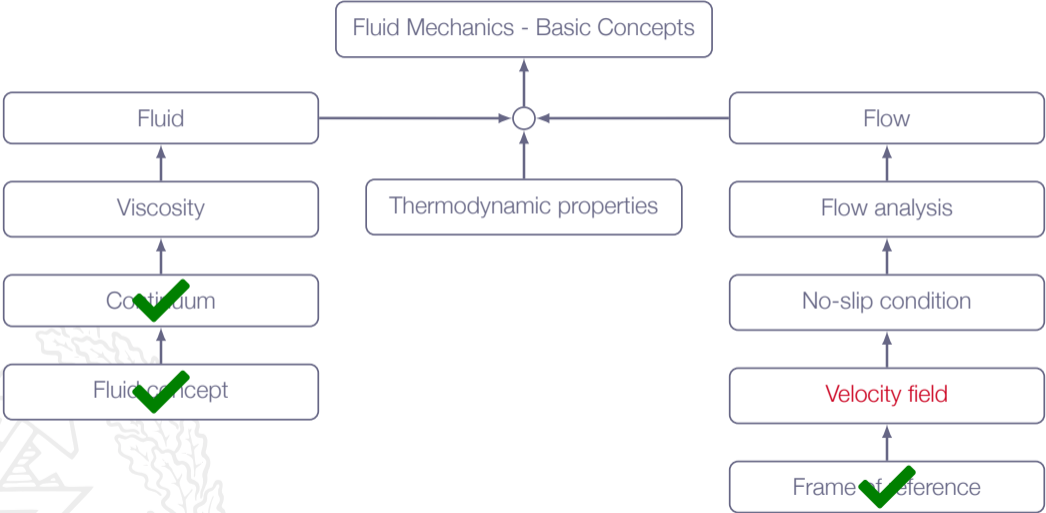
- ▶ Eulerian

- ▶ fluid properties as function of position and time
- ▶ most often used in fluid mechanics

- ▶ Lagrangian

- ▶ follows a system in time and space
- ▶ can be used in fluid mechanics
- ▶ most often used in solid mechanics

Roadmap - Introduction to Fluid Mechanics



Properties of the Velocity Field

- ▶ The fluid velocity is a function of position and time
- ▶ Three components u , v , and w (one in each spatial direction)

$$\mathbf{V}(x, y, z, t) = u(x, y, z, t)\mathbf{e}_x + v(x, y, z, t)\mathbf{e}_y + w(x, y, z, t)\mathbf{e}_z$$



Properties of the Velocity Field

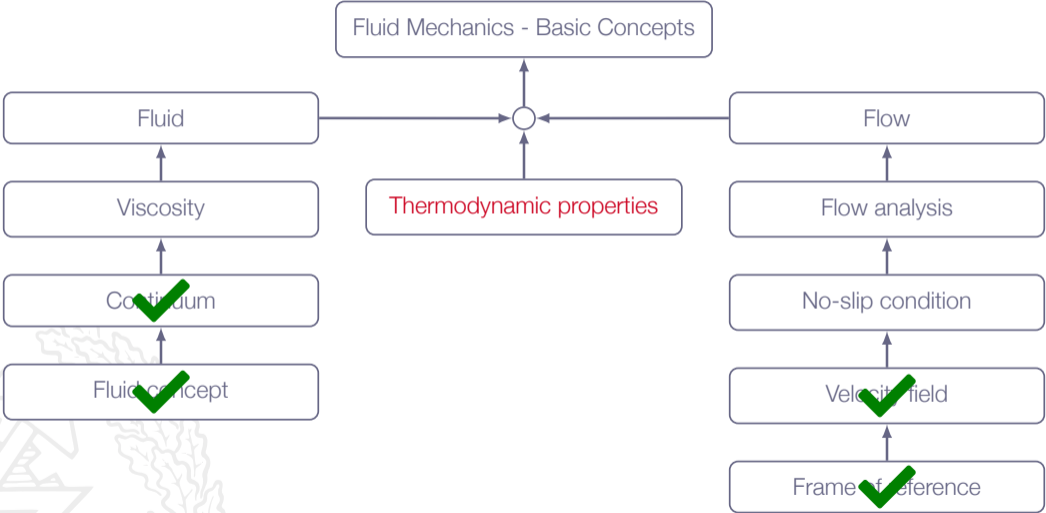
Acceleration:

$$\mathbf{V}(x, y, z, t) = u(x, y, z, t)\mathbf{e}_x + v(x, y, z, t)\mathbf{e}_y + w(x, y, z, t)\mathbf{e}_z$$

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + \left(\frac{\partial \mathbf{V}}{\partial x}\right) \left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial \mathbf{V}}{\partial y}\right) \left(\frac{\partial y}{\partial t}\right) + \left(\frac{\partial \mathbf{V}}{\partial z}\right) \left(\frac{\partial z}{\partial t}\right)$$

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z}$$

Roadmap - Introduction to Fluid Mechanics



Thermodynamic Properties

- ▶ Thermodynamic properties describe the state of a system, i.e., a collection of matter of fixed identity which interacts with its surroundings
- ▶ In this course, the system will be a small fluid element, and all properties will be assumed to be continuum properties of the flow field



Thermodynamic Properties

- ▶ Pressure p Pa
- ▶ Density ρ kg/m³
- ▶ Temperature T K

most common properties



Thermodynamic Properties

- ▶ Pressure p Pa
- ▶ Density ρ kg/m³
- ▶ Temperature T K

- ▶ Internal energy \hat{u}
- ▶ Enthalpy $h = \hat{u} + p/\rho$
- ▶ Entropy s
- ▶ Specific heats C_p and C_v

most common properties

work, heat, and energy balances



Thermodynamic Properties

- ▶ Pressure p Pa
- ▶ Density ρ kg/m³
- ▶ Temperature T K

- ▶ Internal energy \hat{u}
- ▶ Enthalpy $h = \hat{u} + p/\rho$
- ▶ Entropy s
- ▶ Specific heats C_p and C_v

- ▶ Viscosity μ
- ▶ Thermal conductivity k

most common properties

work, heat, and energy balances

friction and heat conduction

Thermodynamic Properties

- ▶ For a single-phase substance, two basic properties are sufficient to get the values of all others

$$\rho = \rho(p, T), h = h(p, T), \mu = \mu(p, T)$$

- ▶ In the following it will be assumed that all thermodynamic properties exists as point functions in a flowing fluid
 - ▶ large enough number of molecules
 - ▶ any changes are slower than the fluid time scale \Rightarrow equilibrium

Thermodynamic Properties

Pressure: $p[\text{Pa}]$

- ▶ the compression stress at a point in a static fluid
- ▶ a fluid flow is often driven by pressure gradients
- ▶ if the pressure drops below the vapor pressure in a liquid, vapor bubbles will form

Temperature: $T[\text{K}]$

- ▶ related to internal energy
- ▶ large temperature differences \Rightarrow heat transfer may be important

Density: $\rho[\text{kg}/\text{m}^3]$

- ▶ mass per unit volume
- ▶ nearly constant in liquids (incompressible) - for water, the density increases about one percent for a pressure increase by a factor of 220
- ▶ not constant for gases

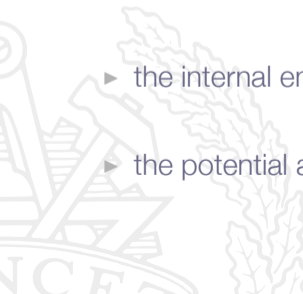
$$\rho = \frac{p}{RT}$$

Potential and Kinetic Energies

The total stored energy per unit mass:

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

- ▶ the internal energy is a function of temperature
- ▶ the potential and kinetic energies are kinematic quantities



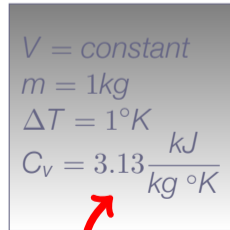
State Relations for Gases

The perfect gas law:

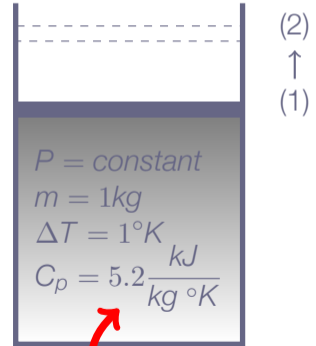
$$p = \rho RT$$

where R is the gas constant

$$R = C_p - C_v$$



3.13kJ



5.2kJ

State Relations for Gases

The ideal gas law requires: $\hat{u} = \hat{u}(T)$ and thus

specific heat (constant volume):

$$C_v = \left(\frac{\partial \hat{u}}{\partial T} \right)_\rho = \frac{d\hat{u}}{dT} = C_v(T)$$



State Relations for Gases

specific heat (constant pressure):

$$h = \hat{u} + \frac{p}{\rho} = \hat{u} + RT = h(T)$$

$$C_p = \left(\frac{\partial h}{\partial T} \right)_p = \frac{dh}{dT} = C_p(T)$$

ratio of specific heats:

$$\gamma = \frac{C_p}{C_v} \geq 1$$



State Relations for Gases

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$



Speed of Sound

Speed of sound plays an important role when compressible effects are important (Chapter 9)

$$a^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$$

$$\tau_s = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_s \Rightarrow a = \sqrt{\frac{1}{\rho \tau_s}}$$

where τ_s is the fluid compressibility

for an ideal gas:

$$a = \sqrt{\gamma RT}$$

Vapor Pressure

"the pressure at which a liquid boils and is in equilibrium with its own vapor"

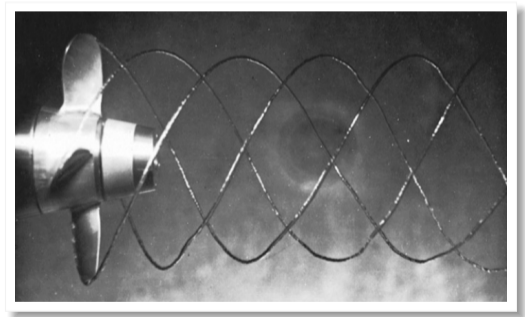
Vapor pressure for water:

$T[^\circ\text{C}]$	vapor pressure [Pa]
20	2340
100	101300

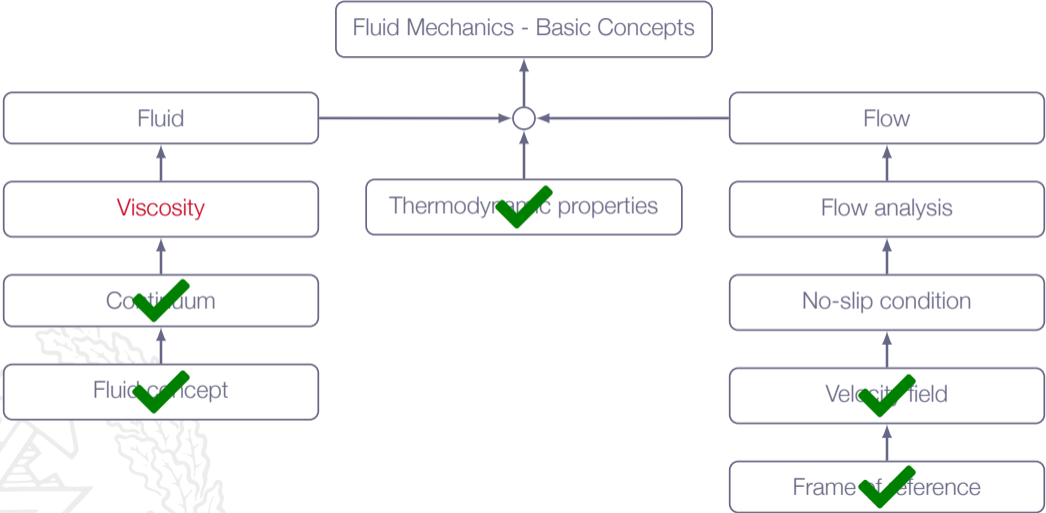


Vapor Pressure

- ▶ If the pressure in a liquid gets lower than the vapor pressure, vapor bubbles will appear in the liquid
- ▶ If the pressure drops below the vapor pressure due to the flow itself we get cavitation



Roadmap - Introduction to Fluid Mechanics





Viscosity

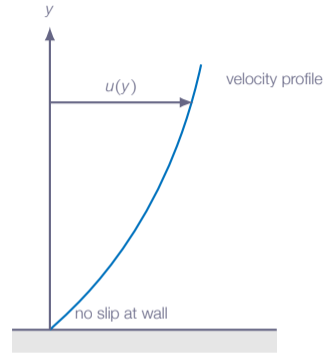
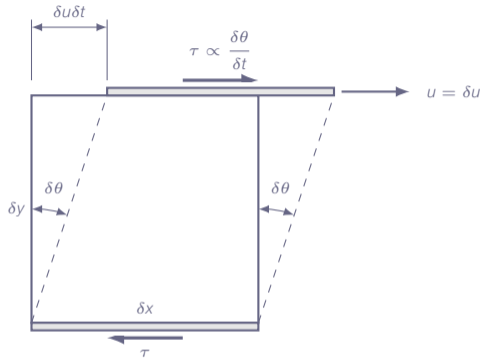
Viscosity

"relates the local stresses in a moving fluid to the strain rate of the fluid element"

"a quantitative measure of the fluid's resistance to flow"



Viscosity



$$\tau \propto \frac{\delta \theta}{\delta t}, \quad \tan \delta \theta = \frac{\delta u \delta t}{\delta y}$$

Viscosity

for infinitesimal changes:

$$\frac{d\theta}{dt} = \frac{du}{dy}$$

from before we know that $\tau \propto \frac{\delta\theta}{\delta t}$ and thus $\tau \propto \frac{d\theta}{dt}$

For newtonian fluids:

$$\tau = \mu \frac{d\theta}{dt} = \mu \frac{du}{dy}$$

where μ is the fluid viscosity

Viscosity

- ▶ Liquids have high viscosity that decreases with temperature
 - ▶ intermolecular forces decreases with temperature
- ▶ Gases have low viscosity that increases with temperature
 - ▶ increased temperature means increased molecular movement



Viscosity

Fluid	μ , kg/(m · s) [†]	Ratio $\mu/\mu(\text{H}_2)$	ρ , kg/m ³	ν m ² /s [†]	Ratio $\nu/\nu(\text{Hg})$
Hydrogen	8.8 E-6	1.0	0.084	1.05 E-4	920
Air	1.8 E-5	2.1	1.20	1.51 E-5	130
Gasoline	2.9 E-4	33	680	4.22 E-7	3.7
Water	1.0 E-3	114	998	1.01 E-6	8.7
Ethyl alcohol	1.2 E-3	135	789	1.52 E-6	13
Mercury	1.5 E-3	170	13,580	1.16 E-7	1.0
SAE 30 oil	0.29	33,000	891	3.25 E-4	2,850
Glycerin	1.5	170,000	1,264	1.18 E-3	10,300

Note! there are two different viscosities in the table (dynamic viscosity μ and kinematic viscosity $\nu = \mu/\rho$)

Viscosity

Inviscid flows: flows where viscous forces are negligible

Viscous flows: flows where viscous forces are important



Reynolds number

$$Re = \frac{\rho V L}{\mu}$$

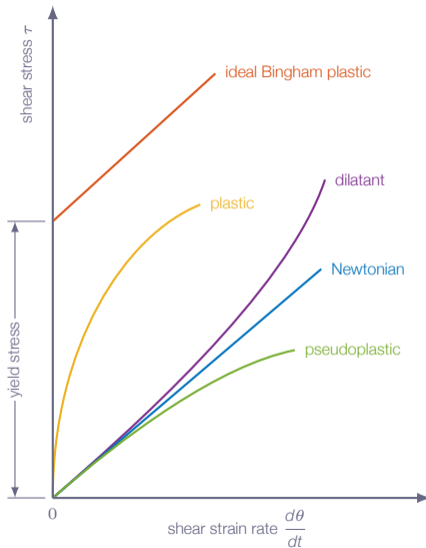
- ▶ Dimensional number that relates viscous forces to inertial forces
- ▶ Very important parameter in fluid mechanics
- ▶ V and L are characteristic velocity and length scales of the flow

Reynolds number

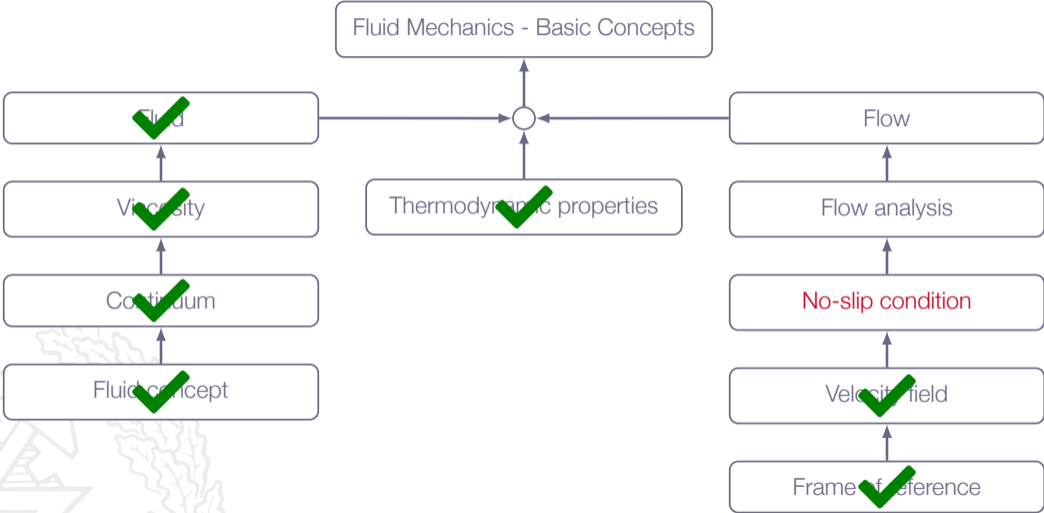
Reynolds number	flow description
low	viscous, creeping motion (inertial forces negligible)
moderate	laminar flow
high	turbulent flow



Nonnewtonian Fluids



Roadmap - Introduction to Fluid Mechanics



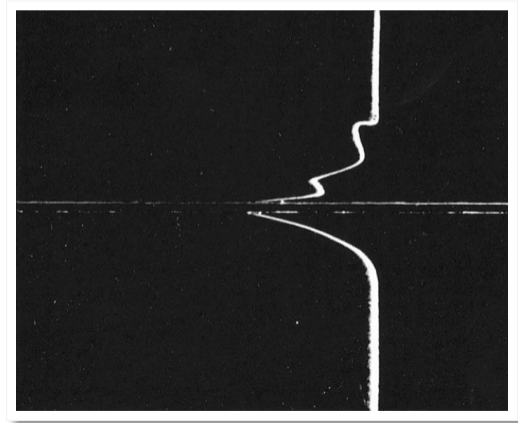
No Slip/No Temperature Jump

"When a fluid flow is bounded by a solid surface, molecular interactions cause the fluid in contact with the surface to seek momentum and energy equilibrium with that surface"

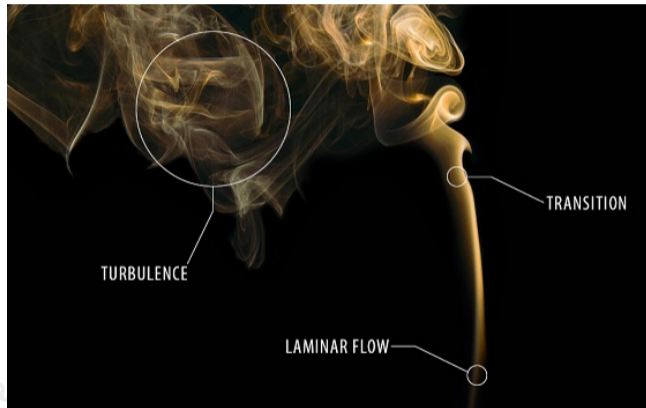


No Slip/No Temperature Jump

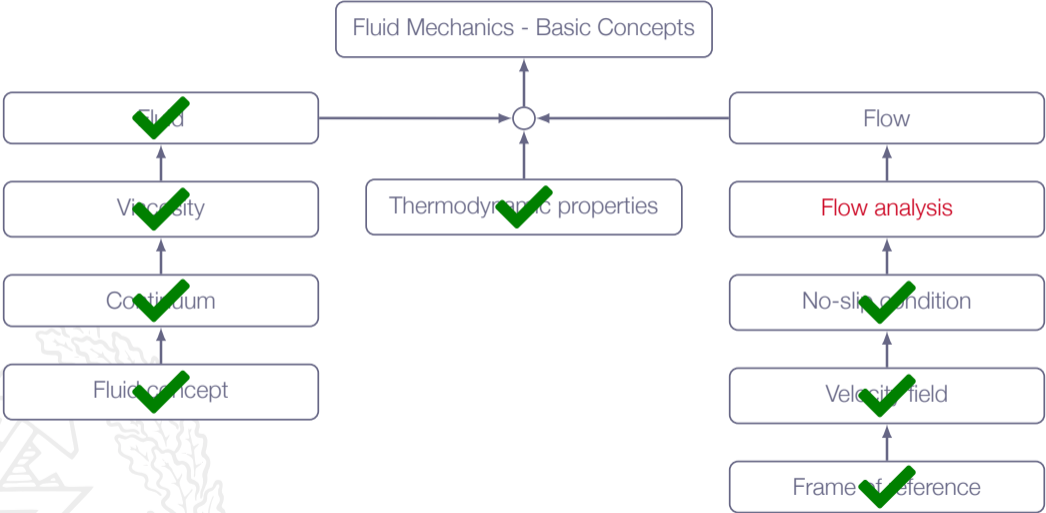
At a solid wall, the fluid will have the velocity and temperature of the wall



Laminar/Turbulent Flow



Roadmap - Introduction to Fluid Mechanics

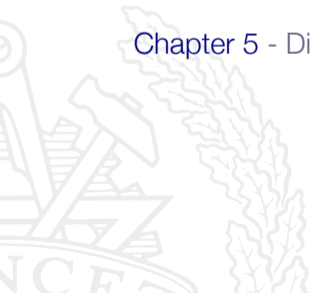


Flow Analysis

Chapter 3 - Control-volume (integral) approach

Chapter 4 - Infinitesimal system (differential) approach

Chapter 5 - Dimensional analysis approach



Flow Analysis

- ▶ Conservation of mass (continuity)
- ▶ Conservation of momentum (Newton's second law)
- ▶ Conservation of energy (first law of thermodynamics)
- ▶ State relation (for example the ideal gas law)
- ▶ Second law of thermodynamics
- ▶ Boundary conditions



Flow Visualization

Streamline

- ▶ a line that is tangent to the velocity vector everywhere at an instant in time

Pathline

- ▶ the actual path traversed by a fluid particle

Streakline

- ▶ the locus of particles that have earlier passed through a prescribed point

Timeline

- ▶ a line formed by a set of particles at a given instant



Flow Visualization

Streamline

- ▶ a line that is tangent to the velocity vector everywhere at an instant in time

Pathline

- ▶ the actual path traversed by a fluid particle

Streakline

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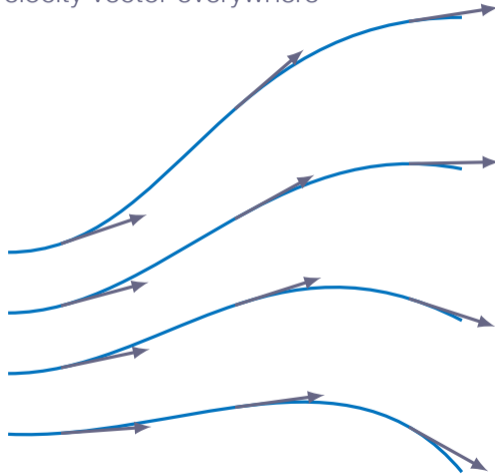
Timeline

- ▶ a line formed by a set of particles at a given instant

Note! In a steady-state flow, streamlines, pathlines and streaklines are identical

Streamline

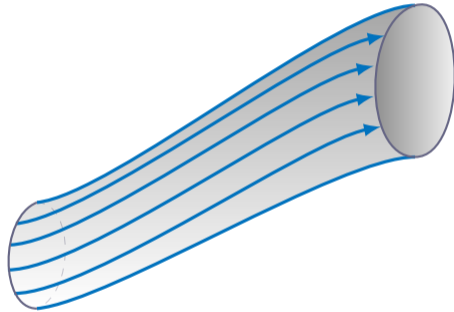
Tangent to flow velocity vector everywhere



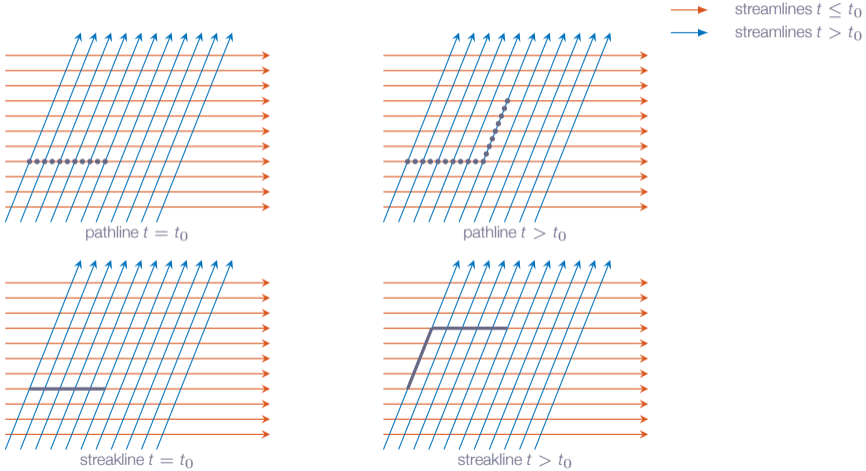
Streamtube

"Constructed" from individual streamlines

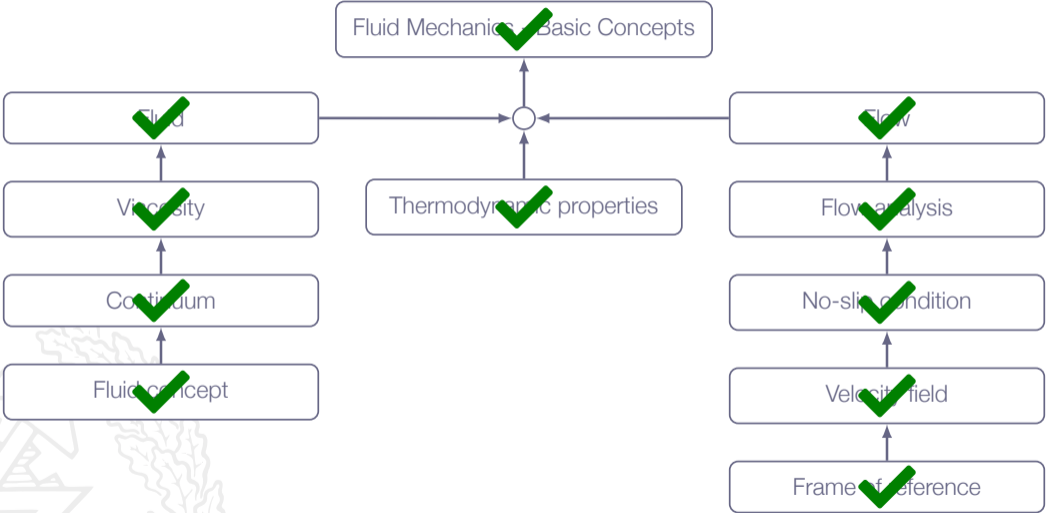
No flow across streamtube walls (by definition)



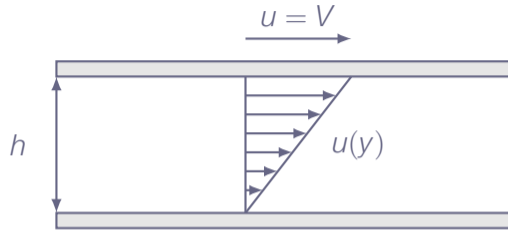
Pathline vs Streakline



Roadmap - Introduction to Fluid Mechanics

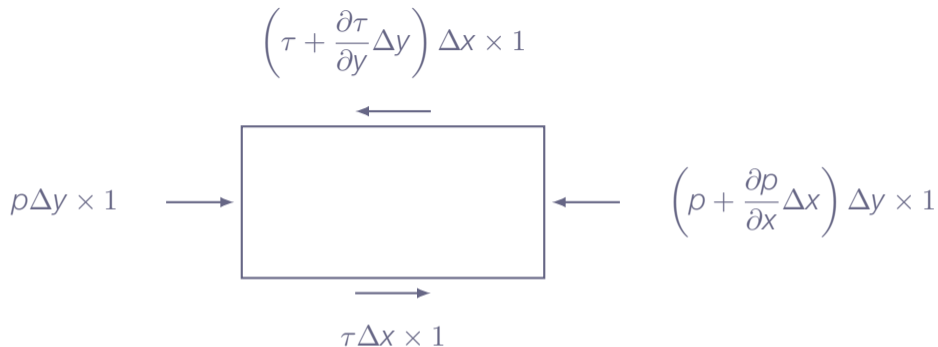


Example - Flow Between Plates



- ▶ No acceleration
- ▶ No pressure gradients
- ▶ two-dimensional flow

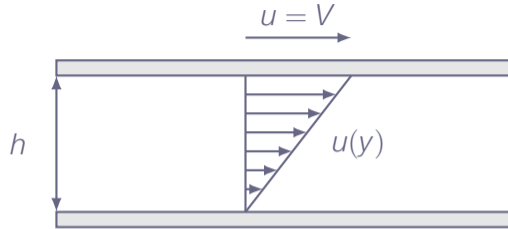
Example - Flow Between Plates



$$\sum F_x = p \Delta y - \left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y + \tau \Delta x - \left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right) \Delta x = 0$$

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} = 0 \Rightarrow \tau = \text{const}$$

Example - Flow Between Plates



$$\frac{du}{dy} = \frac{\tau}{\mu} = \text{const}$$

$$u = a + by$$

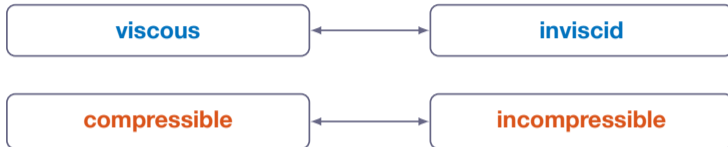
$$\begin{cases} y = 0 \Rightarrow u = 0 \\ y = h \Rightarrow u = V \end{cases}$$

$$u = \frac{V}{h}y$$

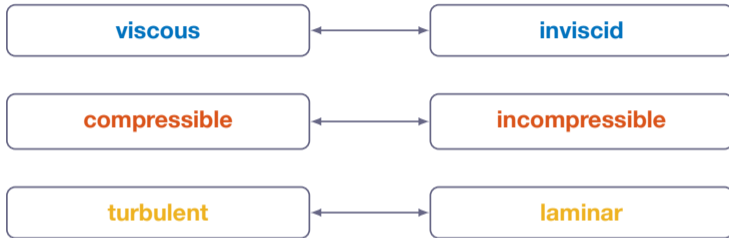
Flow Categories



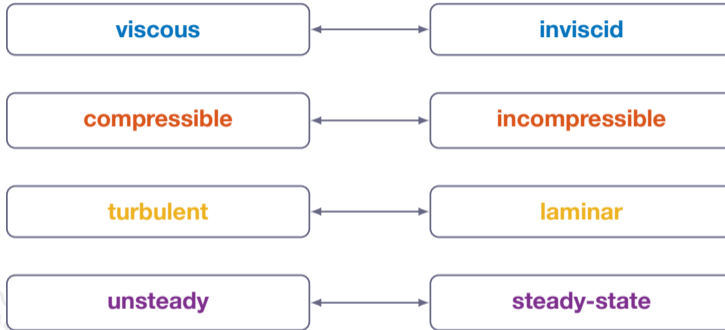
Flow Categories



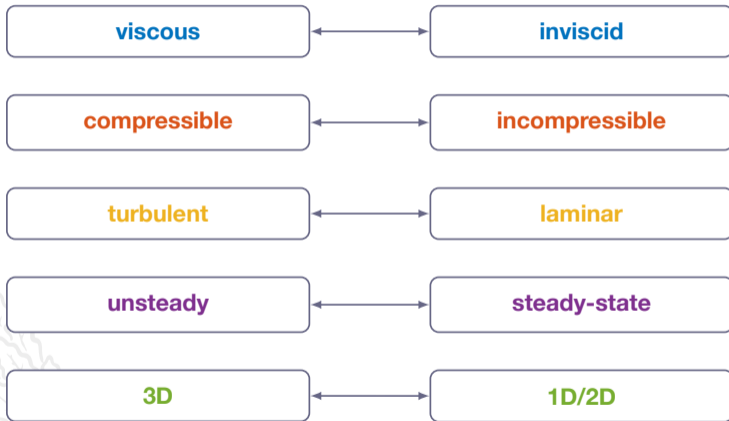
Flow Categories

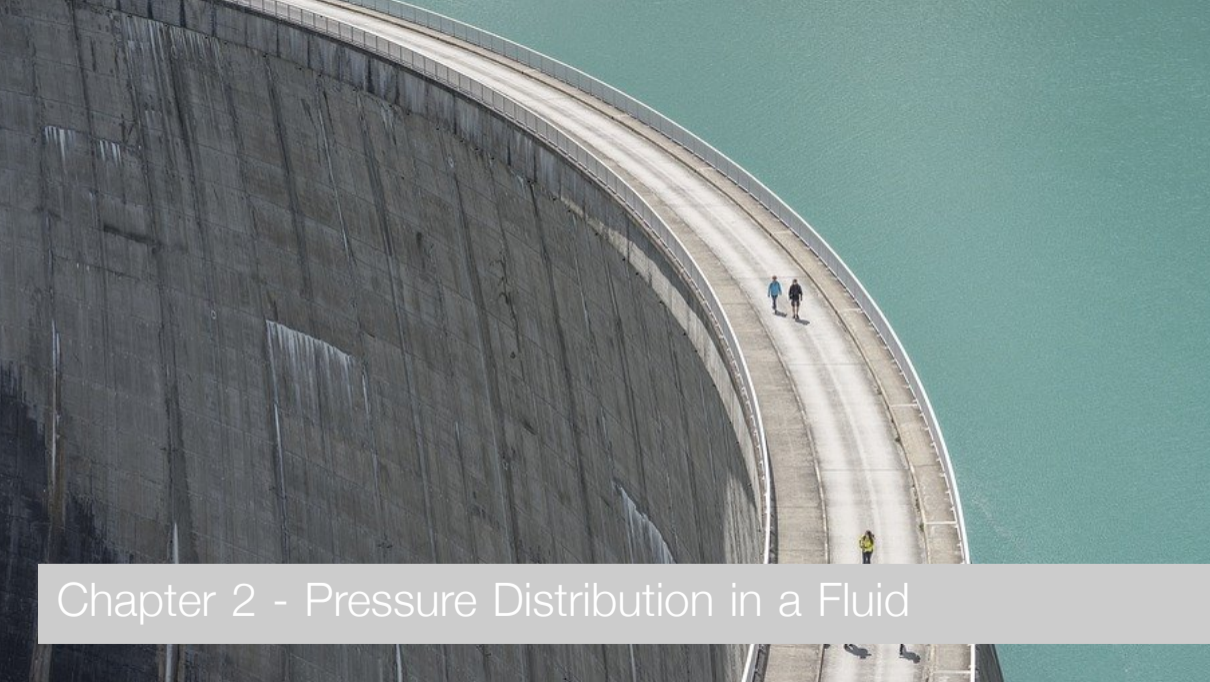


Flow Categories



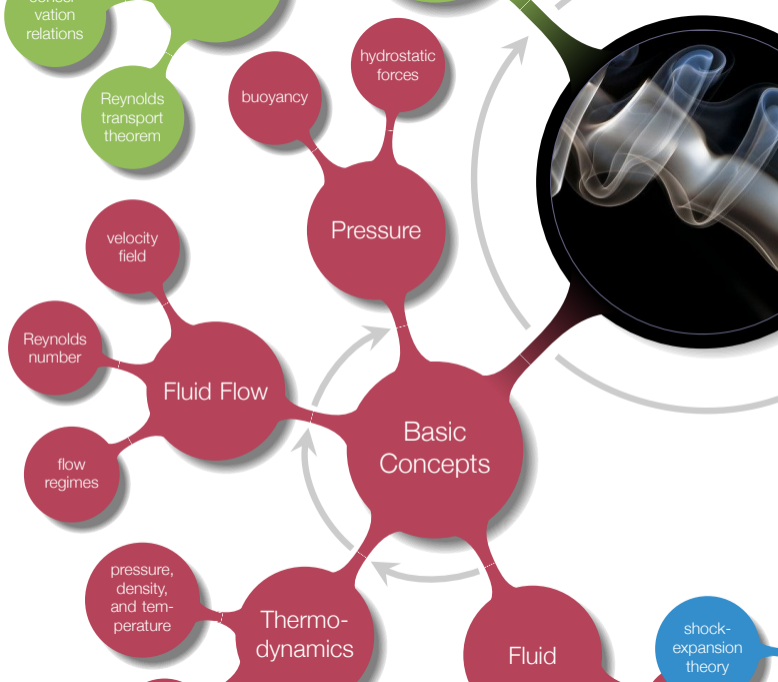
Flow Categories





Chapter 2 - Pressure Distribution in a Fluid

Overview



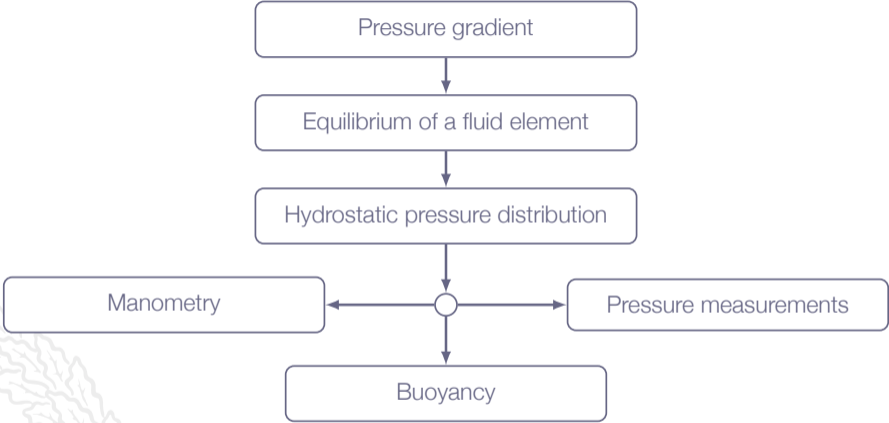
Learning Outcomes

- 9 **Explain** how to do a force balance for fluid element (forces and pressure gradients)
- 10 **Understand and explain** buoyancy and cavitation
- 11 **Solve** problems involving hydrostatic pressure and buoyancy

we will have a look at the pressure distribution in a fluid at rest, i.e. no flow yet...

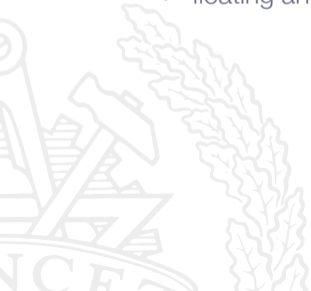


Roadmap - Pressure Distribution in a Fluid



Motivation

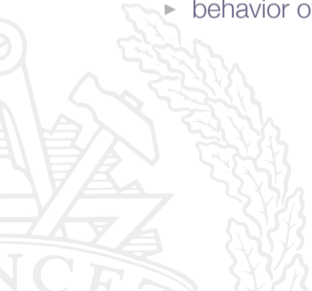
- ▶ Many problems does not include fluid motion
 - ▶ pressure distribution in a static fluid
 - ▶ pressure on solid surfaces due to presence of static fluid
 - ▶ floating and submerged bodies



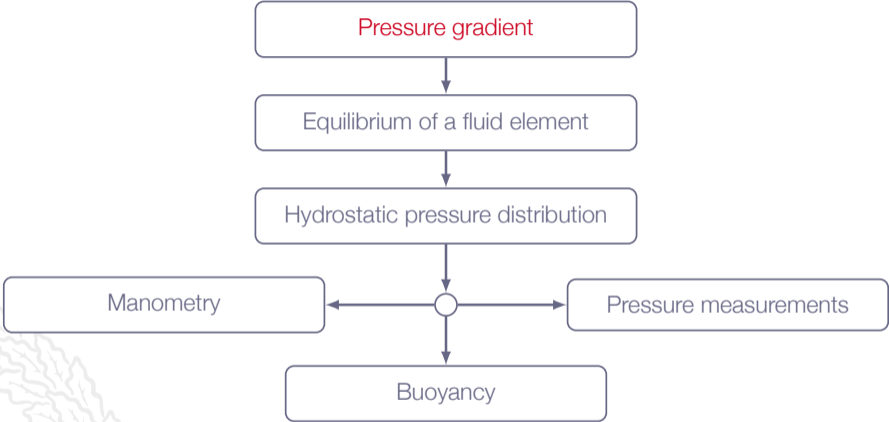
Motivation

Examples:

- ▶ pressure distribution in the atmosphere and in oceans
- ▶ design of pressure measurement devices
- ▶ buoyancy on a submerged body
- ▶ behavior of floating bodies

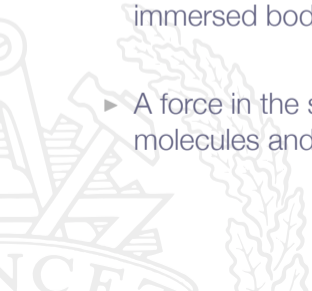


Roadmap - Pressure Distribution in a Fluid



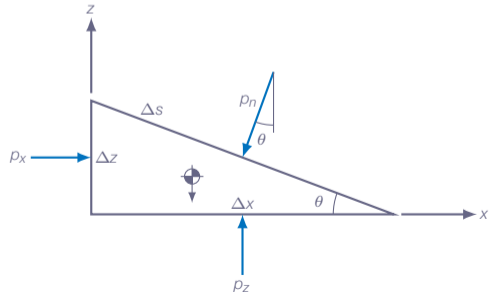
Pressure

- ▶ Pressure is a thermodynamic property
- ▶ Pressure is not a force and has no direction
- ▶ Forces arise when the molecules of the fluid interacts with the surface of an immersed body
 - ▶ A force in the surface-normal direction is generated due to the collision of fluid molecules and the surface



Pressure Variation in a Fluid at Rest

- ▶ Fluid at rest - no shear (by definition)
- ▶ Pressures p_x , p_z , and p_n may be different
- ▶ Small element \Rightarrow constant pressure on each face

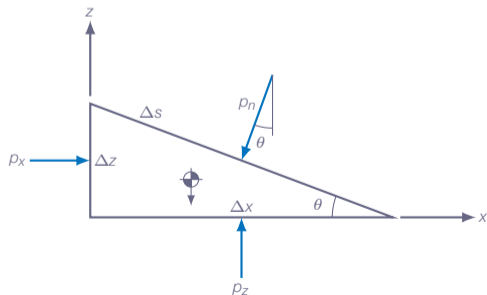


Pressure Variation in a Fluid at Rest

$$\sum F_x = 0 = p_x b \Delta z - p_n b \Delta s \sin \theta$$

$$\sum F_z = 0 = p_z b \Delta x - p_n b \Delta s \cos \theta - \frac{1}{2} \rho g b \Delta x \Delta z$$

$$\begin{cases} \Delta z = \Delta s \sin \theta \\ \Delta x = \Delta s \cos \theta \end{cases}$$

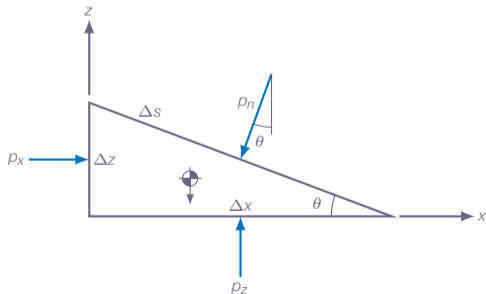


Pressure Variation in a Fluid at Rest

$$\sum F_x = 0 = p_x b \Delta z - p_n b \Delta z$$

$$\sum F_z = 0 = p_z b \Delta x - p_n b \Delta x - \frac{1}{2} \rho g b \Delta x \Delta z$$

$$\begin{cases} p_x = p_n \\ p_z = p_n + \frac{1}{2} \rho g \Delta z \end{cases}$$



Pressure Variation in a Fluid at Rest

- ▶ Since θ is arbitrary, the result is general
- ▶ There is no pressure change in the horizontal direction
- ▶ The pressure change in the vertical direction is proportional to the depth

"The pressure in a static fluid is a point property, independent of orientation"

Pressure Forces on a Fluid Element

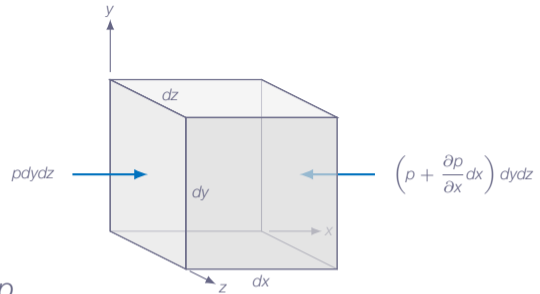
$$\rho = \rho(x, y, z, t)$$

$$dF_x = p dydz - \left(\rho + \frac{\partial \rho}{\partial x} dx \right) dydz = -\frac{\partial \rho}{\partial x} dx dydz$$

$$d\mathbf{F}_p = - \left[\frac{\partial \rho}{\partial x} \mathbf{e}_x + \frac{\partial \rho}{\partial y} \mathbf{e}_y + \frac{\partial \rho}{\partial z} \mathbf{e}_z \right] dx dy dz$$

$$\mathbf{f}_p = -\nabla \rho$$

\mathbf{f} is the net force per unit volume

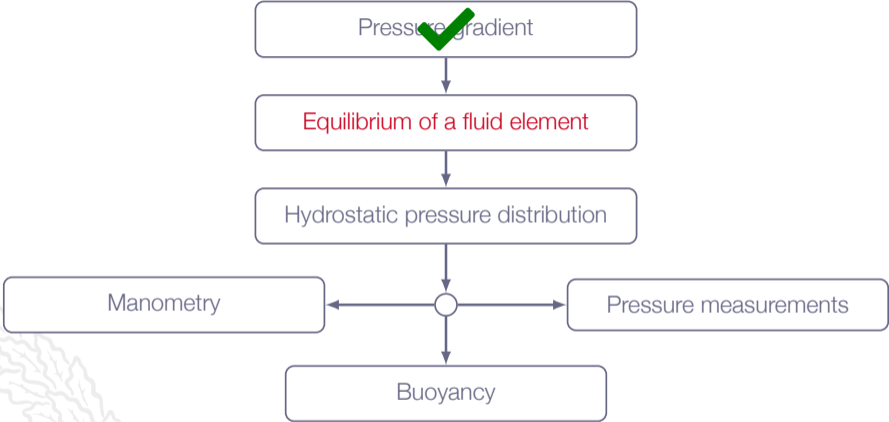


Pressure Forces on a Fluid Element

*"it is not the pressure but the **pressure gradient causing a net force** which must be balanced by gravity or acceleration"*



Roadmap - Pressure Distribution in a Fluid



Equilibrium of a Fluid Element

- ▶ Force balance for a small element
 - ▶ pressure gradients gives surface forces
 - ▶ body forces (electromagnetic or gravitational potentials)
 - ▶ surface forces due to viscous stresses

Newton's second law:

$$\sum \mathbf{f} = \mathbf{f}_p + \mathbf{f}_g + \mathbf{f}_v = -\nabla p + \rho \mathbf{g} + \mathbf{f}_v = \rho \mathbf{a}$$



Equilibrium of a Fluid Element

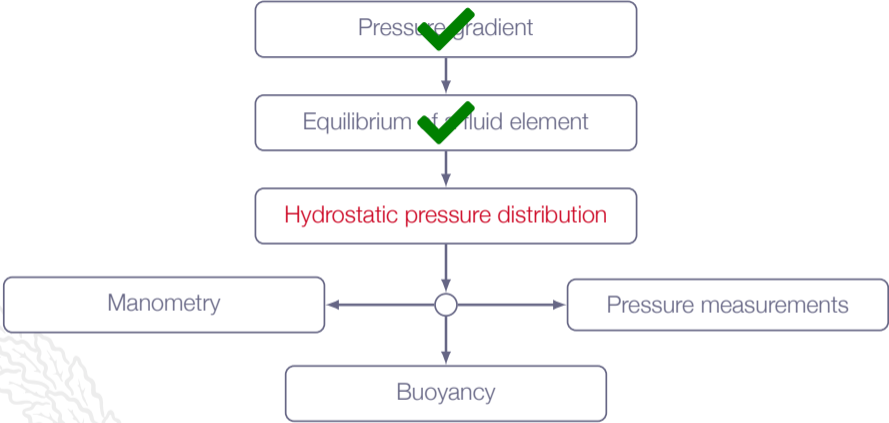
- ▶ Hydrostatic problems:
 - ▶ no viscous forces
 - ▶ no acceleration

Newton's second law reduces to:

$$\nabla p = \rho \mathbf{g}$$

(the general form of Newton's second law will be studied later)

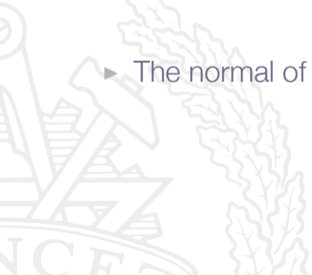
Roadmap - Pressure Distribution in a Fluid



Hydrostatic Pressure in Liquids

$$\nabla p = \rho \mathbf{g}$$

- ▶ ∇p is perpendicular to everywhere to surfaces of constant p
- ▶ The normal of constant-pressure surfaces will be aligned with \mathbf{g}



Hydrostatic Pressure in Liquids

$$\mathbf{g} = -g\mathbf{e}_z$$

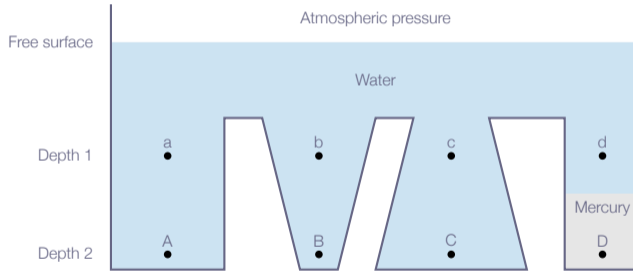
$$\frac{dp}{dz} = -\rho g$$

$$p_2 - p_1 = -\int_1^2 \rho g dz$$



Hydrostatic Pressure in Liquids

for liquids, we assume constant density $\Rightarrow p_2 - p_1 = - \int_1^2 \rho g dz = -\rho g(z_2 - z_1)$



$$p_a = p_b = p_c = p_d$$
$$p_A = p_B = p_C \neq p_D$$

Hydrostatic Pressure in Liquids

- ▶ Is the incompressible assumption for liquids a good assumption?
the density is 4.6 percent higher at the deepest part of the ocean - so yes!

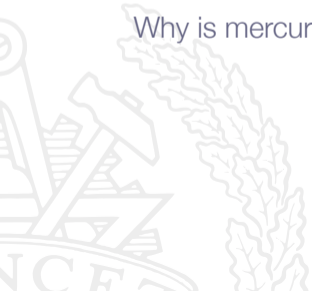
$$p_2 - p_1 = - \int_1^2 \rho g dz = -\rho g(z_2 - z_1)$$



Hydrostatic Pressure in Liquids

$$p_2 - p_1 = - \int_1^2 \rho g dz = -\rho g(z_2 - z_1)$$

Why is mercury used for pressure measurements?



Hydrostatic Pressure in Gases

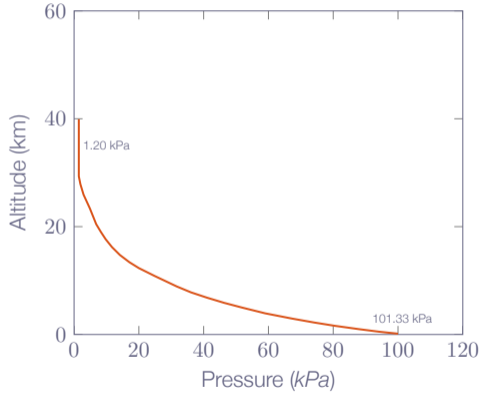
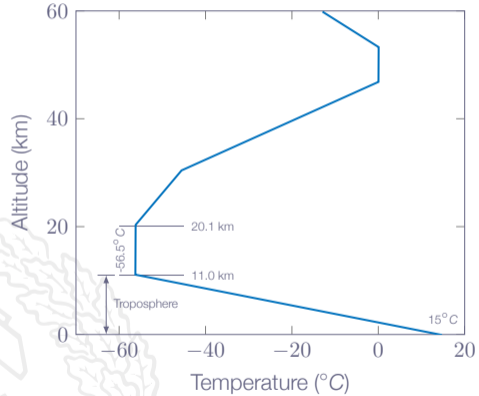
$$\frac{dp}{dz} = -\rho g = -\frac{p}{RT}g$$

both pressure and temperature varies with altitude

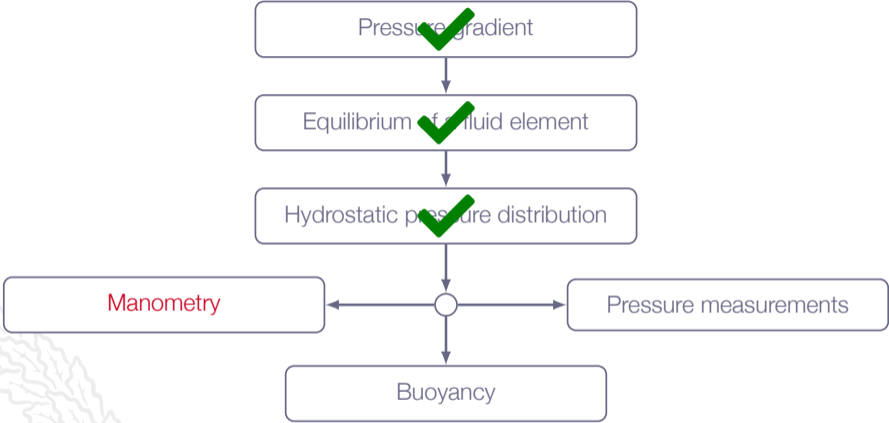
$$\int_1^2 \frac{dp}{p} = \ln \frac{p_2}{p_1} = -\frac{g}{R} \int_1^2 \frac{dz}{T}$$

Temperature variation $T(z)$ needed

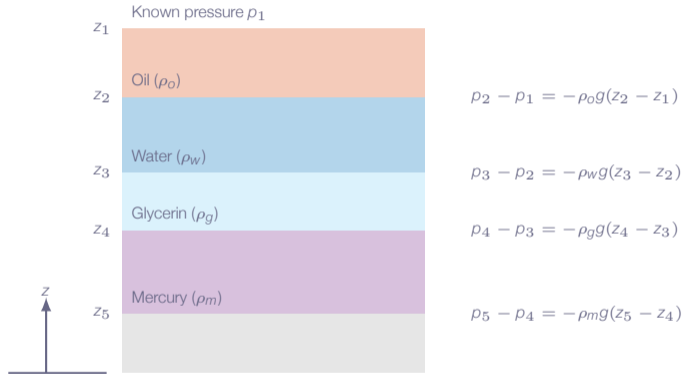
Hydrostatic Pressure in Gases



Roadmap - Pressure Distribution in a Fluid

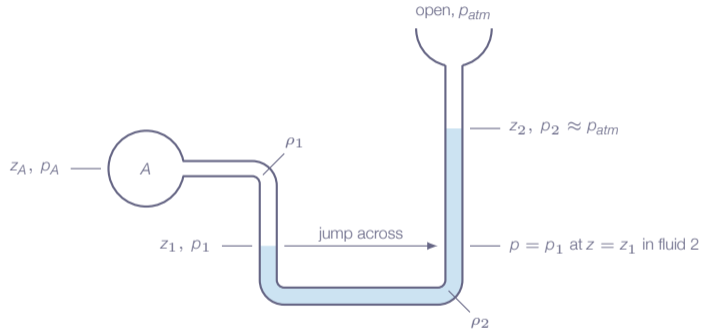


Manometry



$$p_5 - p_1 = -\rho_m g(z_5 - z_4) - \rho_g g(z_4 - z_3) - \rho_w g(z_3 - z_2) - \rho_o g(z_2 - z_1)$$

Manometry

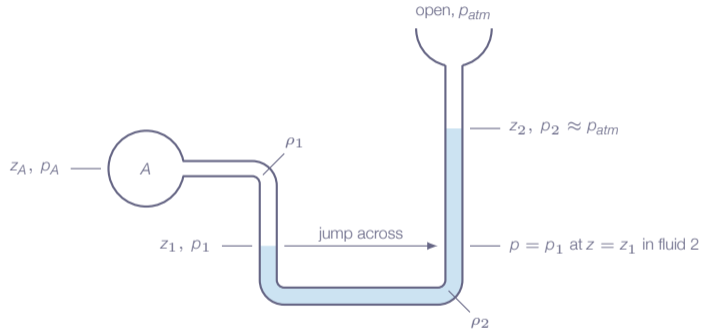


Pascal's law:

"Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure"

Manometry

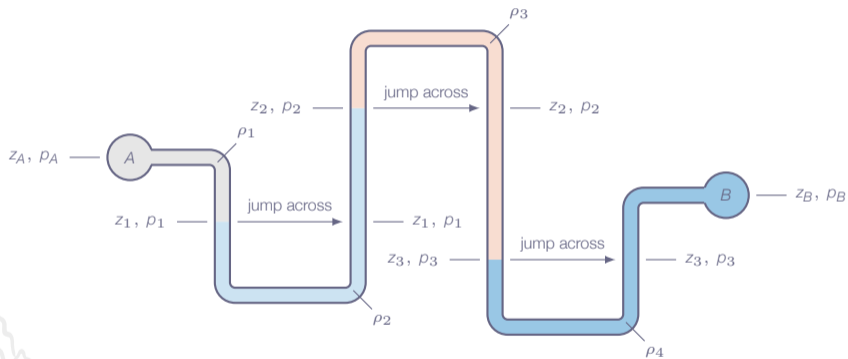
$$\rho_A + \rho_1 g(z_A - z_1) - \rho_2 g(z_2 - z_1) = \rho_2 \approx \rho_{atm}$$



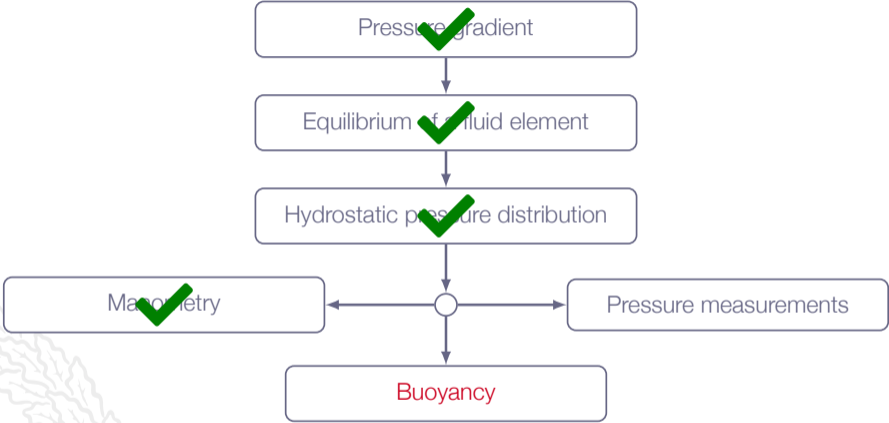
Pascal's law:

"Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure"

Manometry



Roadmap - Pressure Distribution in a Fluid





Buoyancy

Buoyancy

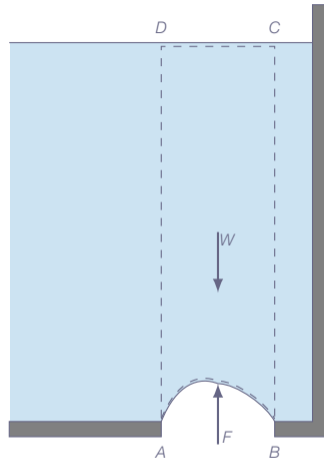
Archimedes:



- ▶ A body immersed in a fluid experiences a vertical buoyant force equal to the **weight of the fluid it displaces**
- ▶ A floating body **displaces its own weight** in the fluid in which it floats

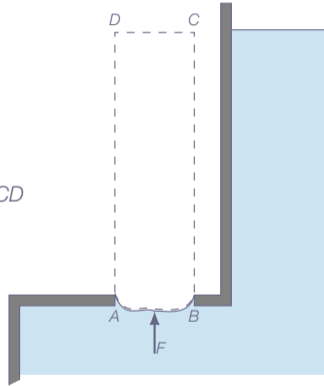
Buoyancy

$$F = \rho g V_{ABCD}$$

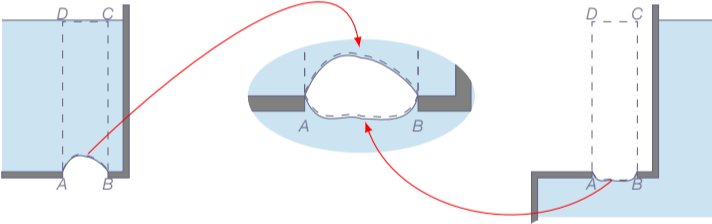


Buoyancy

$$F = \rho g V_{ABCD}$$



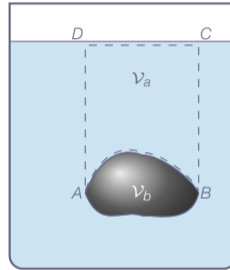
Buoyancy



Buoyancy

$$F_{up} = \rho g(\mathcal{V}_a + \mathcal{V}_b)$$

$$F_{down} = \rho g\mathcal{V}_a$$



Buoyancy

In general

$$\mathbf{F}_B = \sum \rho_i g (\text{displacement volume})_i$$

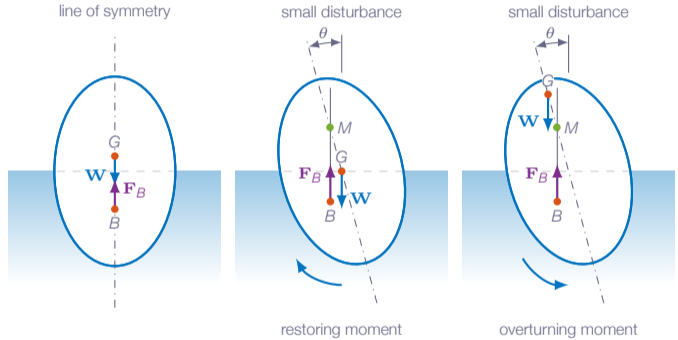
Floating bodies

$$\mathbf{F}_B = \text{body weight}$$



Buoyancy - Stability

- ▶ Center of gravity G
- ▶ Center of buoyancy B
- ▶ Symmetry line
- ▶ Metacenter M

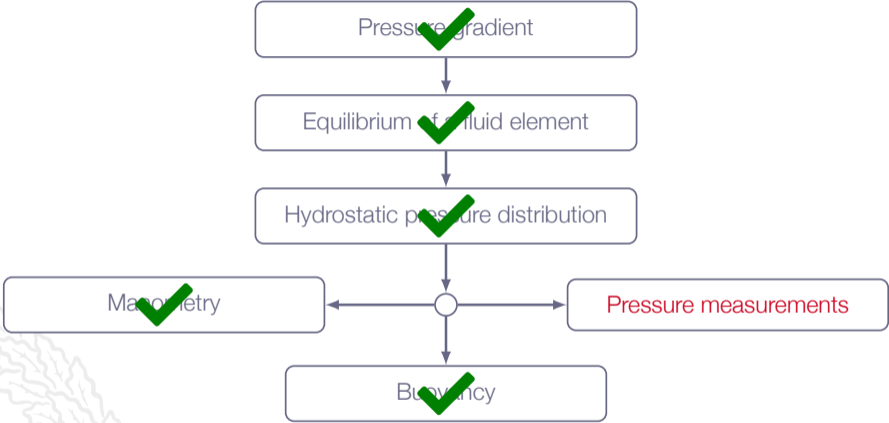


Note! the center of buoyancy (B) is, in this case, the centroid of the displaced volume of liquid

Buoyancy - Stability



Roadmap - Pressure Distribution in a Fluid



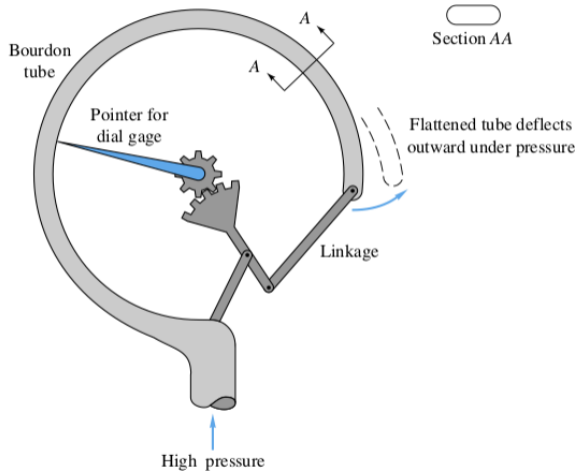
Pressure measurement

Pressure is a derived property

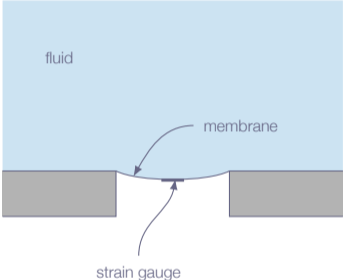
The force per unit area related to fluid molecular bombardment of a surface



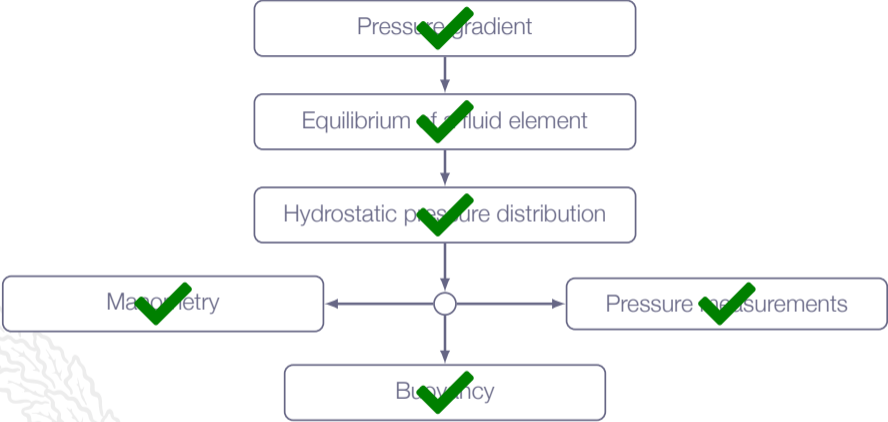
Pressure measurement



Pressure measurement



Roadmap - Pressure Distribution in a Fluid

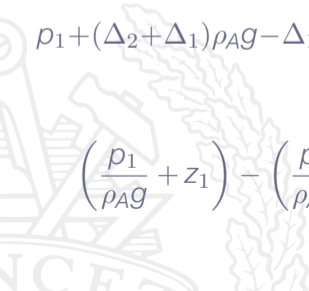
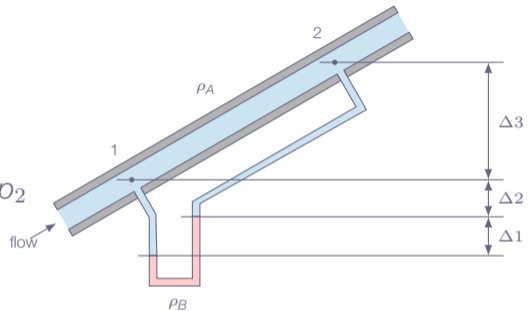


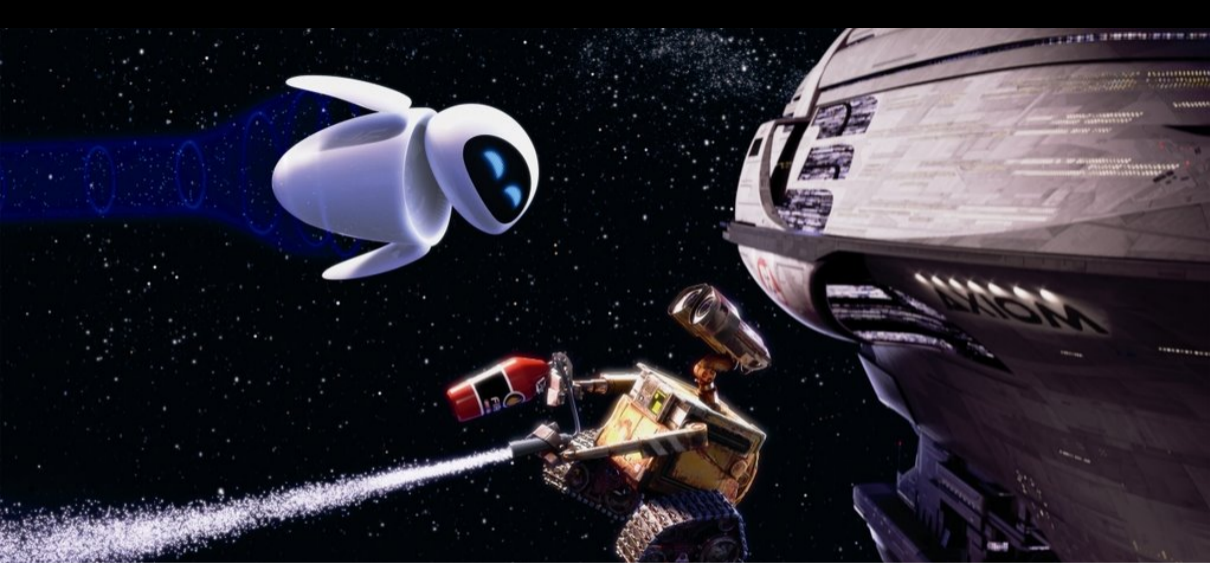
Manometer Example

$$p_1 + \sum_{\text{down}} \rho_i g \Delta_i - \sum_{\text{up}} \rho_i g \Delta_i = p_2$$

$$p_1 + (\Delta_2 + \Delta_1) \rho_A g - \Delta_1 \rho_B g - (\Delta_2 + \Delta_3) \rho_A g = p_2$$

$$p_1 + (\Delta_2 + \Delta_1) \rho_A g - \Delta_1 \rho_B g - (\Delta_2 + z_2 - z_1) \rho_A g = p_2$$





Chapter 3 - Integral Relations for a Control Volume

Overview



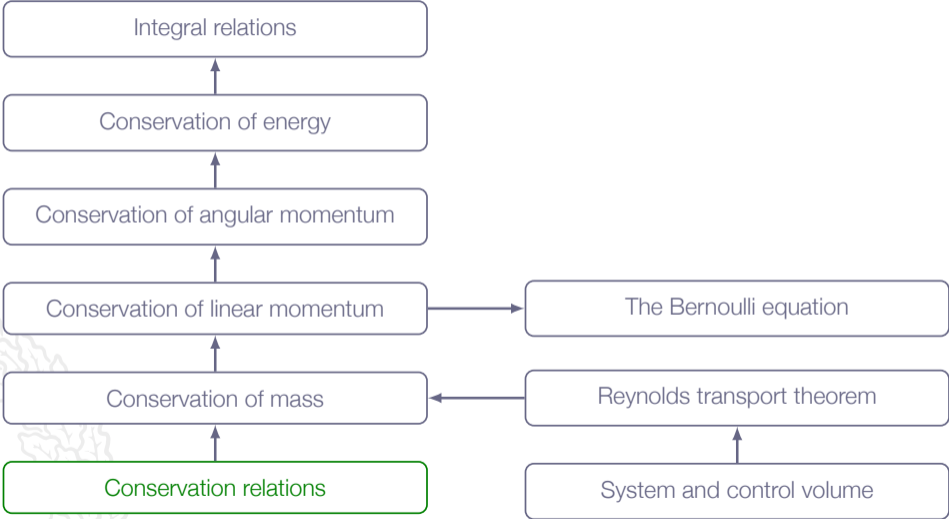
Learning Outcomes

- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 12 **Define** Reynolds transport theorem using the concepts control volume and system
- 13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations
- 15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

we will derive methods suitable for estimation of forces and system analysis

fluid flow finally ...

Roadmap - Integral Relations



Motivation

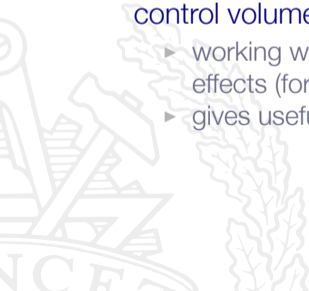
Fluid motion analysis:

differential approach (chapter 4):

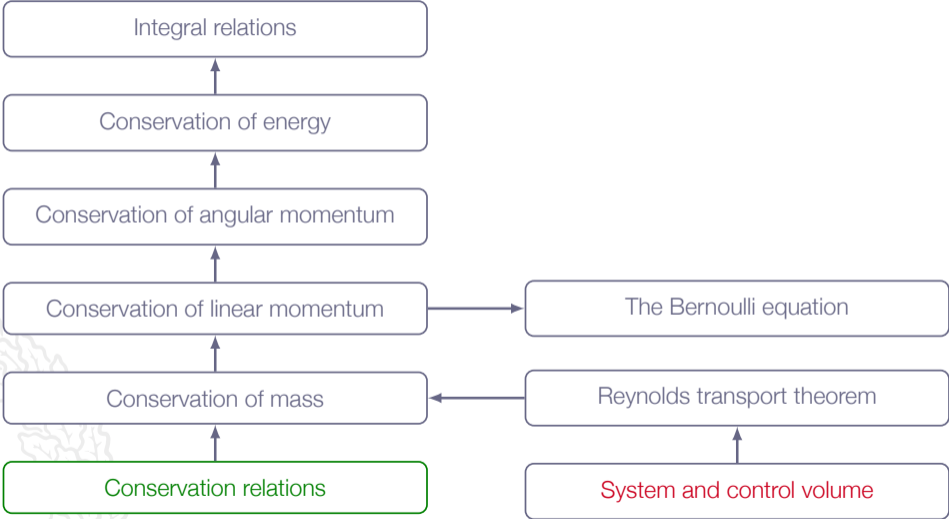
- ▶ describe the detailed flow pattern at every point in the flow

control volume approach (chapter 3):

- ▶ working with a finite region, balance in and out flow and determine gross flow effects (force, torque, energy exchange, ...)
- ▶ gives useful engineering estimates



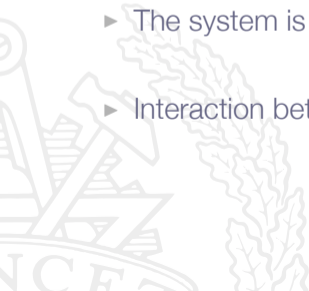
Roadmap - Integral Relations



System vs Control Volume

All laws of mechanics are written for a system:

- ▶ A system is an arbitrary quantity of mass of fixed identity m
- ▶ The system is separated from its surroundings by its boundaries
- ▶ Interaction between the system and its surroundings



System Mass

$$m_{\text{sys}} = \text{const}$$

$$\frac{dm}{dt} = 0$$

- ▶ obvious in solid mechanics
- ▶ needs attention in fluid mechanics



Conservation Relations

- ▶ Mass
- ▶ Linear momentum
- ▶ Angular momentum
- ▶ Energy



Linear Momentum

If the surroundings exert a net force \mathbf{F} on the system, the mass in the system will begin to accelerate

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(m\mathbf{V})$$

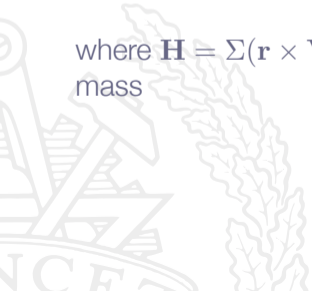


Angular Momentum

If the surroundings exert a net moment \mathbf{M} about the center of mass of the system, there will be a rotation effect

$$\mathbf{M} = \frac{d\mathbf{H}}{dt}$$

where $\mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\delta m$ is the angular momentum of the system about its center of mass



Energy

First law of thermodynamics

$$\delta Q - \delta W = dE$$

Second law of thermodynamics

$$dS \leq \frac{\delta Q}{T}$$



State Relations

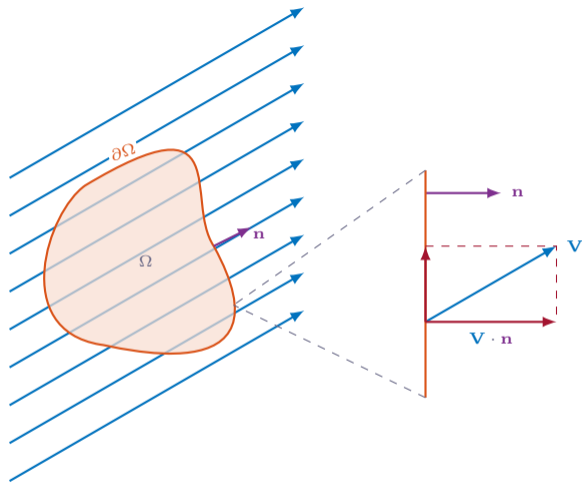
- ▶ The above-listed relations includes thermodynamic properties
- ▶ Needs to be supplemented by a **state relation**
- ▶ Remember: a thermodynamic property can be calculated from **any two other thermodynamics properties**

$$p = p(\rho, T), \quad e = e(\rho, T)$$

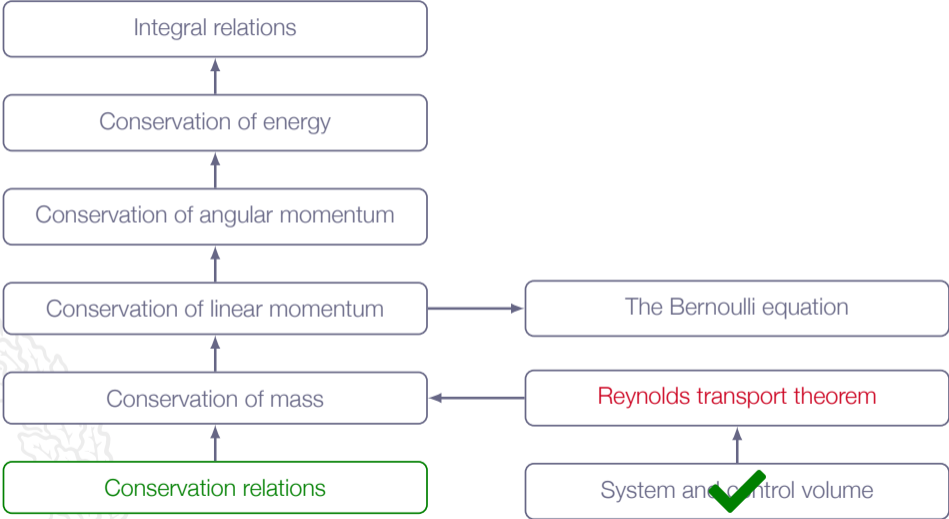
Volume and Mass Flow Rate

$$Q = \int_{\partial\Omega} (\mathbf{V} \cdot \mathbf{n}) dA$$

$$\dot{m} = \int_{\partial\Omega} \rho(\mathbf{V} \cdot \mathbf{n}) dA$$



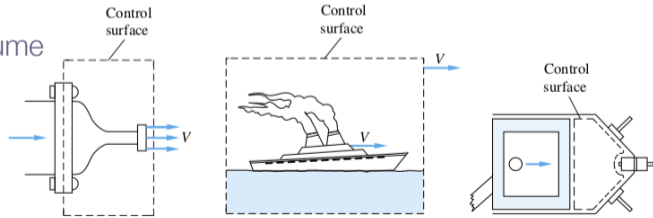
Roadmap - Integral Relations



Reynolds Transport Theorem

Converts mathematical relations for a specific system to relations for a specific region

- ▶ fixed control volume
- ▶ moving control volume
- ▶ deformable control volume



Reynolds Transport Theorem

Let B be any **extensive** property of the fluid (energy, momentum, enthalpy, ...)

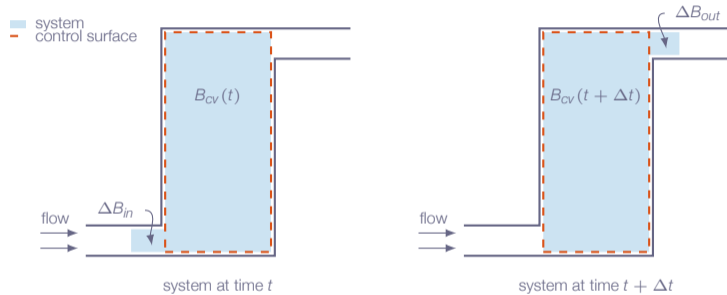
β is the corresponding **intensive** value (*the amount B per unit mass*)

The total amount of B in the control volume is

$$B_{CV} = \int_{CV} \beta dm = \int_{CV} \beta \rho dV$$

where $\beta = \frac{dB}{dm}$

Reynolds Transport Theorem



$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in}$$

$$B_{sys}(t + \Delta t) = B_{cv}(t + \Delta t) + \Delta B_{out}$$

Reynolds Transport Theorem

The rate of change of B for the system:

$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t \rightarrow 0} \left[\frac{B_{\text{sys}}(t + \Delta t) - B_{\text{sys}}(t)}{\Delta t} \right]$$

Apply relations from previous slide \Rightarrow

$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t \rightarrow 0} \left[\frac{B_{\text{cv}}(t + \Delta t) + \Delta B_{\text{out}} - B_{\text{cv}}(t) - \Delta B_{\text{in}}}{\Delta t} \right]$$

Reynolds Transport Theorem

Rewriting \Rightarrow

$$\frac{dB_{\text{sys}}}{dt} = \underbrace{\lim_{\Delta t \rightarrow 0} \left[\frac{B_{\text{cv}}(t + \Delta t) - B_{\text{cv}}(t)}{\Delta t} \right]}_{\frac{dB_{\text{cv}}}{dt}} + \underbrace{\lim_{\Delta t \rightarrow 0} \frac{\Delta B_{\text{out}}}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta B_{\text{in}}}{\Delta t}}_{\dot{B}_{\text{net}}}$$

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{cv}}}{dt} + \dot{B}_{\text{net}}$$

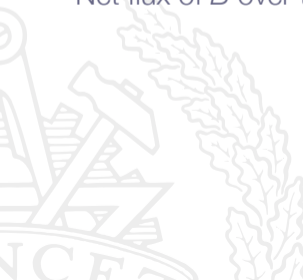
Reynolds Transport Theorem

Rate of change of B within the control volume

$$\frac{d}{dt} \left(\int_{CV} \beta \rho dV \right)$$

Net flux of B over the control volume surface

$$\int_{CS} \beta \rho (\mathbf{V} \cdot \mathbf{n}) dA$$



Reynolds Transport Theorem

$$\underbrace{\frac{d}{dt}(B_{\text{sys}})}_{\text{Lagrange}} = \underbrace{\frac{d}{dt} \left(\int_{\text{CV}} \beta \rho d\mathcal{V} \right) + \int_{\text{CS}} \beta \rho (\mathbf{V} \cdot \mathbf{n}) dA}_{\text{Euler}}$$



Reynolds Transport Theorem

For a fixed control volume (the volume does not change in time)

$$\frac{d}{dt} \left(\int_{CV} \beta \rho dV \right) = \int_{CV} \frac{\partial}{\partial t} (\beta \rho) dV$$



Reynolds Transport Theorem

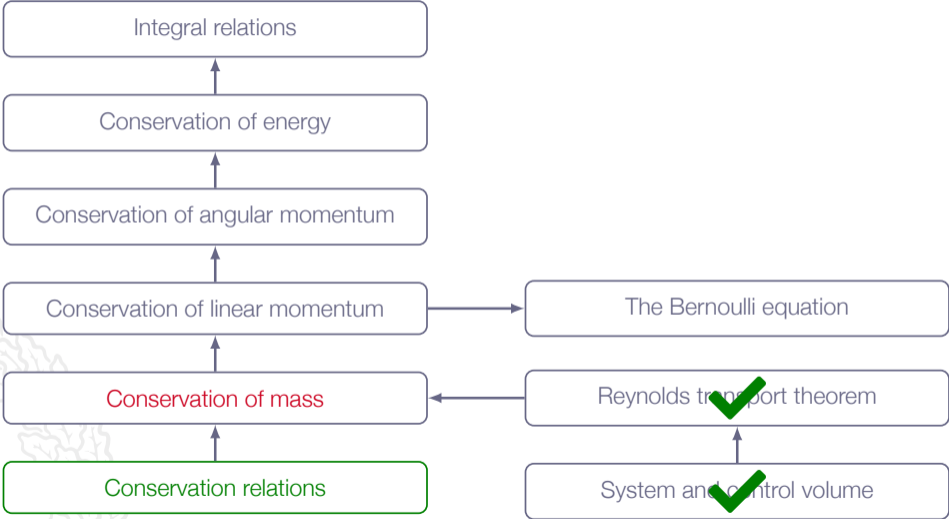
If the control volume moves with the constant velocity \mathbf{V}_s , the relative velocity of the fluid crossing the control volume surface \mathbf{V}_r is

$$\mathbf{V}_r = \mathbf{V} - \mathbf{V}_s$$

and thus

$$\frac{d}{dt}(B_{\text{sys}}) = \frac{d}{dt} \left(\int_{CV} \beta \rho d\mathcal{V} \right) + \int_{CS} \beta \rho (\mathbf{V}_r \cdot \mathbf{n}) dA$$

Roadmap - Integral Relations



Conservation of Mass

Reynolds transport theorem with $B = m$ and $\beta = dB/dm = dm/dm = 1$

$$\frac{d}{dt}(m_{\text{sys}}) = 0 = \frac{d}{dt} \left(\int_{\text{CV}} \rho d\mathcal{V} \right) + \int_{\text{CS}} \rho(\mathbf{V}_{\mathbf{r}} \cdot \mathbf{n}) dA$$

for a fixed control volume

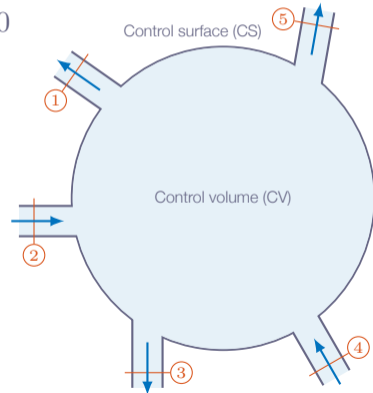
$$\int_{\text{CV}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{\text{CS}} \rho(\mathbf{V}_{\mathbf{r}} \cdot \mathbf{n}) dA = 0$$



Conservation of Mass

for a control volume with a number of one-dimensional inlets and outlets

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$



Conservation of Mass

Steady state $\Rightarrow \partial\rho/\partial t = 0$

$$\int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n})dA = 0$$

or

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$



Conservation of Mass

Incompressible flow $\Rightarrow \partial\rho/\partial t = 0$

$$\int_{CS} (\mathbf{V}_r \cdot \mathbf{n}) dA = 0$$

or

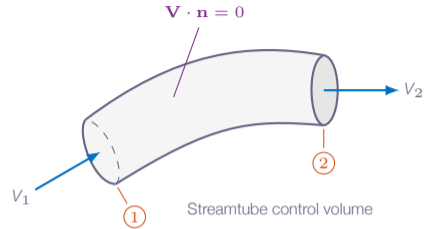
$$\sum_i (A_i V_i)_{out} = \sum_i (A_i V_i)_{in}$$



Conservation of Mass - Example 1

Steady flow through a streamtube

- ▶ steady state \Rightarrow no changes in time
- ▶ streamtube \Rightarrow only flow through the surfaces 1 and 2



$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const}$$

if the density is constant (incompressible flow)

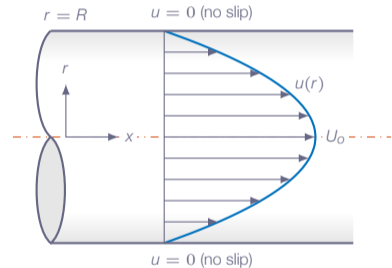
$$Q = A_1 V_1 = A_2 V_2 = \text{const} \Rightarrow V_2 = \frac{A_1}{A_2} V_1$$

Conservation of Mass - Example 2

Compute the average velocity for a steady incompressible viscous flow through a circular tube with given axial velocity profile

$$u = U_o \left(1 - \frac{r}{R}\right)^m$$

- ▶ steady \Rightarrow no changes in time
- ▶ incompressible \Rightarrow constant density
- ▶ laminar flow: $m \approx 1/2$
- ▶ turbulent flow: $m \approx 1/7$



$$V_{av} = \frac{1}{A} \int u dA = \frac{1}{\pi R^2} \int_0^R U_o \left(1 - \frac{r}{R}\right)^m 2\pi r dr = \frac{2U_o}{R^2} \int_0^R \left(1 - \frac{r}{R}\right)^m r dr$$

Conservation of Mass - Example 2

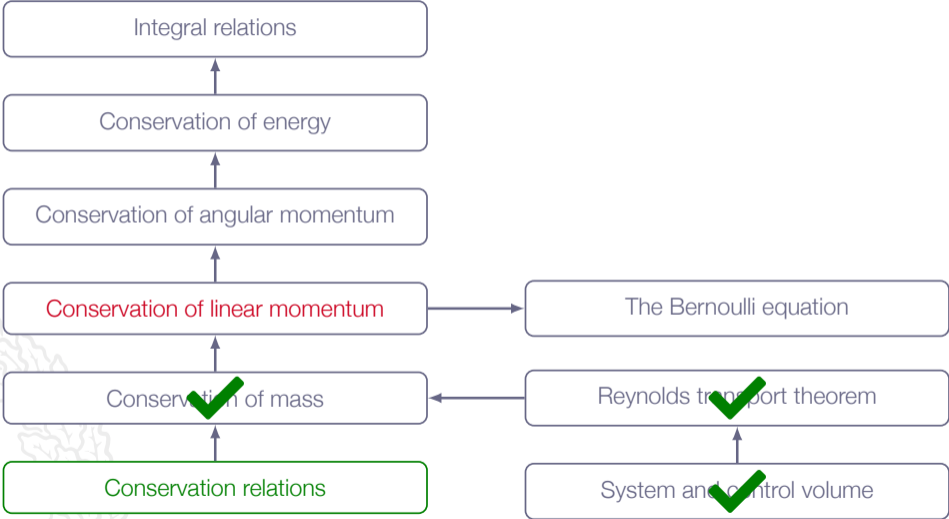
$$V_{av} = \frac{2U_o}{R^2} \int_0^R \left(1 - \frac{r}{R}\right)^m r dr = \frac{2U_o}{R^2} \left[\frac{(r - R) \left(1 - \frac{r}{R}\right)^m (mr + r + R)}{(m + 1)(m + 2)} \right]_0^R$$

$$V_{av} = \frac{2U_o}{(m + 1)(m + 2)}$$

- ▶ laminar flow: $m \approx 1/2 \Rightarrow V_{av} = 8U_o/15 \approx 0.53U_o$
- ▶ turbulent flow: $m \approx 1/7 \Rightarrow V_{av} = 49U_o/60 \approx 0.82U_o$

We will show later that the correct solution for laminar flow is $V_{av} = 0.5U_o$

Roadmap - Integral Relations





Conservation of Linear Momentum

Linear Momentum

Reynolds transport theorem with $B = m\mathbf{V}$ and $\beta = dB/dm = d(m\mathbf{V})/dm = \mathbf{V}$

$$\frac{d}{dt}(m\mathbf{V})_{\text{sys}} = \sum \mathbf{F} = \frac{d}{dt} \left(\int_{CV} \mathbf{V} \rho dV \right) + \int_{CS} \mathbf{V} \rho (\mathbf{V}_r \cdot \mathbf{n}) dA$$

1. \mathbf{V} is the velocity relative to an inertial (nonaccelerating) coordinate system
2. $\sum \mathbf{F}$ is the vector sum of all forces on the system (surface forces and body forces)
3. the relation is a vector relation (three components)

Linear Momentum

Forces:

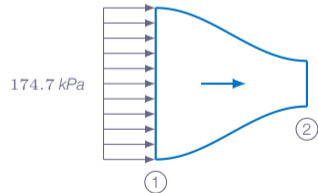
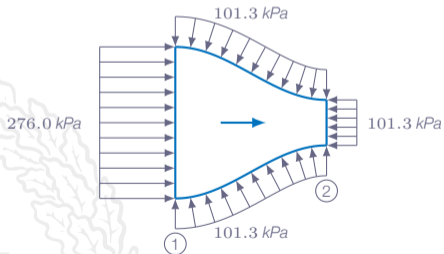
1. solid bodies that protrude through the control volume surface
2. forces due to pressure and viscous stresses of the surrounding fluid



Surface Pressure Force

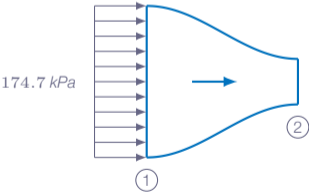
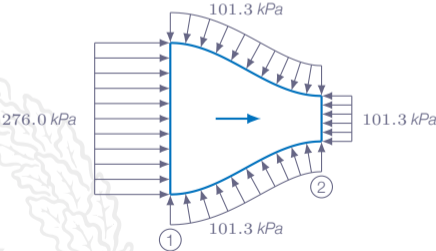
$$\mathbf{F}_p = \int_{CS} p(-\mathbf{n})dA$$

$$\mathbf{F}_p = \int_{CS} (p - p_{atm})(-\mathbf{n})dA = \int_{CS} p_{gage}(-\mathbf{n})dA$$



Surface Pressure Force

A free jet leaving a confined duct and exits into the ambient atmosphere will be at atmospheric pressure



Linear Momentum - Example

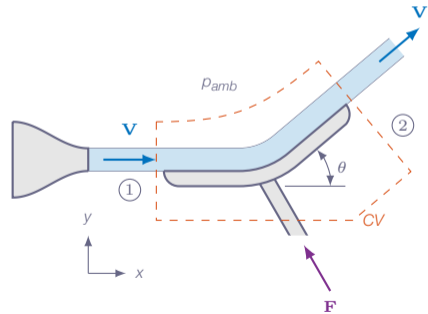
Steady-state flow: deflection of a water jet without changing its velocity magnitude

- ▶ steady-state
- ▶ water \Rightarrow incompressible
- ▶ atmospheric pressure on all control volume surfaces
- ▶ neglect friction

$$\mathbf{F} = \dot{m}_2 \mathbf{V}_2 - \dot{m}_1 \mathbf{V}_1$$

$$|\mathbf{V}_1| = |\mathbf{V}_2| = V$$

- ▶ mass conservation: $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho AV$

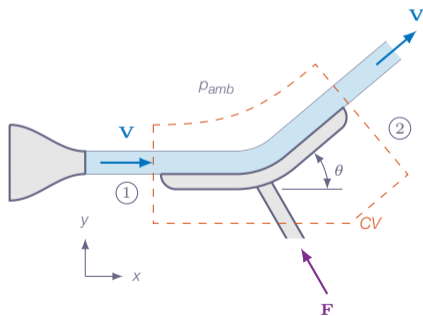


Linear Momentum - Example

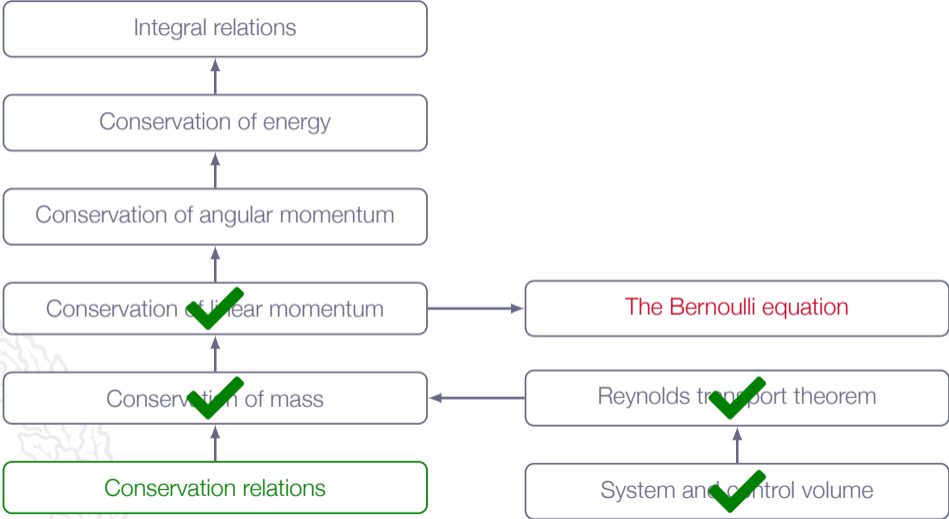
$$F_x = \dot{m}V(\cos \theta - 1)$$

$$F_y = \dot{m}V \sin \theta$$

$$\mathbf{F} = \dot{m}V(\cos \theta - 1, \sin \theta, 0)$$



Roadmap - Integral Relations





Daniel Bernoulli

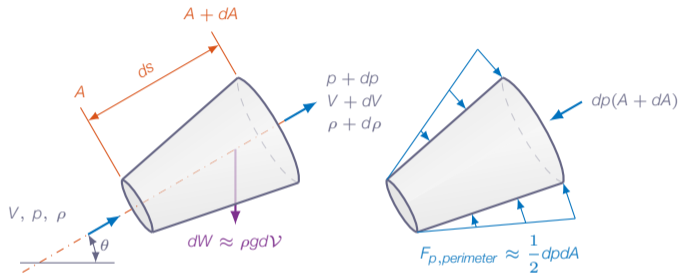
The Bernoulli Equation

The relation between pressure, velocity, and elevation in a frictionless flow



The Bernoulli Equation

Frictionless flow along a streamline (streamtube with infinitesimal cross section area)



conservation of mass:

$$\frac{d}{dt} \left(\int_{CV} \rho dV \right) + \dot{m}_{out} - \dot{m}_{in} = 0 \approx \frac{\partial \rho}{\partial t} dV + d\dot{m}$$

where $\dot{m} = \rho AV$ and $dV \approx A ds$

$$d\dot{m} = d(\rho AV) = -\frac{\partial \rho}{\partial t} A ds$$

The Bernoulli Equation

linear momentum equation in the streamwise direction:

$$\sum dF_s = \frac{d}{dt} \left(\int_{CV} V \rho dV \right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} \approx \frac{\partial}{\partial t} (\rho V) A ds + d(\dot{m}V)$$

frictionless flow means: only pressure and gravity forces

$$dF_{s,p} \approx \frac{1}{2} dp dA - (A + dA) dp \approx -A dp$$

$$dF_{s,grav} = -dW \sin \theta = -(g \rho A) ds \sin \theta = -g \rho A dz$$

$$\sum dF_s = -g \rho A dz - A dp = \frac{\partial}{\partial t} (\rho V) A ds + d(\dot{m}V)$$

The Bernoulli Equation

$$-g\rho Adz - Adp = \frac{\partial \rho}{\partial t} V A ds + \frac{\partial V}{\partial t} \rho A ds + \dot{m} dV + V d\dot{m}$$

the continuity equation gives

$$V \left[\frac{\partial \rho}{\partial t} A ds + d\dot{m} \right] = 0$$

and thus

$$\frac{\partial V}{\partial t} \rho A ds + Adp + \dot{m} dV + g\rho Adz = 0$$

Now, divide by ρA

$$\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + V dV + g dz = 0$$

The Bernoulli Equation

Bernoulli's equation for unsteady **frictionless flow along a streamline** (the relation just derived) can be integrated between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2} (V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$



The Bernoulli Equation

Steady ($\partial V / \partial t = 0$), incompressible (constant density) flow:

$$\rho_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = \rho_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = \text{const}$$



The Bernoulli Equation

Note! the following restrictive assumptions have been made in the derivation

1. steady flow

many flows can be treated as steady at least when doing control volume type of analysis

2. incompressible flow

low velocity gas flow without significant changes in pressure, liquid flow

3. frictionless flow

friction is in general important

4. flow along a single streamline

different streamlines in general have different constants, we shall see later that under specific circumstances all streamlines have the same constant

One should be aware of these restrictions when using the Bernoulli relation

Relation to the Energy Equation

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = \text{const}$$

- ▶ Derived from the momentum equation
- ▶ May be interpreted as a idealized energy equation (changes from 1 to 2)
 - ▶ reversible pressure work
 - ▶ kinetic energy change
 - ▶ potential energy change
 - ▶ no exchange due to viscous dissipation

Stagnation, Static, and Dynamic Pressures

In many flows, elevation changes are negligible

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 = p_o$$

- ▶ Static pressure: p_1 and p_2
- ▶ Dynamic pressure: $\frac{1}{2}\rho V_1^2$ and $\frac{1}{2}\rho V_2^2$
- ▶ Stagnation (total) pressure: p_o

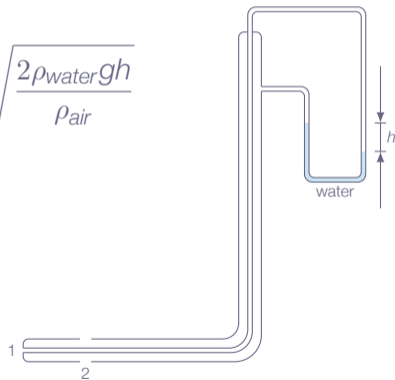
Pitot Static Tube



Pitot Static Tube

$$p_1 + \frac{1}{2}\rho_{air}U_1^2 + \rho gz_1 = p_2 + \frac{1}{2}\rho_{air}U_2^2 + \rho gz_2$$

$$\left. \begin{array}{l} U_1 = 0. \\ U_2 = U \\ z_1 \approx z_2 \\ p_1 - p_2 = \rho_{water}gh \end{array} \right\} \Rightarrow U = \sqrt{\frac{2\rho_{water}gh}{\rho_{air}}}$$



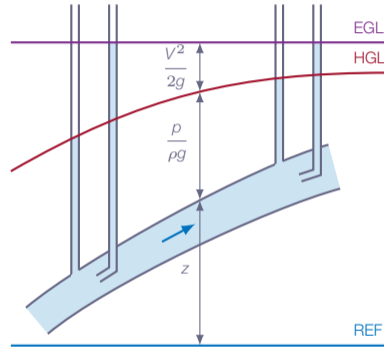
Hydraulic and Energy Grade Lines

$$\text{EGL: } \frac{p}{\rho g} + \frac{V^2}{2g} + z$$

constant if:

- ▶ no friction
- ▶ no heat transfer
- ▶ no work

$$\text{HGL: } \frac{p}{\rho g} + z = \text{EGL} - \frac{V^2}{2g}$$



Venturi Tube

$$\frac{\rho_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{\rho_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$z_1 = z_2$ gives

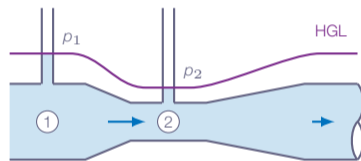
$$V_2^2 - V_1^2 = \frac{2\Delta p}{\rho}$$

continuity:

$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2 = \frac{D_2^2}{D_1^2}V_2$$

inserted in the Bernoulli equation, this gives

$$V_2 = \left[\frac{2D_1^4\Delta p}{\rho(D_1^4 - D_2^4)} \right]^{1/2} \Rightarrow \dot{m} = \rho A_2 V_2 = \frac{\pi D_1^2 D_2^2}{4} \left[\frac{2\rho\Delta p}{D_1^4 - D_2^4} \right]^{1/2}$$



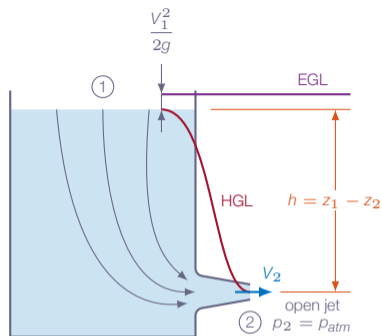
Tank Problem - Solution 2

The outflow is very small in compared to the tank volume and thus the water surface hardly moves at all, *i.e.* $V_1 \approx 0$

Bernoulli:

$$\frac{\rho_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{\rho_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

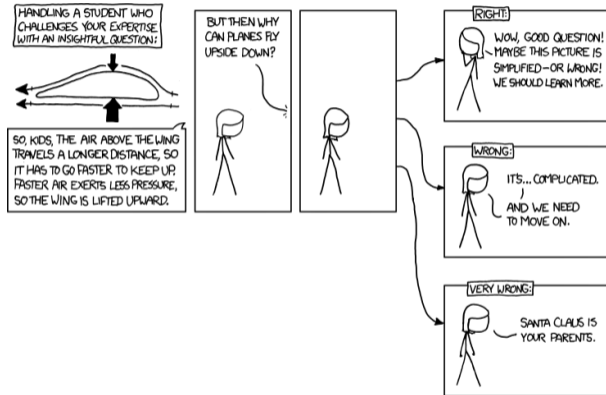
$$V_1 \approx 0, \rho_1 = \rho_2 = \rho_{atm}$$



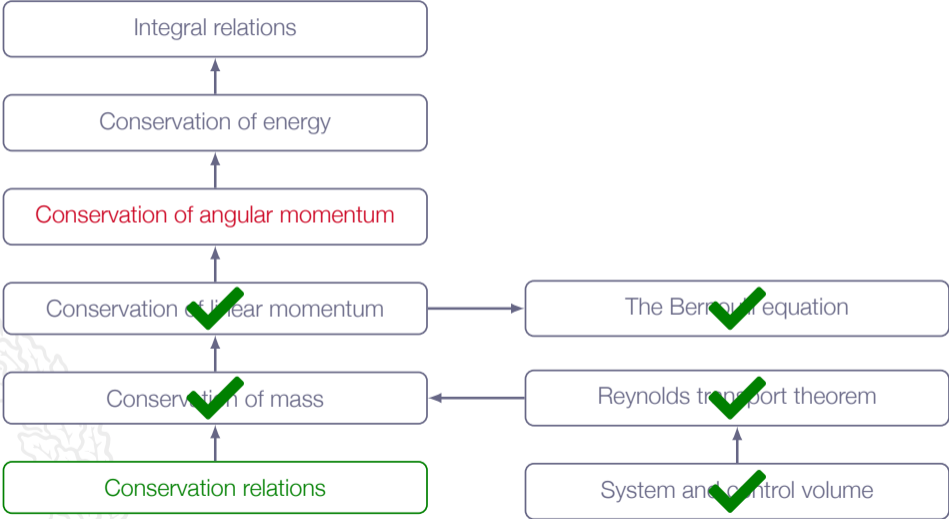
$$V_2^2 = 2g(z_1 - z_2) = 2gh$$

$$V_2 = \sqrt{2gh}$$

Airfoil



Roadmap - Integral Relations





Conservation of Angular Momentum

Angular Momentum

Angular momentum about a point o

$$\mathbf{H}_o = \int_{\text{syst}} (\mathbf{r} \times \mathbf{V}) dm = B$$

where \mathbf{r} is the position vector from o to the element mass dm and \mathbf{V} is the velocity of that element

The amount of angular momentum per unit mass

$$\beta = \frac{d\mathbf{H}_o}{dm} = \mathbf{r} \times \mathbf{V}$$

Angular Momentum

Reynold's transport theorem:

$$\left. \frac{d\mathbf{H}_o}{dt} \right|_{\text{syst}} = \frac{d}{dt} \left[\int_{CV} (\mathbf{r} \times \mathbf{V}) \rho dV \right] + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V}_r \cdot \mathbf{n}) dA$$

for inertial coordinate systems:

$$\frac{d\mathbf{H}_o}{dt} = \sum \mathbf{M}_o = \sum (\mathbf{r} \times \mathbf{F})_o$$

Angular Momentum

Nondeformable inertial control volume:

$$\sum \mathbf{M}_o = \frac{d}{dt} \left[\int_{CV} (\mathbf{r} \times \mathbf{V}) \rho d\mathcal{V} \right] + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

one-dimensional inlets and outlets

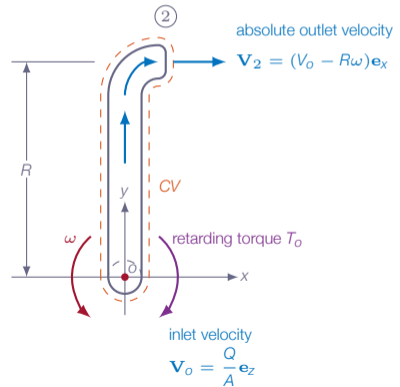
$$\int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V} \cdot \mathbf{n}) dA = \sum (\mathbf{r} \times \mathbf{V})_{out} \dot{m}_{out} - \sum (\mathbf{r} \times \mathbf{V})_{in} \dot{m}_{in}$$

Angular Momentum Example - Lawn Sprinkler

$$\sum \mathbf{M}_o = \frac{d}{dt} \left[\int_{CV} (\mathbf{r} \times \mathbf{V}) \rho dV \right] + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V}_r \cdot \mathbf{n}) dA$$

$$\begin{cases} \mathbf{V}_2 = (V_o - R\omega, 0, 0) \\ \mathbf{V}_o = (0, 0, V_o) \end{cases}$$

$$\begin{cases} \mathbf{r}_2 = (0, R, 0) \\ \mathbf{r}_o = (0, 0, 0) \end{cases}$$



Angular Momentum Example - Lawn Sprinkler

steady-state:

$$\frac{d}{dt} \left[\int_{CV} (\mathbf{r} \times \mathbf{V}) \rho d\mathcal{V} \right] = 0$$



Angular Momentum Example - Lawn Sprinkler

$$\sum \mathbf{M}_o = \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V}_r \cdot \mathbf{n}) dA$$

inlet:

$$(\mathbf{r}_o \times \mathbf{V}_o) = (0, 0, 0) \times (0, 0, V_o) = (0, 0, 0)$$

$$(\mathbf{V}_{o,r} \cdot \mathbf{n}_o) = (0, 0, V_o) \cdot (0, 0, -1) = -V_o$$

$$(\mathbf{r}_o \times \mathbf{V}_o) \rho (\mathbf{V}_o \cdot \mathbf{n}_o) A_o = -(0, 0, 0) \rho V_o A_o = (0, 0, 0)$$

outlet:

$$(\mathbf{r}_2 \times \mathbf{V}_2) = (0, R, 0) \times (V_o - R\omega, 0, 0) = (0, 0, R^2\omega - RV_o)$$

$$(\mathbf{V}_{2,r} \cdot \mathbf{n}_2) = (V_o, 0, 0) \cdot (1, 0, 0) = V_o$$

$$(\mathbf{r}_2 \times \mathbf{V}_2) \rho (\mathbf{V}_2 \cdot \mathbf{n}_2) A_2 = (0, 0, R^2\omega - RV_o) \rho V_o A_2 = \rho Q (0, 0, R^2\omega - RV_o)$$

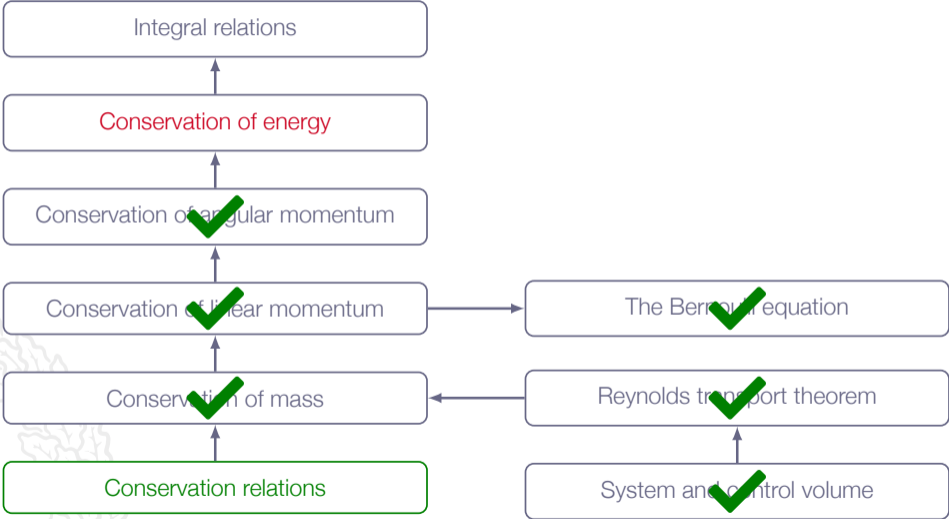
Angular Momentum Example - Lawn Sprinkler

$$\sum \mathbf{M}_o = (0, 0, -T_o) = \rho Q(0, 0, R^2\omega - RV_o)$$

$$\omega = \frac{V_o}{R} - \frac{T_o}{\rho QR^2}$$

Note: with a negligible retarding torque, *i.e.* $T_o \approx 0$, we get $\omega \approx \frac{V_o}{R}$

Roadmap - Integral Relations



The Energy Equation

Reynold's transport theorem applied the the **first law of thermodynamics**
($B = E$, $\beta = dE/dm = e$)

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} = \frac{d}{dt} \left(\int_{CV} e\rho d\mathcal{V} \right) + \int_{CS} e\rho(\mathbf{V} \cdot \mathbf{n})dA$$

Recall:

- ▶ positive Q : heat added to the system
- ▶ positive W : work done by the system on its surroundings

The Energy Equation - Energy per Unit Mass

$$e = e_{internal} + e_{kinetic} + e_{potential} + e_{other}$$

e_{other} could be related to, for example, chemical reactions, nuclear reactions, or magnetic fields and will not be considered here

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$



The Energy Equation - Work

The work term \dot{W} can be divided into **shaft work**, **pressure work**, and work related to **viscous forces**

$$\dot{W} = \dot{W}_s + \dot{W}_p + \dot{W}_\nu$$

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

$$\dot{W}_\nu = - \int_{CS} \boldsymbol{\tau} \cdot \mathbf{V}dA$$



The Energy Equation - Pressure Work

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

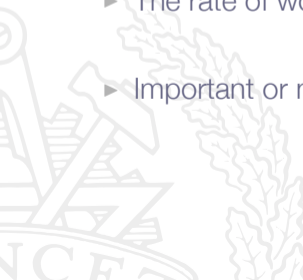
- ▶ The rate of work done by pressure forces on the control volume surfaces
- ▶ Internal forces will always have an opposite force leading to cancelation



The Energy Equation - Viscous Work

$$\dot{W}_\nu = - \int_{CS} \boldsymbol{\tau} \cdot \mathbf{V} dA$$

- ▶ The rate of work related to viscous stresses on the control volume surfaces
- ▶ Important or not depending on flow situation



The Energy Equation - Control Volume Boundaries

Solid walls:

no-slip $\Rightarrow \dot{W}_v = 0$

Machine surfaces:

viscous work included implicitly in shaft work

Inlets/outlets:

flow aligned with surface normal (usually) and normal viscous stress components are in most cases very small

Streamlines:

viscous stresses may be significant *depending on streamline location*



The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt} \left(\int_{CV} e \rho d\mathcal{V} \right) + \int_{CS} \rho e (\mathbf{V} \cdot \mathbf{n}) dA + \int_{CS} p (\mathbf{V} \cdot \mathbf{n}) dA$$

collecting surface integrals gives

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt} \left(\int_{CV} e \rho d\mathcal{V} \right) + \int_{CS} \left(e + \frac{p}{\rho} \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA$$



The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt} \left(\int_{CV} e \rho dV \right) + \int_{CS} \left(e + \frac{p}{\rho} \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

or

$$\begin{aligned} \dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt} \left[\int_{CV} \left(\hat{u} + \frac{1}{2} V^2 + gz \right) \rho dV \right] + \\ + \int_{CS} \left(\hat{h} + \frac{1}{2} V^2 + gz \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA \end{aligned}$$

where \hat{h} is the enthalpy defined as $\hat{h} = \hat{u} + p/\rho$

The Energy Equation

Steady state:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \int_{CS} \left(\hat{h} + \frac{1}{2}V^2 + gz \right) \rho(\mathbf{V} \cdot \mathbf{n})dA$$

Special case: one inlet and one outlet (both one-dimensional)

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = -\dot{m}_1 \left(\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 \right) + \dot{m}_2 \left(\hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 \right)$$

continuity $\Rightarrow \dot{m}_1 = \dot{m}_2 = \dot{m}$, divide by \dot{m} gives

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

The Energy Equation

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

all terms has the dimension $[m^2/s^2]$, divide by $g [m/s^2]$ to get dimension $[m]$

$$\frac{p_1}{\rho g} + \frac{\hat{u}_1}{g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\hat{u}_2}{g} + \frac{V_2^2}{2g} + z_2 - h_q + h_s + h_\nu$$

- ▶ $p/(\rho g)$: pressure head
- ▶ $V^2/2g$: velocity head

The Energy Equation

- ▶ steady-state flow
- ▶ incompressible (low speed)
- ▶ pipe/duct that may or may not include turbines and pumps
- ▶ solid walls $\Rightarrow h_\nu = 0$

$$\underbrace{\left(\frac{\rho_1}{\rho g} + \frac{V_1^2}{2g} + z_1 \right)}_{h_{o1}} = \underbrace{\left(\frac{\rho_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \right)}_{h_{o2}} + \frac{\hat{u}_2 - \hat{u}_1 - q}{g}$$

where h_o is available head or total head

The Energy Equation

$$\frac{\hat{u}_2 - \hat{u}_1 - q}{g} = h_{o1} - h_{o2}$$

- ▶ pump head input h_p
- ▶ turbine head extraction h_t
- ▶ friction head losses h_f (always positive)

$$\left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 \right)_{in} = \left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \right)_{out} + h_f - h_p + h_t$$

Kinetic Energy Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the correction factor α

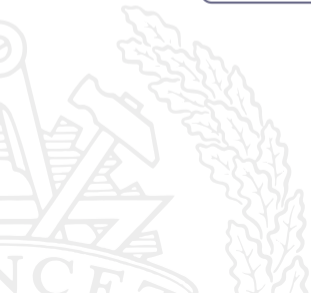
$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{\alpha V_{av}^2}{2} \dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A} \int u dA$$

Kinetic Energy Correction Factor

$$\left(\frac{p_1}{\rho g} + \frac{\alpha_1 V_1^2}{2g} + z_1 \right)_{in} = \left(\frac{p_2}{\rho g} + \frac{\alpha_2 V_2^2}{2g} + z_2 \right)_{out} + h_f - h_p + h_t$$

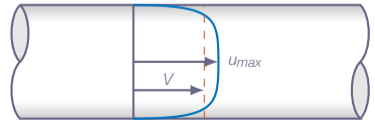
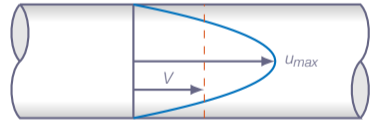


Kinetic Energy Correction Factor

Laminar pipe flow:

$$u(r) = U_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

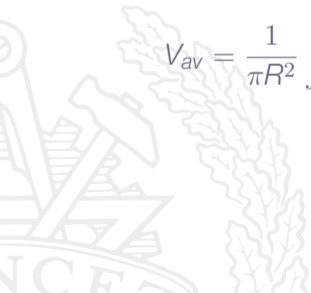
which gives $V_{av} = \frac{1}{2} U_{max}$ and $\alpha = 2.0$



Kinetic Energy Correction Factor

$$u(r) = U_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right], \quad V_{av} = \frac{1}{A} \int u dA$$

$$V_{av} = \frac{1}{\pi R^2} \int_0^R U_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right] 2\pi r dr = \frac{2U_{max}}{R^2} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R = \frac{U_{max}}{2}$$



Kinetic Energy Correction Factor

$$u(r) = U_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \int \frac{1}{2} U_{max}^2 \left[1 - \left(\frac{r}{R} \right)^2 \right]^2 \rho U_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right] 2\pi r dr$$

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \rho \pi U_{max}^3 \int \left[1 - \left(\frac{r}{R} \right)^2 \right]^3 r dr$$

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{1}{8} \rho \pi R^2 U_{max}^3$$

Kinetic Energy Correction Factor

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{1}{8} \rho \pi R^2 U_{max}^3$$

the massflow \dot{m} can be obtained as:

$$\dot{m} = V_{av} \rho \pi R^2 = \frac{1}{2} U_{max} \rho \pi R^2$$

which gives

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{U_{max}^2}{4} \dot{m}$$

from the definition of α

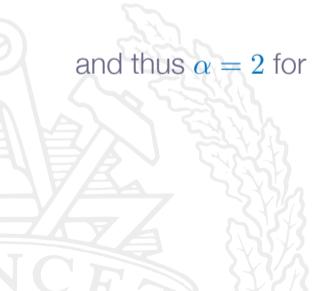
$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{\alpha V_{av}^2}{2} \dot{m} = \left\{ V_{av} = \frac{1}{2} U_{max} \right\} = \frac{\alpha U_{max}^2}{8} \dot{m}$$

Kinetic Energy Correction Factor

comparing the two expressions we have that

$$\frac{U_{max}^2}{4} \dot{m} = \frac{\alpha U_{max}^2}{8} \dot{m}$$

and thus $\alpha = 2$ for laminar incompressible flow



Kinetic Energy Correction Factor

Turbulent flow:

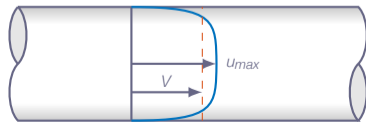
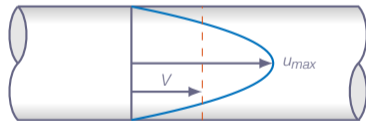
$$u(r) \approx U_{max} \left(1 - \frac{r}{R}\right)^m, \quad m \approx \frac{1}{7}$$

From one of the continuity equation examples we have:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$

Performing the integration and comparing terms gives:

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$



Kinetic Energy Correction Factor

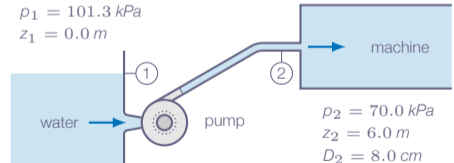
m	1/5	1/6	1/7	1/8	1/9
α	1.106	1.077	1.058	1.046	1.037

- ▶ For turbulent flows, $\alpha = 1$ is often a good assumption
- ▶ For laminar flows, $\alpha = 2$ should be used



The Energy Equation - Pump Example

- ▶ Calculate:
 - ▶ Pump power if $\eta = 0.8$
- ▶ Given:
 - ▶ Geometry and pressures from figure
 - ▶ The pump delivers water at a flow rate $Q = 0.04 \text{ m}^3/\text{s}$
 - ▶ Friction losses between 1 and 2 are given by $h_f = KV_2^2/(2g)$ where $K \approx 7.5$
 - ▶ $\alpha \approx 1.07$
- ▶ Assumptions:
 - ▶ steady-state flow
 - ▶ negligible viscous work
 - ▶ large reservoir ($V_1 \approx 0$)



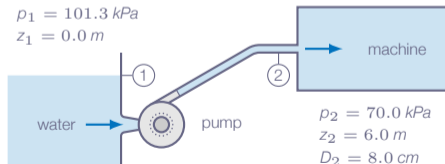
The Energy Equation - Pump Example

$$V_2 = \frac{Q}{A_2} = 7.96 \text{ m/s}$$

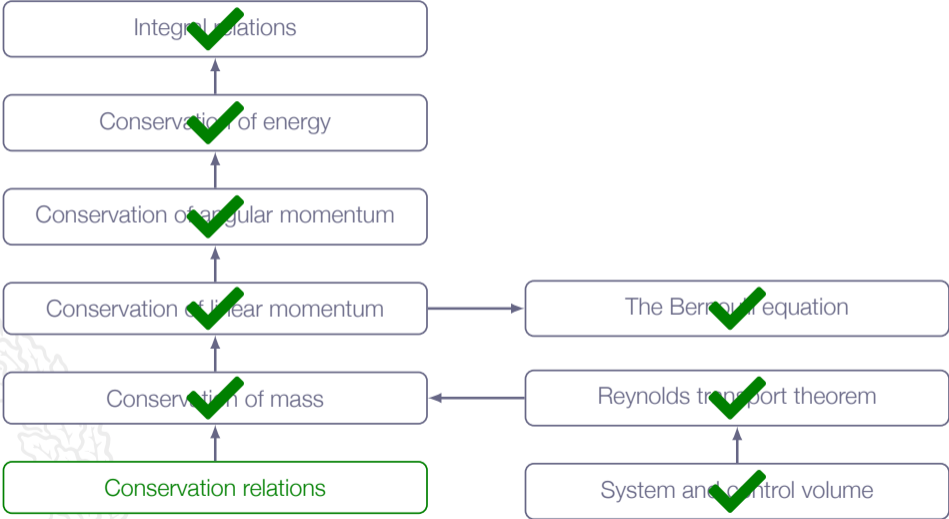
$$\left(\frac{p_1}{\rho g} + \frac{\alpha V_1^2}{2g} + z_1 \right)_{in} = \left(\frac{p_2}{\rho g} + \frac{\alpha V_2^2}{2g} + z_2 \right)_{out} + h_f - h_p + h_t$$

$$h_p = \frac{p_2 - p_1}{\rho g} + (z_2 - z_1) + (\alpha_2 + K) \frac{V_2^2}{2g} = 30.5 \text{ m}$$

$$P_{\text{pump}} = \frac{\rho g Q h_p}{\eta} = 14960 \text{ W (or 20 hp)}$$



Roadmap - Integral Relations



Integral Relations - Considerations

1. control volume type: **nondeforming?**, **nonaccelerating?**



Integral Relations - Considerations

1. control volume type: **nondeforming?**, **nonaccelerating?**
2. **steady** flow? if not can the frame of reference be changed?



Integral Relations - Considerations

1. control volume type: **nondeforming?**, **nonaccelerating?**
2. **steady** flow? if not can the frame of reference be changed?
3. can **friction** be neglected?



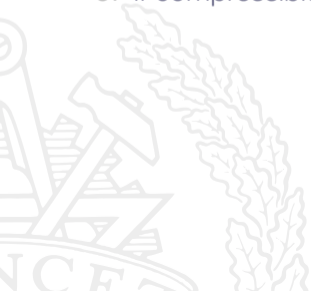
Integral Relations - Considerations

1. control volume type: **nondeforming?**, **nonaccelerating?**
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3. can **friction** be neglected?
4. can the fluid be assumed to be **incompressible?**



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Integral Relations - Considerations

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6. do we need to account for **body forces** (gravity etc)?



Integral Relations - Considerations

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6. do we need to account for **body forces** (gravity etc)?
7. is there **heat transfer**, **shaft work** or **viscous work**



Integral Relations - Considerations

1. control volume type: **nondeforming?**, **nonaccelerating?**
2. **steady** flow? if not can the frame of reference be changed?
3. can **friction** be neglected?
4. can the fluid be assumed to be **incompressible**?
5. if compressible, can the **ideal gas law** be used?
6. do we need to account for **body forces** (gravity etc)?
7. is there **heat transfer**, **shaft work** or **viscous work**
8. can inlets/outlets be assumed to be **onedimensional**

$$\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right]$$

$$\frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right]$$

$$\frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re} \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right]$$

$$\frac{\partial(\rho u \epsilon_0)}{\partial x} + \frac{\partial(\rho v \epsilon_0)}{\partial y} + \frac{\partial(\rho w \epsilon_0)}{\partial z} = -\frac{\partial(\rho u p)}{\partial x} - \frac{\partial(\rho v p)}{\partial y} - \frac{\partial(\rho w p)}{\partial z} + \dots$$

Chapter 4 - Differential Relations for Fluid Flow

Overview



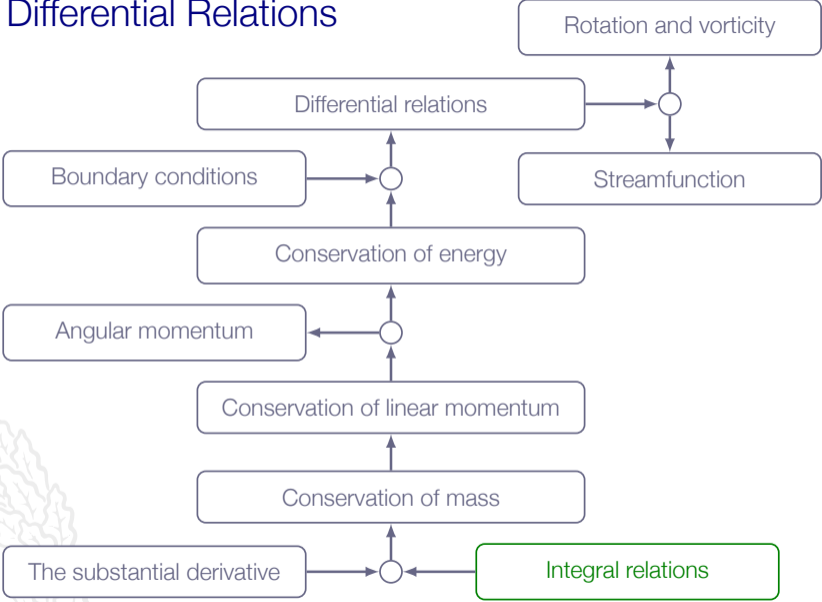
Learning Outcomes

- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 14 **Derive** the continuity, momentum and energy equations on differential form
- 36 **Define** and explain vorticity

let's push the control volume approach to the limit ...



Roadmap - Differential Relations



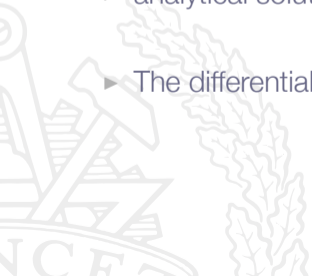
Differential Relations

seeking the point-by-point details of a flow pattern by analyzing an infinitesimal region of the flow



Differential Relations

- ▶ Apply the four basic conservation laws to an infinitesimally small control volume
- ▶ The differential relations are in general very difficult to solve
- ▶ analytical solutions exists for a few cases
- ▶ The differential relations form the basis for CFD software



High-Speed Nozzle Flow

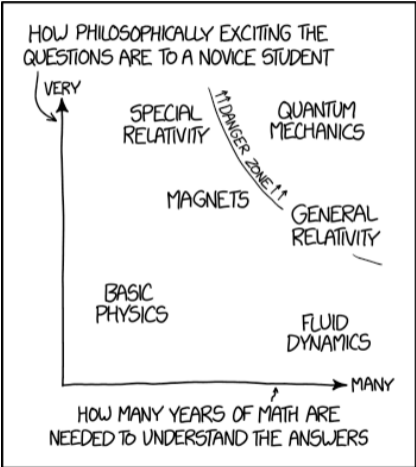


The Acoustic Signature of a Supersonic Jet

Screeching rectangular supersonic jet

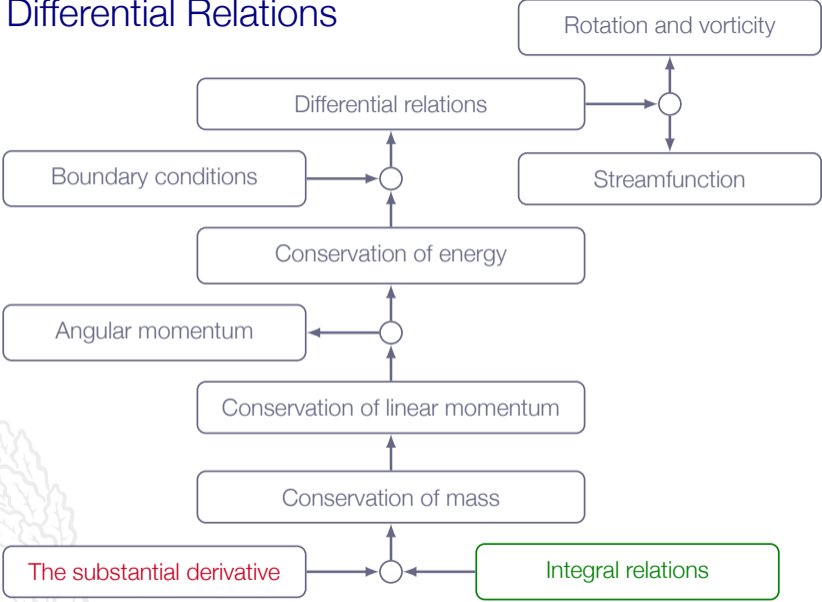


Lots of Equations Ahead



WHY SO MANY PEOPLE HAVE WEIRD IDEAS ABOUT QUANTUM MECHANICS

Roadmap - Differential Relations



Frame of Reference

Eulerian: observer fixed in space

Lagrangian: observer follows a fluid particle

recall the speedometer/traffic-camera analogy



Acceleration Field

In order to get to Newton's second law, we need the acceleration vector

Velocity field:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{e}_x u(x, y, z, t) + \mathbf{e}_y v(x, y, z, t) + \mathbf{e}_z w(x, y, z, t)$$

Acceleration field:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{e}_x \frac{du}{dt} + \mathbf{e}_y \frac{dv}{dt} + \mathbf{e}_z \frac{dw}{dt}$$



Acceleration Field

Each scalar component of the velocity vector (u, v, w) is a function of four variables (x, y, z, t) and thus

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

By definition $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V} \cdot \nabla)u$$

Acceleration Field

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V} \cdot \nabla)u$$

$$a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V} \cdot \nabla)v$$

$$a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V} \cdot \nabla)w$$

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{local}} + \underbrace{\left(u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z} \right)}_{\text{convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{D\mathbf{V}}{Dt}$$

Acceleration Field

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{local}} + \underbrace{\left(u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z} \right)}_{\text{convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{D\mathbf{V}}{Dt}$$

local acceleration: only in unsteady flows

convective acceleration: fluid particle that moves through regions of spatially varying velocity

- ▶ nozzle flow
- ▶ diffuser flow

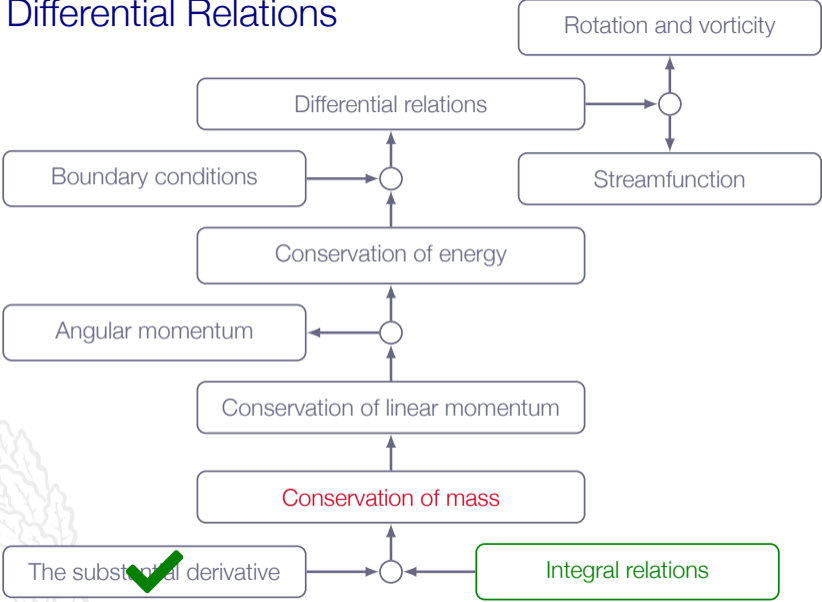
Substantial derivative

- ▶ The total temporal derivative
 - ▶ the sum of the local derivative and the convective derivative
 - ▶ often referred to as the **substantial derivative**
- ▶ The substantial derivative **follows a fluid particle** but is expressed in an **Eulerian frame of reference**

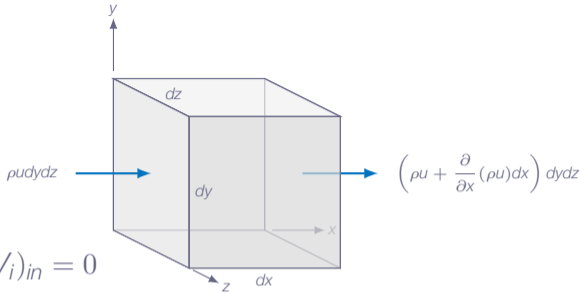
The substantial derivative is an operator that can be applied to any variable

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p$$

Roadmap - Differential Relations



Mass Conservation



$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

$$\int_{CV} \frac{\partial \rho}{\partial t} dV \approx \frac{\partial \rho}{\partial t} dx dy dz$$

$$\frac{\partial \rho}{\partial t} dx dy dz + \frac{\partial}{\partial x} (\rho u) dx dy dz + \frac{\partial}{\partial y} (\rho v) dx dy dz + \frac{\partial}{\partial z} (\rho w) dx dy dz = 0$$

Mass Conservation

The result is the continuity equation - conservation of mass for an infinitesimal control volume

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

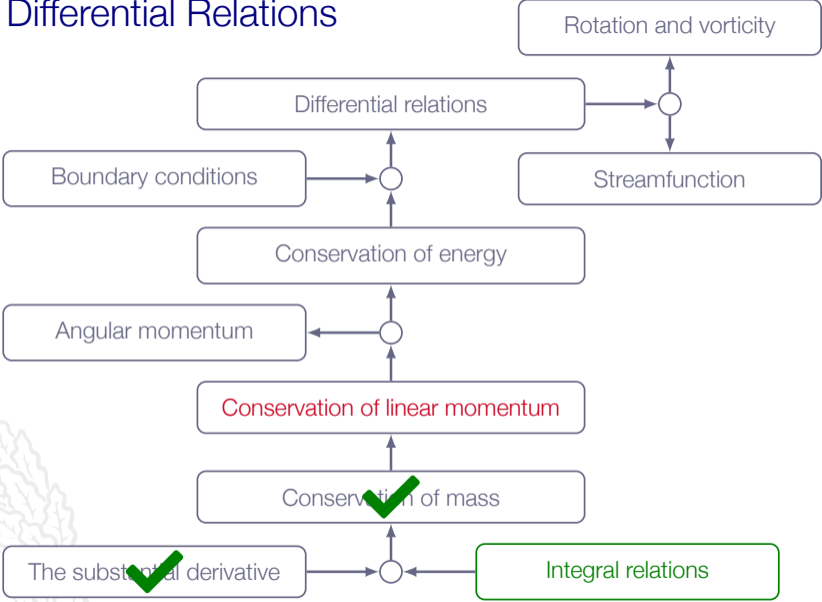
or in more compact form using vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Incompressible flow (constant density)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} = 0$$

Roadmap - Differential Relations



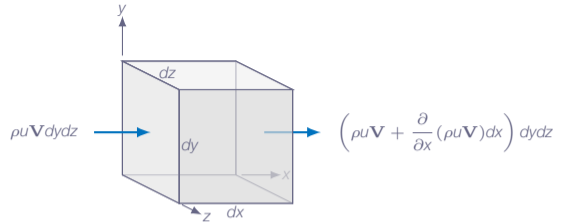
Linear Momentum

$$\sum \mathbf{F} = \int_{CV} \frac{\partial}{\partial t} (\mathbf{V}\rho) d\mathcal{V} + \sum (\dot{m}_i \mathbf{V}_i)_{out} - \sum (\dot{m}_i \mathbf{V}_i)_{in}$$

$$\frac{\partial}{\partial t} (\mathbf{V}\rho) d\mathcal{V} \approx \frac{\partial}{\partial t} (\mathbf{V}\rho) dx dy dz$$



Linear Momentum



Face	Inlet momentum flux	Outlet momentum flux
------	---------------------	----------------------

x	$\rho u \mathbf{V} dy dz$	
---	---------------------------	--

		$\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V}) dx \right] dy dz$
--	--	--

y	$\rho v \mathbf{V} dx dz$	
---	---------------------------	--

		$\left[\rho v \mathbf{V} + \frac{\partial}{\partial y}(\rho v \mathbf{V}) dy \right] dx dz$
--	--	--

z	$\rho w \mathbf{V} dx dy$	
---	---------------------------	--

		$\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V}) dz \right] dx dy$
--	--	--

Linear Momentum

$$\sum \mathbf{F} = \left[\frac{\partial}{\partial t}(\mathbf{V}\rho) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho v \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V}) \right] dx dy dz$$



Linear Momentum

$$\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\rho)}_{\mathbf{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\mathbf{V}}{\partial t}} + \underbrace{\frac{\partial}{\partial x}(\rho u\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial x}(\rho u)+\rho u\frac{\partial\mathbf{V}}{\partial x}} + \underbrace{\frac{\partial}{\partial y}(\rho v\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial y}(\rho v)+\rho v\frac{\partial\mathbf{V}}{\partial y}} + \underbrace{\frac{\partial}{\partial z}(\rho w\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial z}(\rho w)+\rho w\frac{\partial\mathbf{V}}{\partial z}}$$

can be rewritten as

$$\mathbf{V} \underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) \right]}_{\text{continuity equation}} + \rho \underbrace{\left(\frac{\partial\mathbf{V}}{\partial t} + u\frac{\partial\mathbf{V}}{\partial x} + v\frac{\partial\mathbf{V}}{\partial y} + w\frac{\partial\mathbf{V}}{\partial z} \right)}_{\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{D\mathbf{V}}{Dt}}$$

and thus

$$\sum \mathbf{F} = \rho \frac{D\mathbf{V}}{Dt} dx dy dz$$

Linear Momentum - Forces

$$\sum \mathbf{F} = \rho \frac{D\mathbf{V}}{Dt} dx dy dz$$

$\sum \mathbf{F}$:

body forces: gravity and other field forces

surface forces: pressure and viscous stresses

Linear Momentum - Gravity Force

$$d\mathbf{F}_{gravity} = \rho \mathbf{g} dx dy dz$$

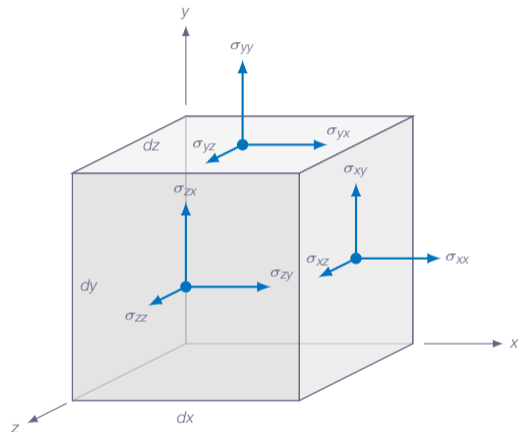
if gravity is aligned with the negative z-direction

$$d\mathbf{F}_{gravity} = -\mathbf{e}_z \rho g dx dy dz$$

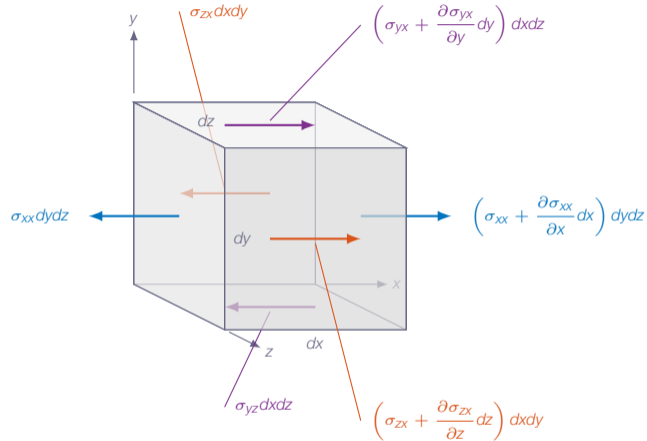


Linear Momentum - Surface Forces

$$\sigma_{ij} = \begin{vmatrix} -\rho + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -\rho + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -\rho + \tau_{zz} \end{vmatrix}$$



Linear Momentum - Surface Forces



$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx}) \right] dx dy dz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx}) \right] dx dy dz$$

$$\sigma_{xx} = \tau_{xx} - p, \quad \sigma_{yx} = \tau_{yx}, \quad \sigma_{zx} = \tau_{zx}$$

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \right] dx dy dz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \right] dx dy dz$$

$$dF_{y,surf} = \left[-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy}) \right] dx dy dz$$

$$dF_{z,surf} = \left[-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz}) \right] dx dy dz$$



Linear Momentum - Surface Forces

$$d\mathbf{F} = [-\nabla p + \nabla \cdot \boldsymbol{\tau}_{ij}] dx dy dz$$

where

$$\boldsymbol{\tau}_{ij} = \begin{vmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{vmatrix}$$

is the **viscous stress tensor**

Linear Momentum - Forces

Now, inserting the forces into the momentum equation gives

$$\rho \mathbf{g} - \nabla \rho + \nabla \cdot \boldsymbol{\tau}_{ij} = \rho \frac{D\mathbf{V}}{Dt}$$



Linear Momentum

vector notation is powerfull, tensor notation is even better ...

$$\rho g_x - \frac{\partial \rho}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y - \frac{\partial \rho}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z - \frac{\partial \rho}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$

Note! the convective term (RHS) is nonlinear

Linear Momentum

Recall:

"For a Newtonian fluid, the viscous stresses are proportional to the element strain and the viscosity"

For incompressible flow:

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y}, \quad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

The incompressible Navier-Stokes equations

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

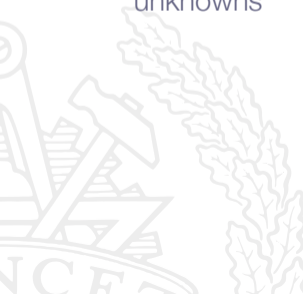
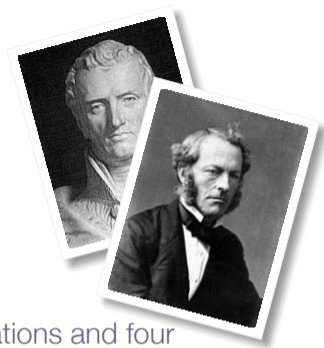
$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right)$$



The incompressible Navier-Stokes equations

- ▶ Non-linear equations
- ▶ Three equations and four unknowns (p, u, v, w)
- ▶ Combined with the continuity equations we have four equations and four unknowns



Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang–Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part $1/2$.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

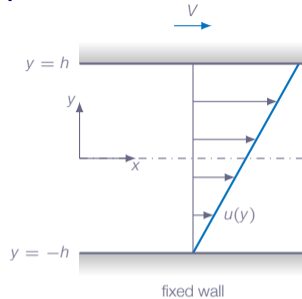
Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Example - Couette Flow



- ▶ incompressible ($\rho = \text{const}$)
- ▶ steady-state
- ▶ lower plate fixed, upper plate moving with the velocity V
- ▶ flow only in the x -direction $v = w = 0$, $u \neq 0$
- ▶ no pressure gradient

Example - Couette Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0, \frac{\partial p}{\partial x} = 0\} \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = 0$$

Example - Couette Flow

$$\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u = ay + b$$

boundary conditions:

$$\left. \begin{array}{l} u(h) = V \\ u(-h) = 0 \end{array} \right\} \Rightarrow$$

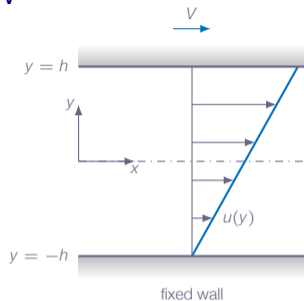
$$\begin{array}{r} V = ah + b \\ + \quad 0 = -ah + b \\ \hline V = 2b \end{array}$$

$$b = \frac{V}{2}$$

$$\begin{array}{r} V = ah + b \\ - \quad 0 = -ah + b \\ \hline V = 2ah \end{array}$$

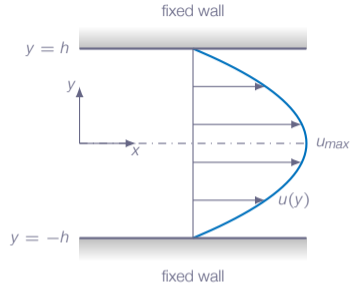
$$a = \frac{V}{2h}$$

Example - Couette Flow



$$u = \frac{V}{2} \left(\frac{y}{h} - 1 \right)$$

Example - Poiseuille Flow



- ▶ incompressible ($\rho = const$)
- ▶ steady-state
- ▶ lower and upper plate fixed
- ▶ flow only in the x -direction $v = w = 0$, $u \neq 0$
- ▶ pressure gradient driven

Example - Poiseuille Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0\} \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

Example - Poiseuille Flow

momentum equation (y-direction and z-direction):

$$\left. \begin{aligned} \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial p}{\partial z} &= 0 \end{aligned} \right\} \Rightarrow p = p(x)$$



Example - Poiseuille Flow

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = \text{const} < 0$$

- ▶ Why constant?
 - ▶ RHS function of x only
 - ▶ LHS function of y only
 - ▶ RHS=LHS \Rightarrow must be a constant
- ▶ Why < 0 ?
 - ▶ pressure must decrease in the flow direction

Example - Poiseuille Flow

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = \text{const} < 0$$

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + ay + b$$



Example - Poiseuille Flow

boundary conditions:

$$\left. \begin{aligned} u(h) &= 0 \\ u(-h) &= 0 \end{aligned} \right\} \Rightarrow$$

$$0 = \frac{1}{2\mu} \frac{dp}{dx} h^2 + ah + b$$

$$+ 0 = \frac{1}{2\mu} \frac{dp}{dx} h^2 - ah + b$$

$$0 = \frac{1}{\mu} \frac{dp}{dx} h^2 + 2b$$

$$b = -\frac{1}{2\mu} \frac{dp}{dx} h^2$$

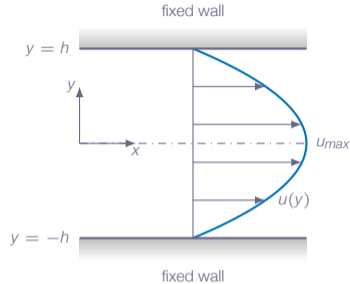
$$0 = \frac{1}{2\mu} \frac{dp}{dx} h^2 + ah + b$$

$$- 0 = \frac{1}{2\mu} \frac{dp}{dx} h^2 - ah + b$$

$$0 = 2ah$$

$$a = 0$$

Example - Poiseuille Flow

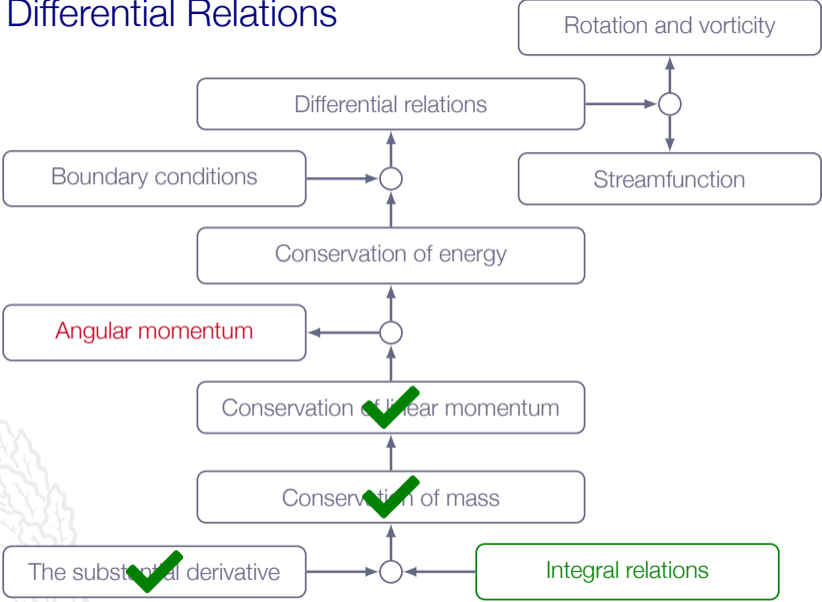


$$u = -\frac{h^2}{2\mu} \frac{dp}{dx} \left(1 - \left(\frac{y}{h} \right)^2 \right)$$

$$\frac{du}{dy} = \frac{dp}{dx} \frac{y}{\mu} \Rightarrow \left. \frac{du}{dy} \right|_{y=0} = 0 \Rightarrow u_{max} = u(0) = -\frac{dp}{dx} \frac{h^2}{2\mu}$$

(remember: $dp/dx < 0$)

Roadmap - Differential Relations



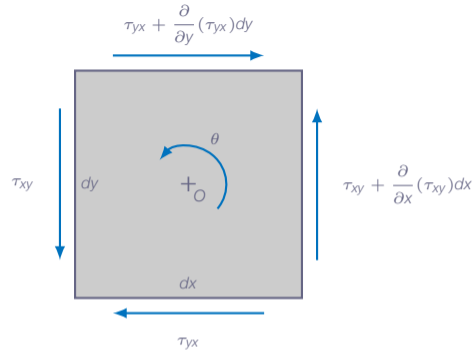
Angular Momentum

$$\sum \mathbf{M}_o = \frac{d}{dt} \left(\int_{CV} \rho(\mathbf{r} \times \mathbf{V}) d\mathcal{V} \right) + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho(\mathbf{V} \cdot \mathbf{n}) dA$$



Angular Momentum

- ▶ axis through o parallel to the z -axis
- ▶ axis through the centroid of the element
- ▶ θ angle of rotation about o



$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2} \frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2} \frac{\partial}{\partial y}(\tau_{yx})dy \right] dx dy dz =$$

$$\frac{1}{12} \rho(dx dy dz)(dx^2 + dy^2) \frac{d^2\theta}{dt^2}$$

Angular Momentum

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2} \frac{\partial}{\partial x} (\tau_{xy}) dx - \frac{1}{2} \frac{\partial}{\partial y} (\tau_{yx}) dy \right] dx dy dz =$$

$$\frac{1}{12} \rho (dx dy dz) (dx^2 + dy^2) \frac{d^2 \theta}{dt^2}$$

Neglect higher-order differential terms gives

$$\tau_{xy} \approx \tau_{yx}$$

Analogously, we may obtain $\tau_{xz} \approx \tau_{zx}$ and $\tau_{zy} \approx \tau_{yz}$

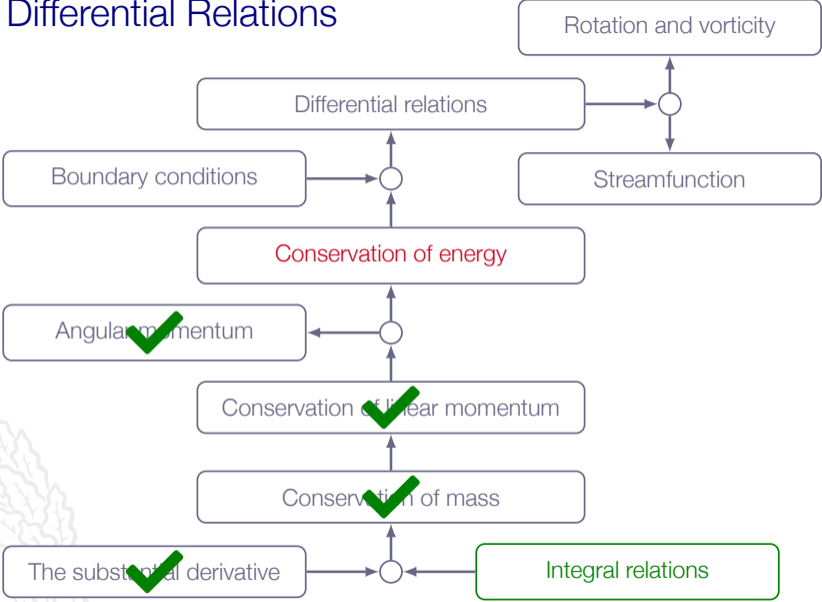
Angular Momentum

Note! there is no differential angular momentum equation ...

the only result from this section is that shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$



Roadmap - Differential Relations



The Energy Equation

Integral formulation:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt} \left(\int_{CV} e \rho dV \right) + \int_{CS} \left(e + \frac{p}{\rho} \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

$$h = e + p/\rho$$

Differential form:

$$\dot{Q} - \dot{W}_\nu = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) \right] dx dy dz$$

$\dot{W}_s = 0$ we can not have a infinitesimal shaft protruding the control volume

The Energy Equation

$$\dot{Q} - \dot{W}_v = \left[\underbrace{\frac{\partial}{\partial t}(\rho e)}_I + \underbrace{\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)}_{II} \right] dx dy dz$$

Part I.

$$\frac{\partial}{\partial t}(\rho e) = e \frac{\partial \rho}{\partial t} + \rho \frac{\partial e}{\partial t}$$

The Energy Equation

Part II.

$$\frac{\partial}{\partial x}(\rho uh) + \frac{\partial}{\partial y}(\rho vh) + \frac{\partial}{\partial z}(\rho wh) =$$
$$\underbrace{\frac{\partial}{\partial x}(\rho ue) + \frac{\partial}{\partial y}(\rho ve) + \frac{\partial}{\partial z}(\rho we)}_* + \underbrace{\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp)}_{**}$$

Part II*

$$\frac{\partial}{\partial x}(\rho ue) + \frac{\partial}{\partial y}(\rho ve) + \frac{\partial}{\partial z}(\rho we) =$$
$$e \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] + \rho \left[u \frac{\partial e}{\partial x} + v \frac{\partial e}{\partial y} + w \frac{\partial e}{\partial z} \right]$$

The Energy Equation

Part II**

$$\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp) =$$

$$\rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} =$$

$$\rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho$$



The Energy Equation

reassemble and collect terms:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) =$$

$$\rho \left[\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} + v \frac{\partial e}{\partial y} + w \frac{\partial e}{\partial z} \right] +$$
$$\underbrace{\hspace{10em}}_{\frac{\partial e}{\partial t} + (\mathbf{V} \cdot \nabla) e = \frac{De}{Dt}}$$

$$e \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] +$$
$$\underbrace{\hspace{10em}}_{\text{continuity equation}}$$

$$\rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho$$

The Energy Equation

$$\dot{Q} - \dot{W}_v = \left[\rho \frac{De}{Dt} + p \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p \right] dx dy dz$$



The Energy Equation - Added Heat

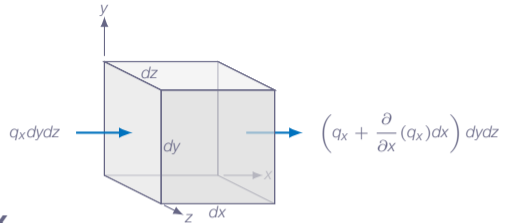
Now, let's have a look at the added heat term \dot{Q}

- ▶ Only **conduction** will be considered (no radiation)
- ▶ According the **Fourier's law** of conduction, the **heat flux** is proportional to the **temperature gradient**

$$\mathbf{q} = -k\nabla T$$

where k is the **thermal conductivity** and \mathbf{q} is heat transfer per unit area

The Energy Equation - Added Heat



Face	Inlet heat flux	Outlet heat flux
x	$q_x dy dz$	$\left[q_x + \frac{\partial q_x}{\partial x} dx \right] dy dz$ where $q_x = -k \frac{\partial T}{\partial x}$
y	$q_y dx dz$	$\left[q_y + \frac{\partial q_y}{\partial y} dy \right] dx dz$ where $q_y = -k \frac{\partial T}{\partial y}$
z	$q_z dx dy$	$\left[q_z + \frac{\partial q_z}{\partial z} dz \right] dx dy$ where $q_z = -k \frac{\partial T}{\partial z}$

The Energy Equation - Added Heat

net added heat:

$$\dot{Q} = - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] dx dy dz = -\nabla \cdot \mathbf{q} dx dy dz$$

or

$$\dot{Q} = \nabla \cdot (k \nabla T) dx dy dz$$

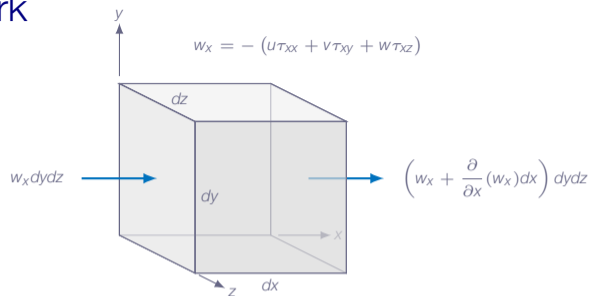


The Energy Equation - Viscous Work

The rate of work done by viscous stresses equals the product of the **stress component**, its corresponding **velocity component** and **surface area**



The Energy Equation - Viscous Work



$$\dot{W}_\nu = - \left[\frac{\partial}{\partial x} (u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \right.$$

$$\left. \frac{\partial}{\partial y} (u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z} (u\tau_{zx} + v\tau_{zy} + w\tau_{zz}) \right] =$$

$$-\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) dx dy dz$$



The Energy Equation

with the derived expressions for heat and viscous work we end up with

$$\nabla \cdot (k \nabla T) + \nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) = \rho \frac{De}{Dt} + \rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho$$



The Energy Equation

Now, introducing the **viscous-dissipation function** ϕ for Newtonian fluids and incompressible flows

$$\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) = \mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) + \phi$$

where

$$\phi = \mu \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right]$$

The Energy Equation

Note!

"All terms in the viscous-dissipation function are quadratic which means that in a viscous flow there will always be losses, which is in line with the second law of thermodynamics"



The Energy Equation

Now, let's eliminate the term $\mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij})$ in the energy equation:

Momentum equation:

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla \rho + \nabla \cdot \boldsymbol{\tau}_{ij}$$

Multiply the momentum equation with the velocity vector (scalar product)

$$\mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) = \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} - \mathbf{V} \cdot \nabla \rho$$

The Energy Equation

Energy equation:

$$\rho \frac{De}{Dt} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) + \phi$$

eliminate $\mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij})$ using the result from previous slide

$$\rho \frac{De}{Dt} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla \rho + \phi$$

The Energy Equation

Finally, write out energy per unit mass (e)

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

Apply the substantial derivative to e

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \frac{1}{2} \frac{D}{Dt}(\mathbf{V} \cdot \mathbf{V}) - \frac{D}{Dt}(\mathbf{g}\mathbf{x}) = \frac{D\hat{u}}{Dt} + \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \underbrace{\left(\frac{\partial}{\partial t}(\mathbf{g}\mathbf{x}) + (\mathbf{V} \cdot \nabla)(\mathbf{g}\mathbf{x}) \right)}_{\frac{D}{Dt}(\mathbf{g}\mathbf{x}) = \mathbf{V} \cdot \mathbf{g}}$$

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \mathbf{V} \cdot \mathbf{g}$$

The Energy Equation

Now, insert

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \mathbf{V} \cdot \mathbf{g}$$

in the energy equation

$$\rho \frac{D\hat{u}}{Dt} + \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + \rho \nabla \cdot \mathbf{V} =$$
$$\nabla \cdot (k \nabla T) + \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + \phi$$

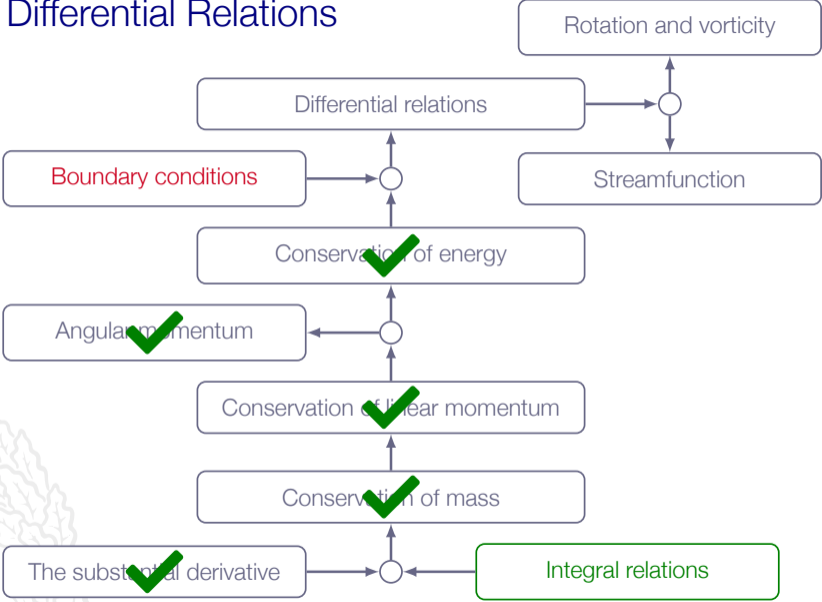
The highlighted terms cancel each other

The Energy Equation

$$\rho \frac{D\hat{u}}{Dt} + \rho \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \phi$$



Roadmap - Differential Relations



Flow Equations on Differential Form

Continuity:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Momentum:
$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Energy:
$$\rho \frac{D\hat{u}}{Dt} + \rho \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \phi$$

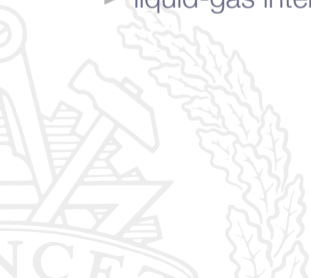
five equations and seven unknowns $(\rho, u, v, w, p, \hat{u}, T) \Rightarrow$ two additional relations needed:

$$\rho = \rho(p, T), \quad \hat{u} = \hat{u}(p, T)$$

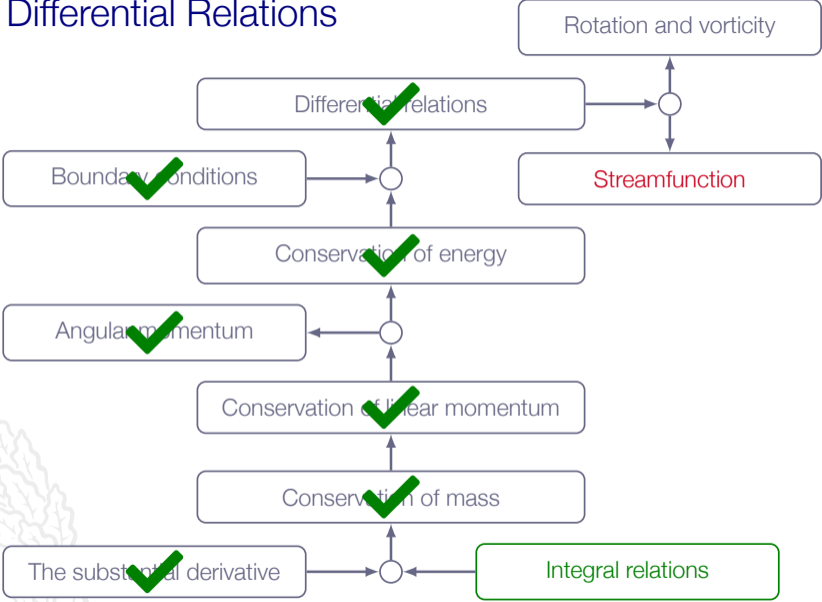
Flow Equations on Differential Form

Boundary conditions:

- ▶ solid wall: no slip, no temperature jump
- ▶ inlet, outlet
- ▶ liquid-gas interface



Roadmap - Differential Relations



The Streamfunction

fulfil the continuity equation and solve the momentum equation directly for the single variable ψ



The Streamfunction

incompressible, two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

define $\psi(x, y)$ such that

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \psi}{\partial x} \right) = 0$$

and thus

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

or

$$\mathbf{V} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right]$$

The Streamfunction (*for the interested*)

The rotation of the flow field is calculated using the curl operator

$$\text{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = -\nabla^2 \psi \mathbf{e}_z$$



The Streamfunction (*for the interested*)

Now, apply the curl operator to the momentum equation

$$\nabla \times \frac{D\mathbf{V}}{Dt} = \underbrace{\nabla \times \mathbf{g}}_{=0} - \frac{1}{\rho} \underbrace{\nabla \times \nabla p}_{=0} + \nu \nabla \times \nabla^2 \mathbf{V} = \nu \nabla \times \nabla^2 \mathbf{V}$$

$$\nabla \times \frac{\partial \mathbf{V}}{\partial t} + \nabla \times (\mathbf{V} \cdot \nabla) \mathbf{V} = \nu \nabla \times \nabla^2 \mathbf{V}$$

$$\left. \begin{array}{l} \frac{\partial \mathbf{V}}{\partial t} = 0 \text{ (steady)} \\ \nu \nabla \times \nabla^2 \mathbf{V} = \nu \nabla^2 (\nabla \times \mathbf{V}) \end{array} \right\} \Rightarrow \nabla \times (\mathbf{V} \cdot \nabla) \mathbf{V} = \nu \nabla^2 (\nabla \times \mathbf{V})$$

The Streamfunction (*for the interested*)

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times (\nabla \times \mathbf{V}) = \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times (\nabla \times \mathbf{V})$$

and thus

$$\nabla \times (\mathbf{V} \cdot \nabla)\mathbf{V} = \underbrace{\nabla \times \nabla\left(\frac{V^2}{2}\right)}_{=0} - \nabla \times \mathbf{V} \times (\nabla \times \mathbf{V}) = \nabla \times (\nabla \times \mathbf{V}) \times \mathbf{V}$$

$$\nabla \times (\nabla \times \mathbf{V}) \times \mathbf{V} =$$

$$(\mathbf{V} \cdot \nabla)(\nabla \times \mathbf{V}) - ((\nabla \times \mathbf{V}) \cdot \nabla)\mathbf{V} + \underbrace{(\nabla \times \mathbf{V})(\nabla \cdot \mathbf{V})}_{=0 \text{ (incompressible)}} + \mathbf{V} \underbrace{(\nabla \cdot (\nabla \times \mathbf{V}))}_{=0} =$$

$$(\mathbf{V} \cdot \nabla)(\nabla \times \mathbf{V}) - ((\nabla \times \mathbf{V}) \cdot \nabla)\mathbf{V}$$

The Streamfunction (*for the interested*)

$$(\mathbf{V} \cdot \nabla)(\nabla \times \mathbf{V}) - ((\nabla \times \mathbf{V}) \cdot \nabla)\mathbf{V} = \nu \nabla^2(\nabla \times \mathbf{V})$$

insert the streamfunction

$$(\mathbf{V} \cdot \nabla)(\nabla \times \mathbf{V}) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(0, 0, -\nabla^2 \psi)$$

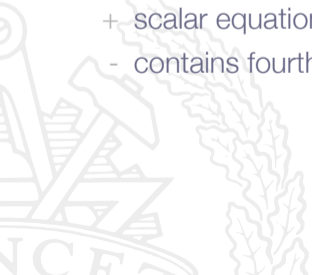
$$((\nabla \times \mathbf{V}) \cdot \nabla)\mathbf{V} = (0, 0, -\nabla^2 \psi) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right) = 0$$

$$\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}(\nabla^2 \psi) = \nu \nabla^2(\nabla^2 \psi)$$

Streamfunction

$$\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \psi) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \psi) = \nu \nabla^2 (\nabla^2 \psi)$$

- + one equation for ψ that fulfils both the momentum and continuity equations
- + scalar equation
- contains fourth-order derivatives



Streamfunction

Definition of a streamline in two dimensions:

$$\frac{dx}{u} = \frac{dy}{v}$$

or

$$udy - vdx = 0$$

and thus

$$\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = d\psi = 0$$

or ψ is constant along a streamline ...

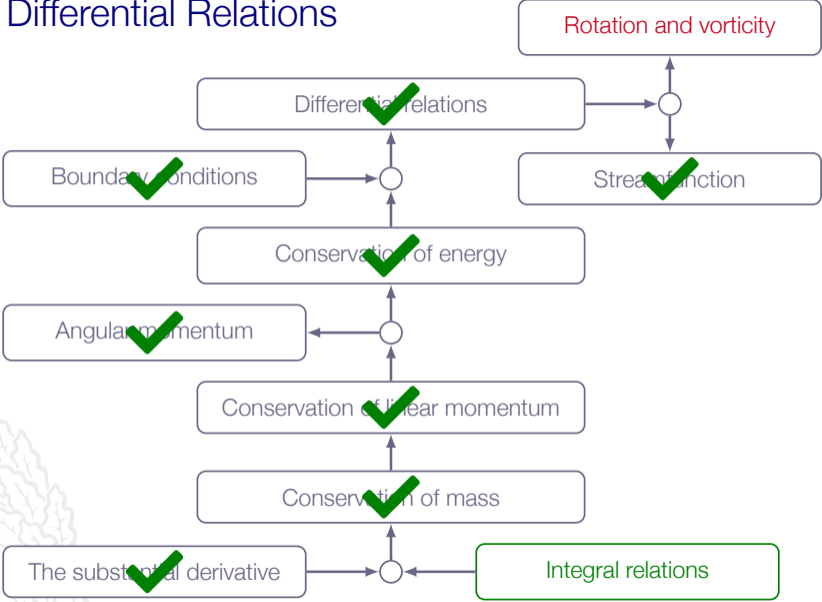
Streamfunction

Implication:

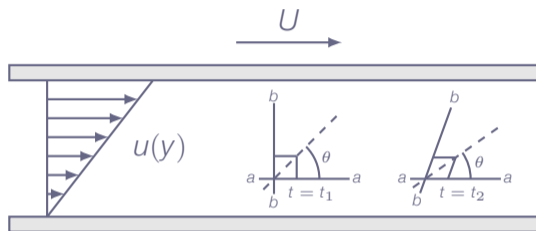
- ▶ Lines of constant ψ are streamlines of the flow
- ▶ If we know $\psi(x, y)$, lines of constant ψ will be streamlines of the flow



Roadmap - Differential Relations

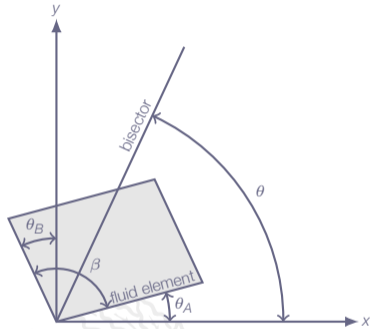


Flow Rotation



- ▶ Is the Couette flow irrotational?
- ▶ Note the change of the fluid element bisector angle θ

Flow Rotation



$$\beta = \frac{\pi}{2} + \theta_B - \theta_A$$

$$\theta = \frac{\beta}{2} + \theta_A = \frac{\pi}{4} + \frac{1}{2}(\theta_A + \theta_B)$$

the angular velocity of the bisector:

$$\dot{\theta} = \frac{1}{2} (\dot{\theta}_A + \dot{\theta}_B)$$

Flow Rotation

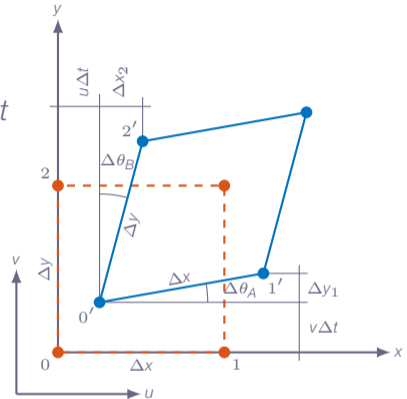
$$\sin(\Delta\theta_A) = \frac{\Delta y_1}{\Delta x} \approx \frac{(v + \frac{\partial v}{\partial x} \Delta x) \Delta t - v \Delta t}{\Delta x} = \frac{\partial v}{\partial x} \Delta t$$

$\sin(\Delta\theta_A) \approx \Delta\theta_A$ for small angles

$$\Rightarrow \underbrace{\frac{\Delta\theta_A}{\Delta t}}_{=\dot{\theta}_A} = \frac{\partial v}{\partial x}$$

in the same way $\dot{\theta}_B \approx -\frac{\partial u}{\partial y}$

the angular velocity of the bisector: $\dot{\theta} = \frac{1}{2} (\dot{\theta}_A + \dot{\theta}_B) = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$



Flow Rotation

From previous slide we get the angular velocity about the z axis

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Using the same reasoning, we can get the angular velocities about the x and y axes

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)$$

$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$



Flow Rotation

$$\boldsymbol{\omega} = \frac{1}{2} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2} \text{curl}(\mathbf{V})$$

The flow **vorticity** ζ is defined as:

$$\zeta = 2\boldsymbol{\omega} = \text{curl}(\mathbf{V})$$

Flows with **zero vorticity** are called **irrotational**

Frictionless Irrotational Flow

If the flow is both **frictionless** and **irrotational**:

1. the momentum equation reduces to Euler's equation

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p$$

2. the acceleration term can be simplified

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V}$$

where we can use the vector identity

$$(\mathbf{V} \cdot \nabla) \mathbf{V} = \nabla \left(\frac{1}{2} V^2 \right) + \zeta \times \mathbf{V}$$

Frictionless Flow

1. combine Euler's equation with the modified acceleration term
2. divide by ρ
3. dot product between the entire equation and an arbitrary displacement vector $d\mathbf{r}$

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla \left(\frac{1}{2} V^2 \right) + \boldsymbol{\zeta} \times \mathbf{V} + \frac{1}{\rho} \nabla p - \mathbf{g} \right] \cdot d\mathbf{r} = 0$$



Frictionless Flow

Now we want to get rid of the term $(\zeta \times \mathbf{V}) \cdot d\mathbf{r}$

1. $\mathbf{V} = 0$; no flow - not interesting
2. $\zeta = 0$; irrotational flow
3. $d\mathbf{r}$ perpendicular to $(\zeta \times \mathbf{V})$; strange
4. $d\mathbf{r}$ parallel to \mathbf{V} ; integrate along a streamline



Frictionless Flow

Fourth alternative: integrate along a streamline:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla \left(\frac{1}{2} V^2 \right) + \frac{1}{\rho} \nabla p - \mathbf{g} \right] \cdot d\mathbf{r} = 0$$

performing the scalar products gives

$$-\mathbf{g} \cdot d\mathbf{r} = \{ \mathbf{g} = -g\mathbf{e}_z \} = g dz$$

$$\nabla p \cdot d\mathbf{r} = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = dp$$

$$\nabla \left(\frac{1}{2} V^2 \right) \cdot d\mathbf{r} = \frac{1}{2} \left(\frac{\partial V^2}{\partial x} dx + \frac{\partial V^2}{\partial y} dy + \frac{\partial V^2}{\partial z} dz \right) = \frac{1}{2} d(V^2)$$

Frictionless Flow

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + \frac{1}{2}d(V^2) + \frac{dp}{\rho} + gdz = 0$$



Frictionless Flow

Integrate between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

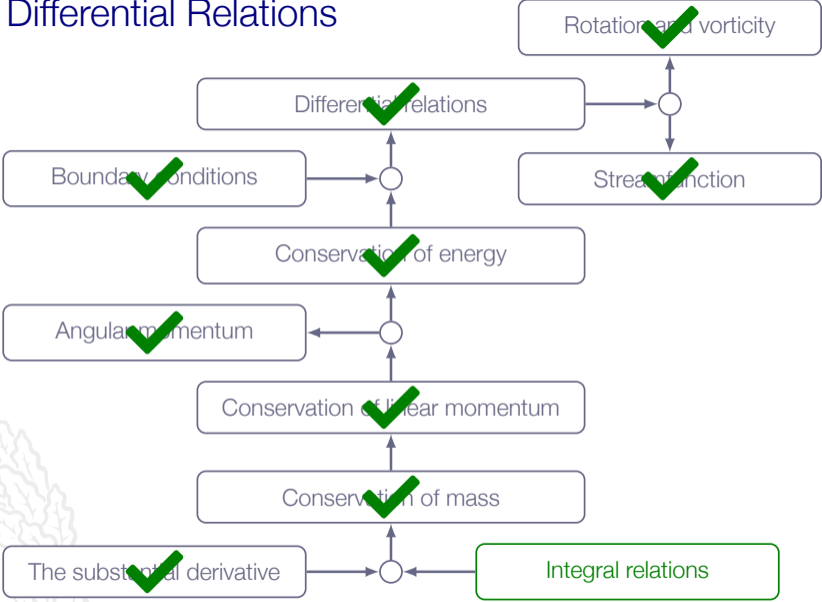
The Bernoulli equation frictionless unsteady flow

Steady incompressible flow gives

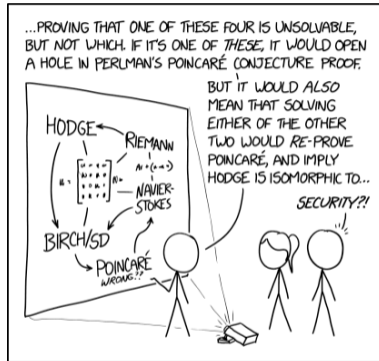
$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant}$$

Note! for irrotational flow this last results holds in the entire flow field with the same constant

Roadmap - Differential Relations



Millennium Problems

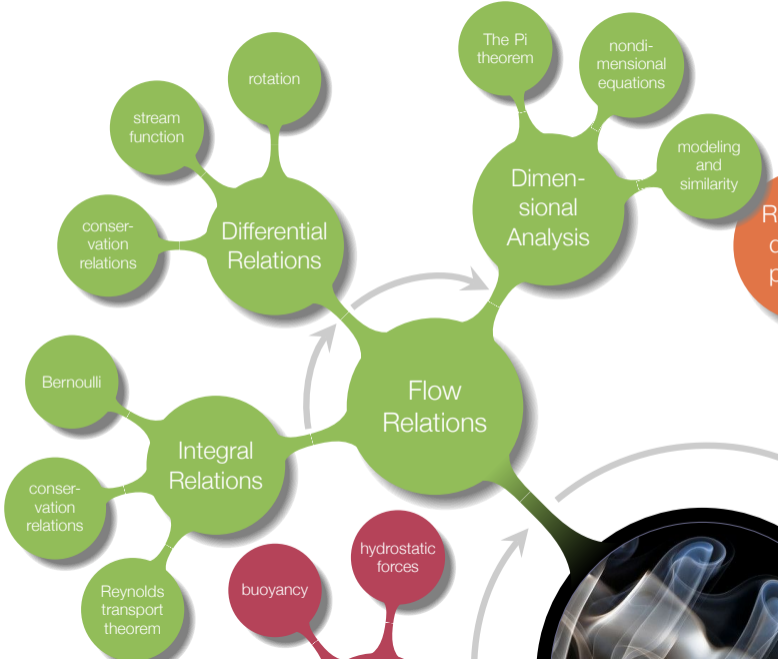


I'M TRYING TO MAKE IT SO THE CLAY MATHEMATICS INSTITUTE HAS TO OFFER AN EIGHTH PRIZE TO WHOEVER FIGURES OUT WHO THEIR OTHER PRIZES SHOULD GO TO.



Chapter 5 - Dimensional Analysis and Similarity

Overview

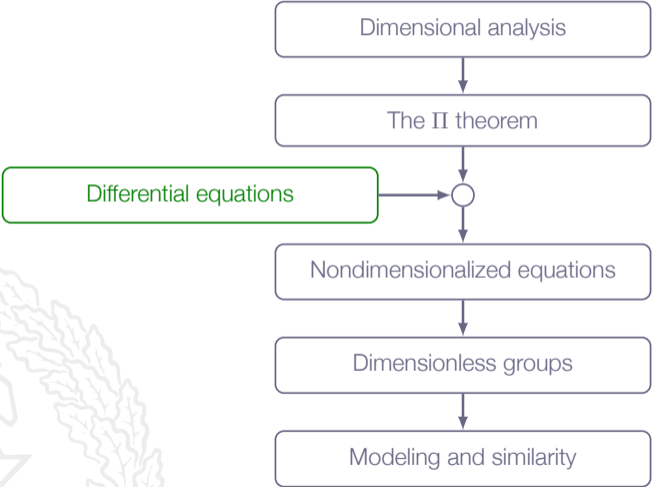


Learning Outcomes

- 3 **Define** the Reynolds number
- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 17 **Explain** about how to use nondimensional numbers and the Π theorem

we will learn about how to plan experiments and compare experimental data using dimensionless numbers

Roadmap - Dimensional Analysis and Similarity



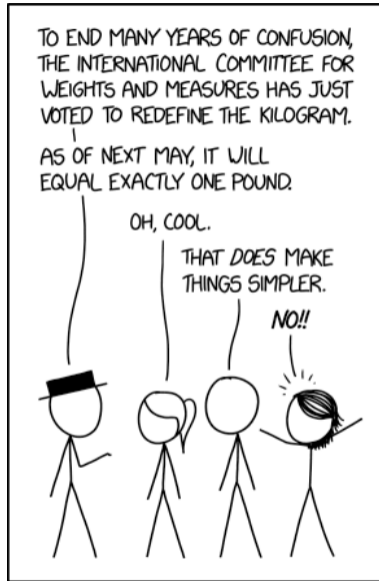
Motivation

"Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiments or approximated by CFD"

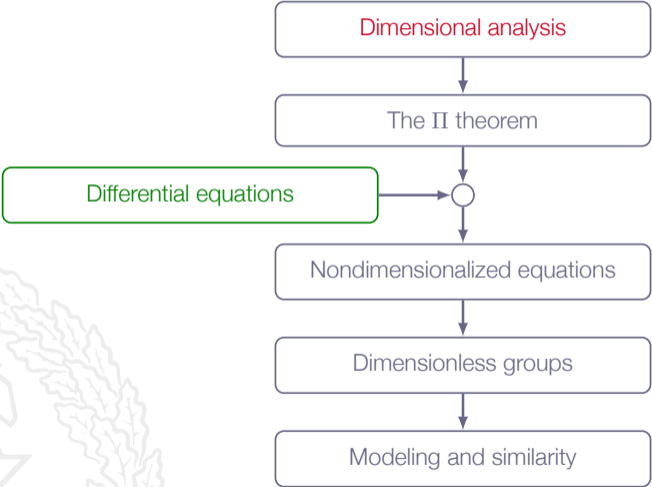
Dimensional analysis:

- ▶ Large data sets may be represented by a **few curves** or even a single curve
- ▶ A systematic tool for **data reduction**
- ▶ Experimental/simulation data are **more general** in **dimensionless** form

Dimensions



Roadmap - Dimensional Analysis and Similarity



Dimensional Analysis

Dimensional analysis is a tool for systematic

planning of experiments

presentation of experimental data

interpretation of measurements



Dimensional Analysis

"If a phenomenon depends on n dimensional variables, dimensional analysis will reduce the problem to only k dimensionless variables, where the reduction $n - k$ depends on the problem complexity"

"Generally, $n - k$ equals the number of primary dimensions"



Dimensional Analysis

Suppose that we know that the force F on a particular body shape in a fluid flow depends on

- ▶ The length of the body L
- ▶ The flow velocity V
- ▶ The fluid density ρ
- ▶ The fluid viscosity μ

$$F = f(L, V, \rho, \mu)$$



Dimensional Analysis

Let's say that we need ten points to define a curve

We need to test 10 lengths and for each of those, 10 velocities,

The result in this case would be 10000 experiments

With dimensional analysis, the problem can be reduced as follows

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \text{ or } C_F = g(Re)$$

Dimensional Analysis



Dimensional analysis:

- ▶ Gives insight into physical relationships
- ▶ Helps in identifying important and unimportant parameters for a specific problem
- ▶ Provides scaling laws
 - ▶ convert data from model-scale to prototype-scale
 - ▶ similarity between model and prototype

Similarity

Let's go back to the force example from before

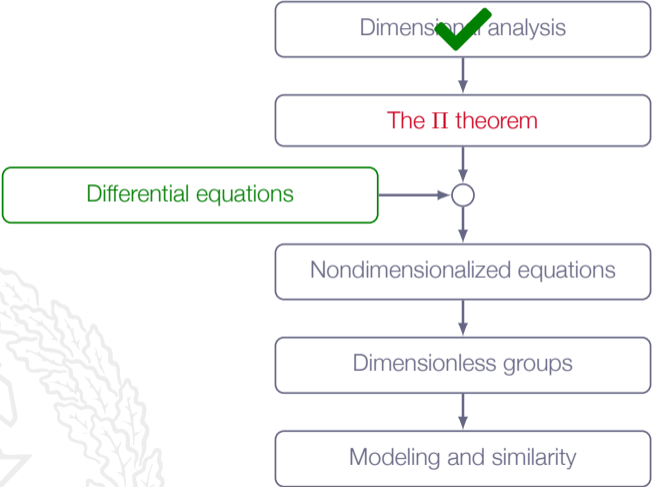
$$C_F = g(Re)$$

so if $Re_m = Re_p$ that means that $C_{F,m} = C_{F,p}$ (where m is model and p prototype)

$$C_{F,m} = \frac{F_m}{\rho_m V_m^2 L_m^2} \text{ and } C_{F,p} = \frac{F_p}{\rho_p V_p^2 L_p^2}$$

$$\frac{F_m}{\rho_m V_m^2 L_m^2} = \frac{F_p}{\rho_p V_p^2 L_p^2} \Rightarrow \frac{F_p}{F_m} = \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m} \right)^2 \left(\frac{L_p}{L_m} \right)^2$$

Roadmap - Dimensional Analysis and Similarity



The Buckingham Π -theorem

"If there is a physically meaningful equation involving a certain number n of physical variables, then the original equation can be rewritten in terms of a set of k dimensionless parameters $\Pi_1, \Pi_2, \dots, \Pi_k$. The reduction $j = n - k$ equals the number of variables that do not form a Π among themselves and is always less than or equal to the number of physical dimensions involved"



The Buckingham Π -theorem

Systematic identification of Π groups:

1. List and count the **number of variables** in the problem n
2. List the **dimensions** for each of the n variables
3. Find the **reduction j**
 - ▶ initial guess: j equals the number of dimensions
 - ▶ look for j variables that do not form a Π
 - ▶ if not possible reduce j by one and start over
4. Select **j scaling parameters**
5. Add one of the other variables to your j repeating variables and form a power product
6. Algebraically, find exponents that make the product dimensionless

The Buckingham Π -theorem - Example

$$F = f(L, U, \rho, \mu)$$

number of variables: $n = 5$

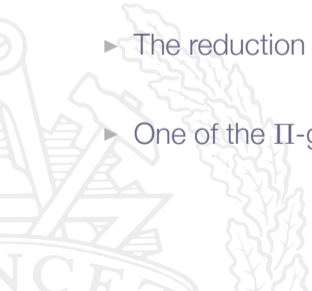
F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

number of dimensions is 3 and thus the reduction is: $j \leq 3$

number of dimensionless groups (Π s): $k = n - j \geq 2$

The Buckingham Π -theorem - Example

- ▶ Inspecting the variables, we see that L , U , and ρ cannot form a Π -group
 - ▶ only ρ contains M (mass)
 - ▶ only U contains T (time)
- ▶ L , U , and ρ are selected as the j repeating variables
- ▶ The reduction will be $j = 3$ and thus $k = n - j = 2$
- ▶ One of the Π -groups will contain F and the other will contain μ



The Buckingham Π -theorem - Example

$$\Pi_1 = L^a U^b \rho^c F \Rightarrow (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

$$\begin{array}{l} L: a + b - 3c + 1 = 0 \\ M: \quad \quad \quad c + 1 = 0 \\ T: \quad \quad -b \quad \quad -2 = 0 \end{array}$$

which gives

$$a = -2, b = -2, c = -1$$

and thus

$$\Pi_1 = \frac{F}{\rho U^2 L^2} = C_F$$

The Buckingham Π -theorem - Example

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} \Rightarrow (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

$$\begin{array}{l} L: a + b - 3c + 1 = 0 \\ M: \quad \quad \quad c - 1 = 0 \\ T: \quad \quad -b \quad \quad + 1 = 0 \end{array}$$

which gives

$$a = b = c = 1$$

and thus

$$\Pi_2 = \frac{\rho UL}{\mu} = Re$$

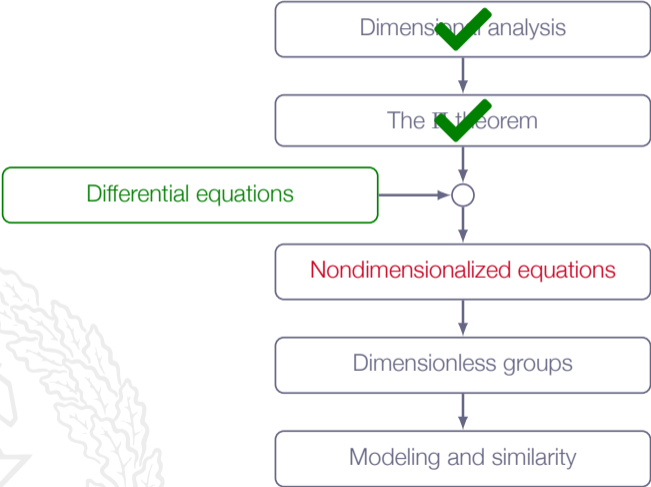
The Buckingham Π -theorem - Example

If $F = f(L, V, \rho, \mu)$, the theorem guaranties that, in this case, $\Pi_1 = g(\Pi_2)$

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho UL}{\mu}\right) \text{ or } C_F = g(Re)$$



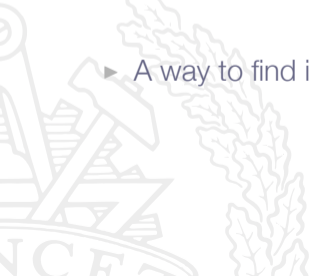
Roadmap - Dimensional Analysis and Similarity



Nondimensionalized Equations

Why would one want to make the governing equations nondimensional?

- ▶ Understand flow physics
- ▶ Gives information about under what conditions terms are negligible
- ▶ A way to find important nondimensional groups for a specific flow



Nondimensionalized Equations

The incompressible flow continuity and momentum equations and corresponding boundary conditions:

Continuity: $\nabla \cdot \mathbf{V} = 0$

Navier-Stokes: $\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V}$

Solid surface: no-slip ($\mathbf{V} = 0$ if fixed surface)

Inlet/outlet: known velocity and pressure



Nondimensionalized Equations

- ▶ The variables in the continuity and momentum equations contain three primary dimensions; M , L , and T
- ▶ All variables included ($\rho, \mathbf{V}, p, x, y, z$) can be made nondimensional using three constants:
 - ▶ density: ρ
 - ▶ reference velocity: U
 - ▶ reference length: L

reference properties are *constants characteristic for a specific flow*

Nondimensionalized Equations

nondimensional variables are denoted by an asterisk:

$$\mathbf{V}^* = \frac{\mathbf{V}}{U}$$

$$\nabla^* = L \nabla$$

$$(x^*, y^*, z^*) = \frac{1}{L} (x, y, z)$$

$$t^* = \frac{tU}{L}$$

$$p^* = \frac{\rho + \rho g z}{\rho U^2}$$

Check:

$$\begin{aligned} \nabla^* p^* &= \frac{L}{\rho U^2} \nabla (\rho + \rho g z) = \\ &= \frac{L}{\rho U^2} [\nabla \rho - \rho \mathbf{g}] \end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{\partial (U u^*)}{\partial (L x^*)} = \frac{U}{L} \frac{\partial u^*}{\partial x^*}$$

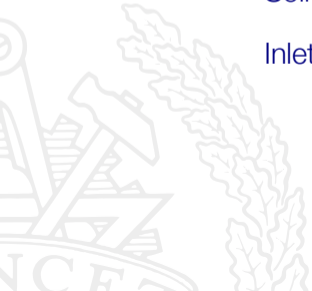
Nondimensionalized Equations

Continuity: $\nabla^* \cdot \mathbf{V}^* = 0$

Navier-Stokes: $\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho UL} \nabla^{*2} \mathbf{V}^*$

Solid surface: no-slip ($\mathbf{V}^* = 0$ if fixed surface)

Inlet/outlet: known velocity and pressure (\mathbf{V}^* , p^*)



Nondimensionalized Equations

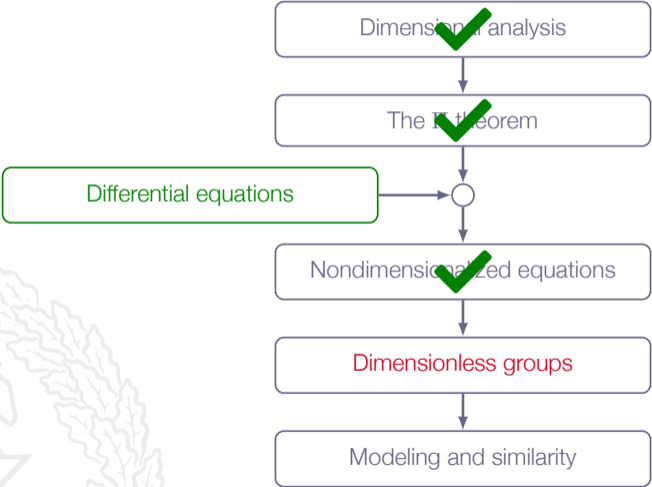
The Reynolds number appears in the nondimensional Navier-Stokes equations

$$\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho UL} \nabla^{*2} \mathbf{V}^*$$

$$Re = \frac{\rho UL}{\mu}$$

Reynolds number - ratio of inertia and viscosity

Roadmap - Dimensional Analysis and Similarity



Dimensionless Groups

parameter	definition	interpretation	importance
Reynolds number	$Re = \frac{\rho UL}{\mu}$	$\frac{\text{inertia}}{\text{viscosity}}$	almost always
Mach number	$M = \frac{U}{a}$	$\frac{\text{flowspeed}}{\text{speedofsound}}$	compressible flow
Froude number	$Fr = \frac{U^2}{gL}$	$\frac{\text{inertia}}{\text{gravity}}$	free-surface flow
Weber number	$We = \frac{\rho U^2 L}{\gamma}$	$\frac{\text{inertia}}{\text{surfacetension}}$	free-surface flow
Prandtl number	$Pr = \frac{\mu C_p}{k}$	$\frac{\text{dissipation}}{\text{conduction}}$	heat convection
specific heat ratio	$\gamma = \frac{C_p}{C_v}$	$\frac{\text{enthalpy}}{\text{internalenergy}}$	compressible flow
Strouhal number	$St = \frac{\omega L}{U}$	$\frac{\text{oscillation}}{\text{meanspeed}}$	oscillating flow
roughness ratio	$\frac{\epsilon}{L}$	$\frac{\text{wallroughness}}{\text{bodylength}}$	turbulent flow
pressure coefficient	$C_p = \frac{p - p_\infty}{0.5 \rho U^2}$	$\frac{\text{staticpressure}}{\text{dynamicpressure}}$	aerodynamics
lift coefficient	$C_L = \frac{F_L}{0.5 \rho U^2 A}$	$\frac{\text{liftforce}}{\text{dynamicforce}}$	aerodynamics
drag coefficient	$C_D = \frac{F_D}{0.5 \rho U^2 A}$	$\frac{\text{dragforce}}{\text{dynamicforce}}$	aerodynamics
skin friction coefficient	$C_f = \frac{\tau_{wall}}{0.5 \rho U^2}$	$\frac{\text{wallshearstress}}{\text{dynamicpressure}}$	boundary layers

The Reynolds Number

$$Re = \frac{\rho UL}{\mu} = \frac{UL}{\nu}$$



Compressible Flow

<https://www.youtube.com/watch?v=wRaDPnpx04>

$$Ma = \frac{U}{a} = \frac{U}{\sqrt{\gamma RT}}$$

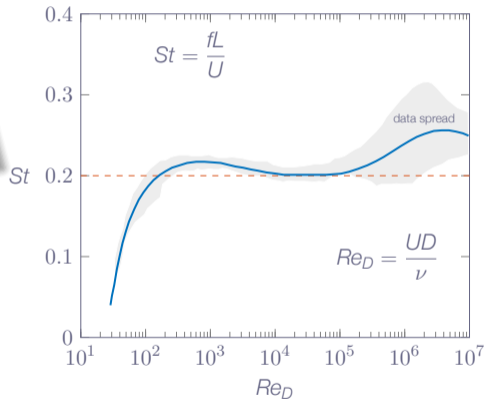
$$\gamma = \frac{C_p}{C_v}$$



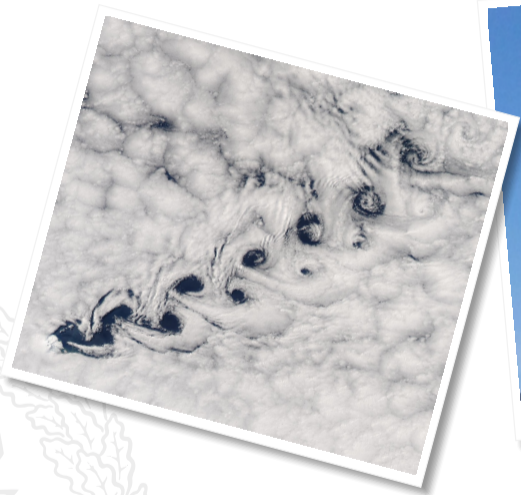
Oscillating Flows



Von Kármán vortex street



Oscillating Flows



Oscillating Flows

<https://www.youtube.com/watch?v=XggxeuFDaDU>

Tacoma bridge collapse 1940



oscillating frequency close to the natural vibration frequency of the bridge structure

Oscillating Flows

<https://www.youtube.com/watch?v=awtgPCWD58w>



Example of Successful Dimensional Analysis

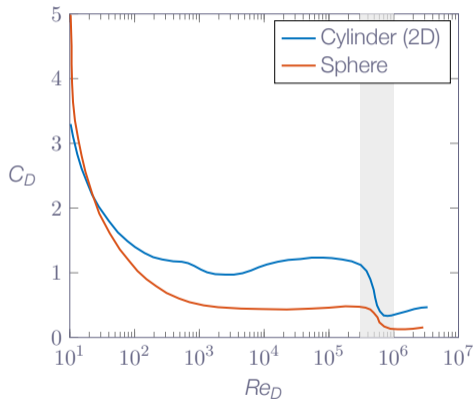
collection of data from a large number of experiments

cylinder: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L d}$

sphere: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2}$

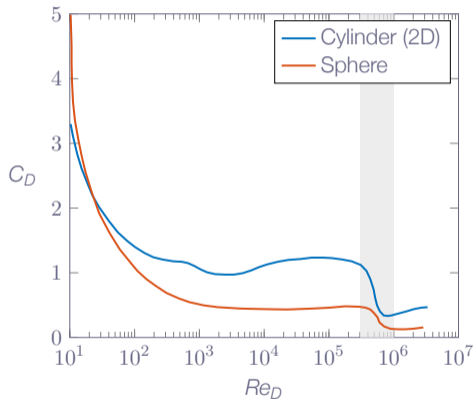
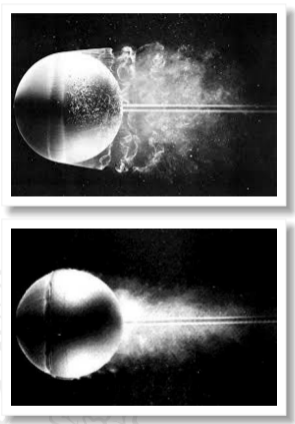
general: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A_p}$

A_p is the projected area



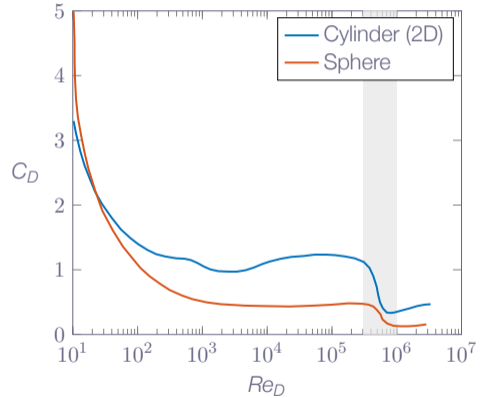
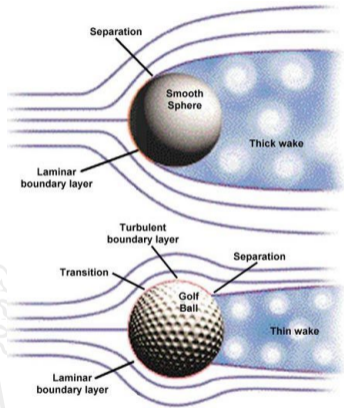
Example of Successful Dimensional Analysis

collection of data from a large number of experiments



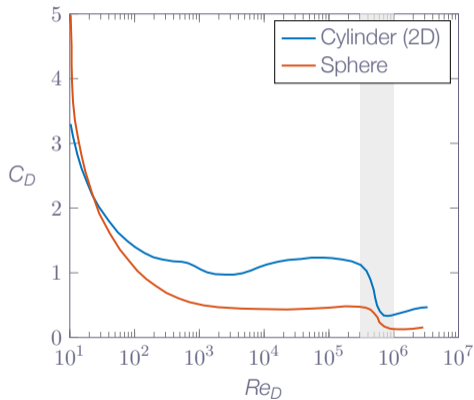
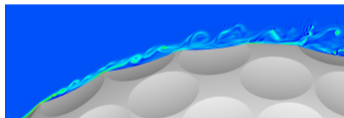
Example of Successful Dimensional Analysis

collection of data from a large number of experiments

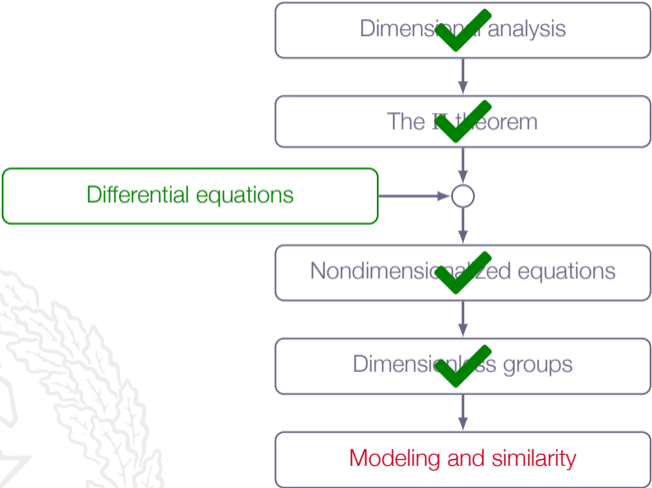


Example of Successful Dimensional Analysis

collection of data from a large number of experiments



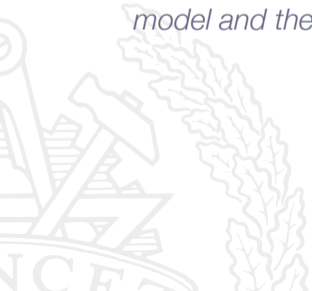
Roadmap - Dimensional Analysis and Similarity



Modeling and Similarity

Scaling of experimental results from model scale to prototype scale:

"Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype"



Geometric Similarity

"A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio"

"All angles are preserved in geometric similarity. All flow directions are preserved. The orientations of model and prototype with respect to the surroundings must be identical"

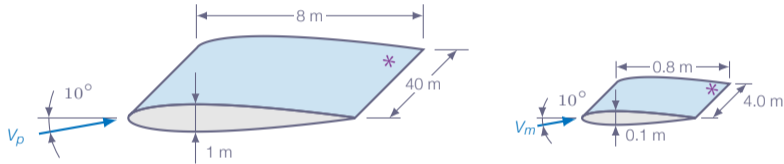


Geometric Similarity



Geometric Similarity

Homologous points - points that with the same relative location



1. all dimensions should be scaled with the same linear scaling ratio
2. angle of attach should be the same
3. scaled nose radius
4. scaled surface roughness

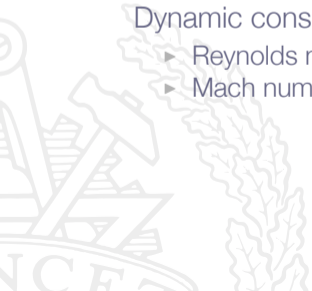
Kinematic Similarity

"The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times"

Geometric similarity is probably not sufficient to establish time-scale equivalence

Dynamic considerations:

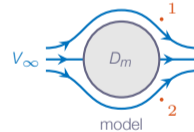
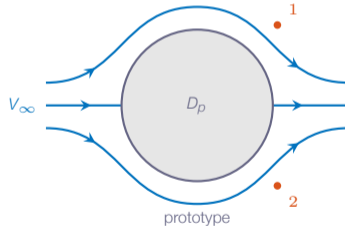
- ▶ Reynolds number equivalence
- ▶ Mach number equivalence



Kinematic Similarity

"Incompressible frictionless low-speed flows without free surfaces are kinematically similar with independent length and time scales"

$$\begin{aligned} D_m &= \alpha D_p \\ V_{\infty m} &= \beta V_{\infty p} \\ V_{1m} &= \beta V_{1p} \\ V_{2m} &= \beta V_{2p} \end{aligned}$$



Dynamic Similarity

"Dynamic similarity is achieved when the model and prototype have the same length scale ratio, time scale ratio, and force scale ratio"

Compressible flow:

1. Reynolds number equivalence
2. Mach number equivalence
3. specific-heat ratio equivalence

Incompressible flow without free surfaces:

1. Reynolds number equivalence

Incompressible flow with free surfaces:

1. Reynolds number equivalence
2. Froude number equivalence (*and if necessary Weber number and/or cavitation number*)

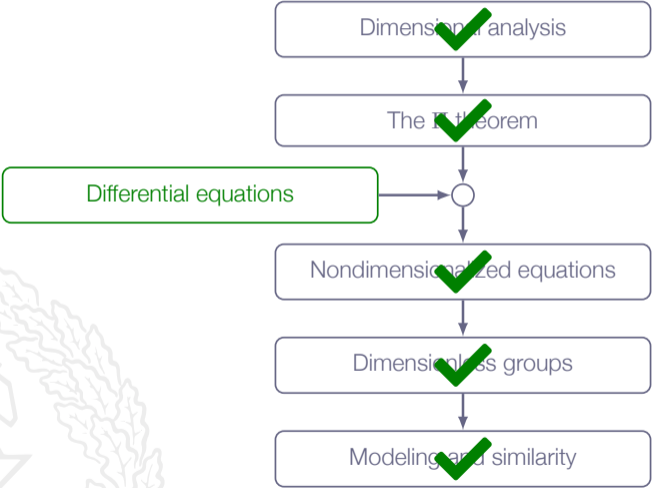
Dynamic Similarity

$$\mathbf{F}_{inertia} = \mathbf{F}_{pressure} + \mathbf{F}_{gravity} + \mathbf{F}_{friction}$$

"Dynamic similarity ensures that each of the force components will be in the same ratio and have the same directions for model and prototype"



Roadmap - Dimensional Analysis and Similarity





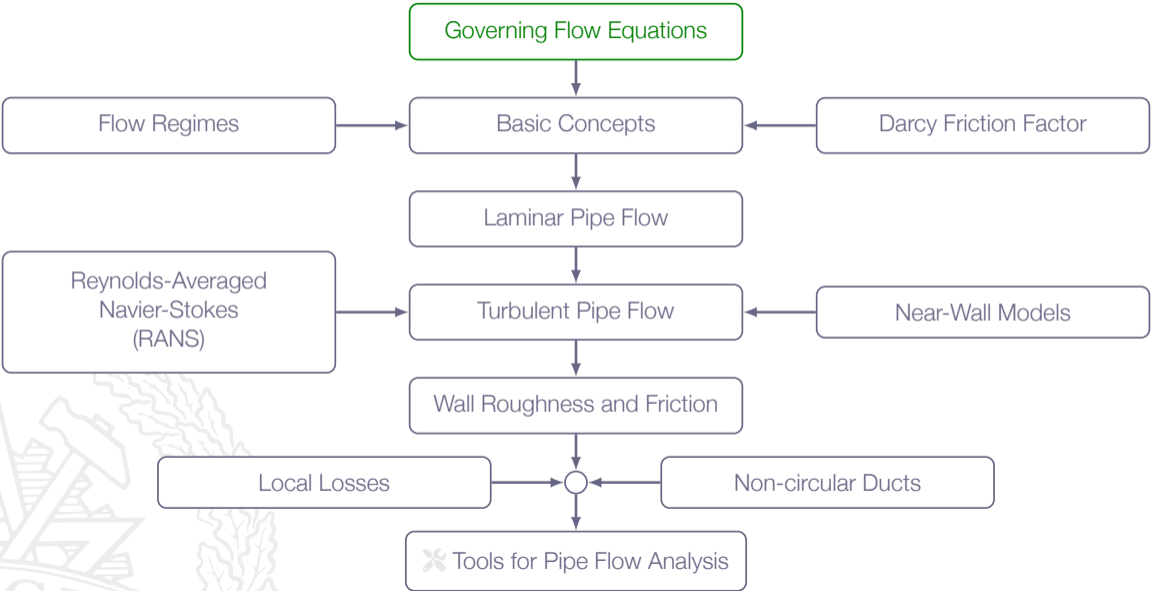
Chapter 6 - Viscous Flow in Ducts

Learning Outcomes

- 3 **Define** the Reynolds number
- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 6 **Explain** what a boundary layer is and when/where/why it appears
- 8 **Understand** and be able to **explain** the concept shear stress
- 18 **Explain** losses appearing in pipe flows
- 19 **Explain** the difference between laminar and turbulent pipe flow
- 20 **Solve** pipe flow problems using Moody charts
- 24 **Explain** what is characteristic for a turbulent flow
- 25 **Explain** Reynolds decomposition and derive the RANS equations
- 26 **Understand** and **explain** the Boussinesq assumption and turbulent viscosity
- 27 **Explain** the difference between the regions in a boundary layer and what is characteristic for each of the regions (viscous sub layer, buffer region, log region)

if you think about it, pipe flows are everywhere (a pipe flow is not a flow of pipes)

Roadmap - Viscous Flow in Ducts





"Piping systems are encountered in almost every engineering design and thus have been studied extensively"

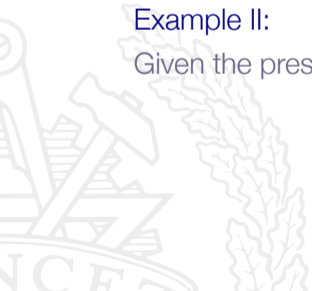
Typical Pipe-Flow Problems

Example I:

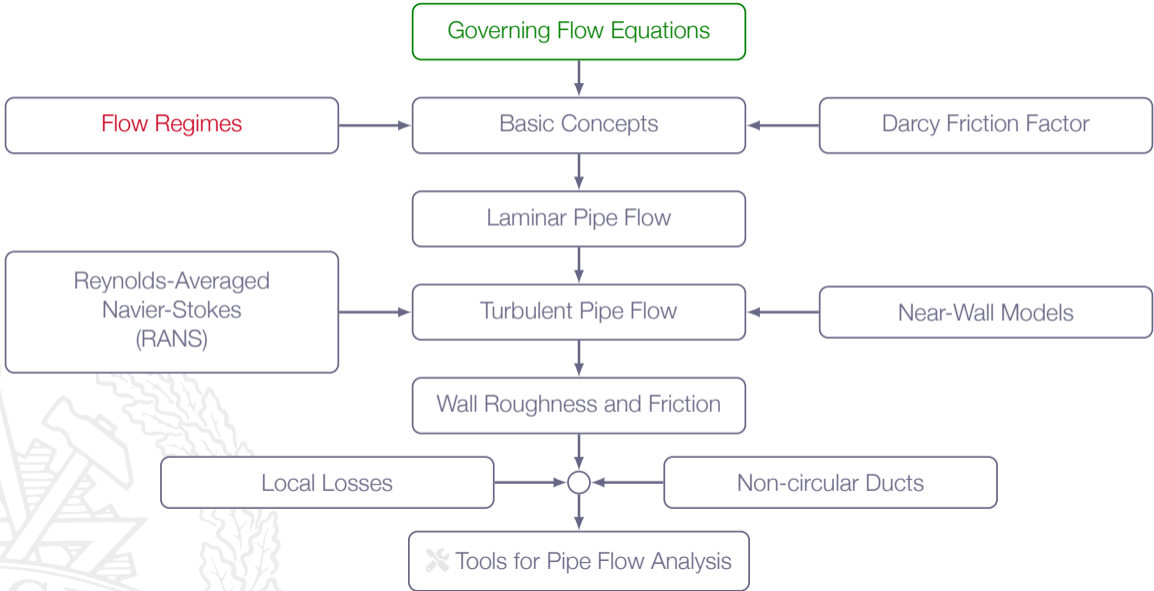
Given pipe geometry, fluid properties, flow rate, and locations of valves, bends, diffusers etc - estimate the pressure drop needed to drive the flow

Example II:

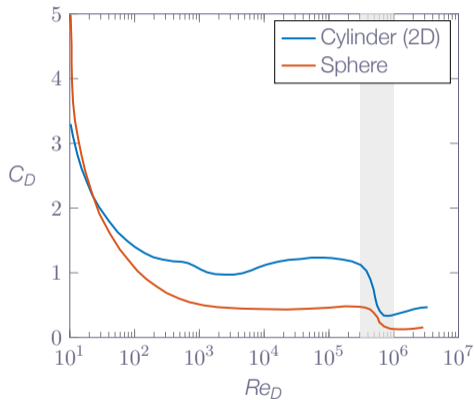
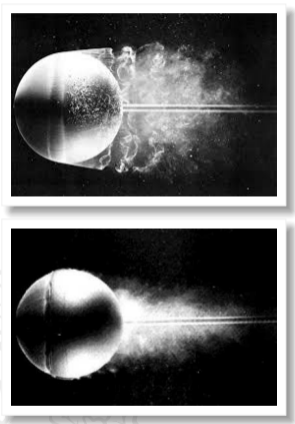
Given the pressure drop available from a pump - what flow rate can be expected



Roadmap - Viscous Flow in Ducts



Transition to Turbulence



Transition to Turbulence

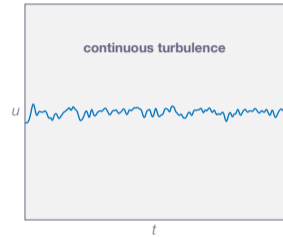
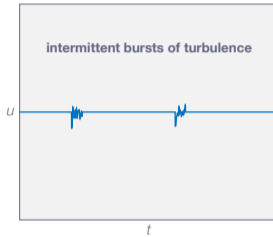
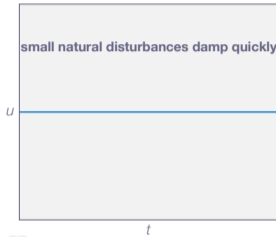
Factors that effects transition:

- ▶ Wall roughness
- ▶ Fluctuations in incoming flow
- ▶ Reynolds number



Transition to Turbulence

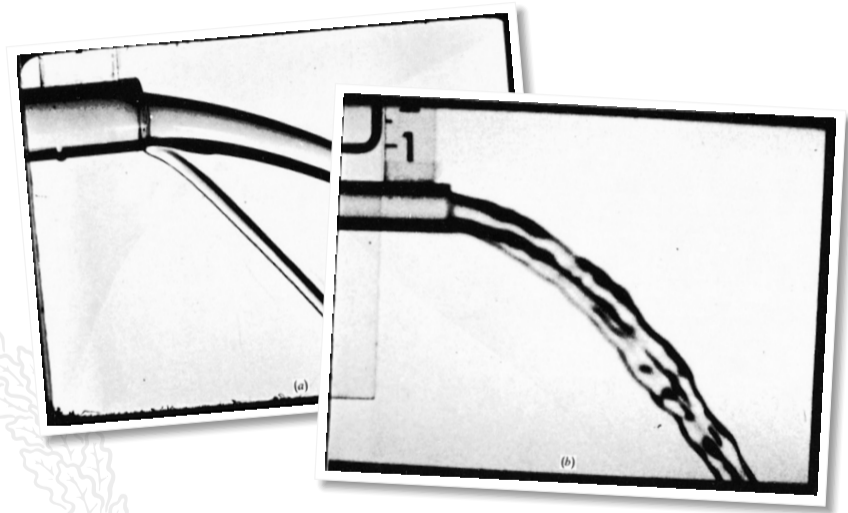
Reynolds number



Fluctuations in the fully turbulent flow velocity signal:

- ▶ typically 1% to 20% of the average velocity
- ▶ not periodic
- ▶ random
- ▶ continuous range (spectrum) of frequencies

Transition to Turbulence



Transition to Turbulence

0	$< Re <$	1	highly viscous laminar "creeping" motion
1	$< Re <$	100	laminar, strong Reynolds number dependence
100	$< Re <$	10^3	laminar, boundary layer theory useful
10^3	$< Re <$	10^4	transition to turbulence
10^4	$< Re <$	10^6	turbulent, moderate Reynolds number dependence
10^6	$< Re <$	∞	turbulent, slight Reynolds number dependence

Note! The ranges will vary somewhat with geometry and surface roughness

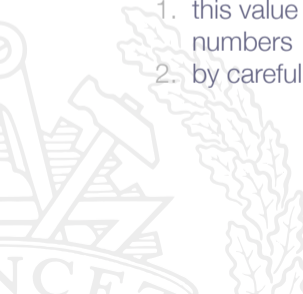
Transition to Turbulence

An accepted design value for pipe flow transition is

$$Re_{d,crit} \approx 2300$$

Note!

1. this value is for pipe flows, other applications have different transition Reynolds numbers
2. by careful design the Reynolds number can be pushed to higher values



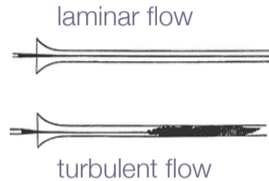
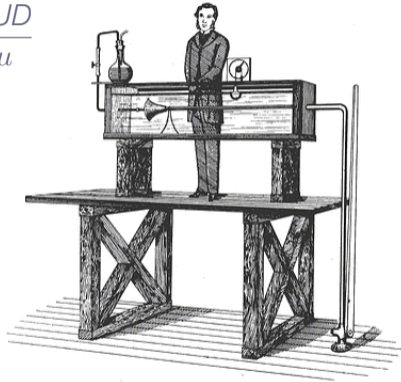
Transition to Turbulence

The great majority of our analyses are concerned with laminar flow or with turbulent flow, and one should not normally design a flow operation in the transition region.



Transition to Turbulence

$$Re = \frac{\rho U D}{\mu}$$

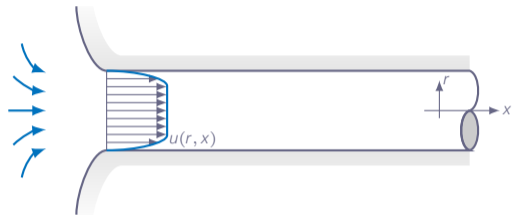


Internal Flows

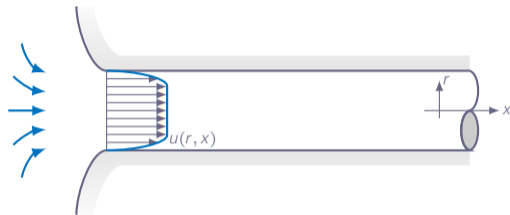
- ▶ Constrained by bounding walls - **wall-bounded**
- ▶ Boundary layers grows and meet at the center



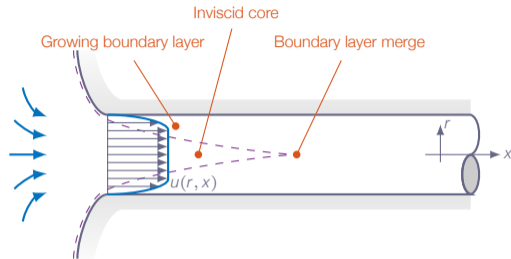
Velocity Profile Development



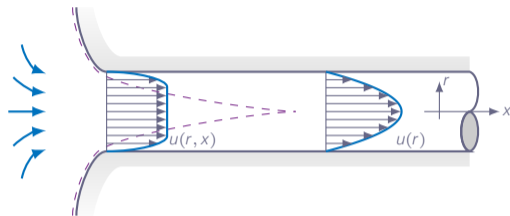
Velocity Profile Development



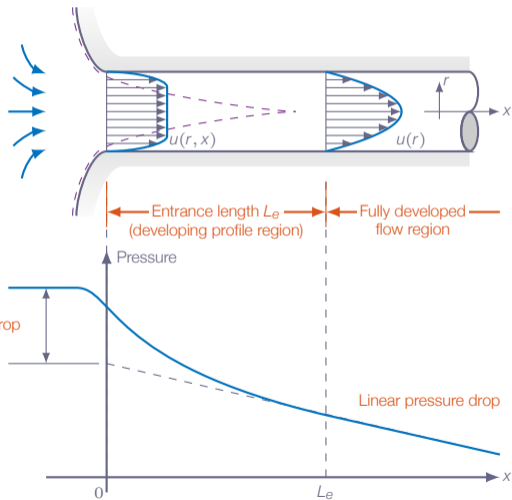
Velocity Profile Development



Velocity Profile Development



Velocity Profile Development



Velocity Profile Development

$$L_e = f(d, V, \rho, \mu), \quad V = \frac{Q}{A}, \quad Q = \int u dA = \text{const}$$

Dimensional analysis gives:

$$\frac{L_e}{d} = g\left(\frac{\rho V d}{\mu}\right) = g(Re_d)$$



Velocity Profile Development

Laminar flow:

$$\frac{L_e}{d} \approx 0.06Re_d$$

The maximum laminar entrance length, at $Re_d = Re_{d,crit} = 2300$, is $L_e = 138d$, which is the longest development length possible



Velocity Profile Development

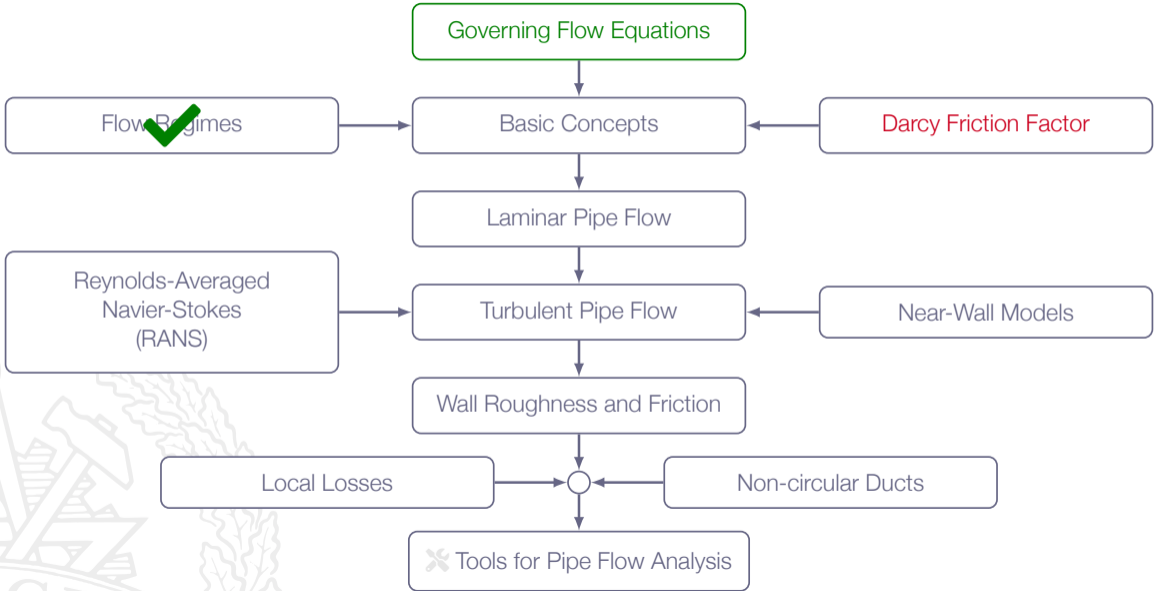
Turbulent flow ($Re_d \leq 10^7$):

$$\frac{L_e}{d} \approx 1.6Re_d^{1/4}$$

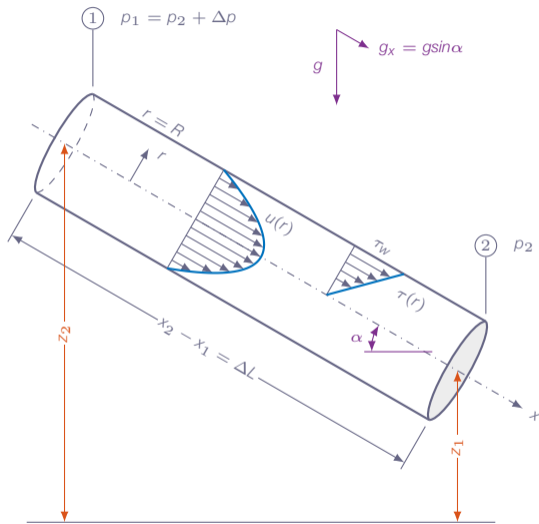
Re_d	4.0×10^3	1.0×10^4	1.0×10^5	1.0×10^6	1.0×10^7
L_e/d	13	16	28	51	90



Roadmap - Viscous Flow in Ducts



Head Loss



Head Loss

$$Q_1 = Q_2 = Q, \quad V_1 = V_2 = V$$

Steady-flow energy equation:

$$\left(\frac{p}{\rho g} + \alpha \frac{V^2}{2g} + z \right)_1 = \left(\frac{p}{\rho g} + \alpha \frac{V^2}{2g} + z \right)_2 + h_f$$

- ▶ No pumps or turbines between 1 and 2
- ▶ Fully developed flow ($\alpha_1 = \alpha_2$)

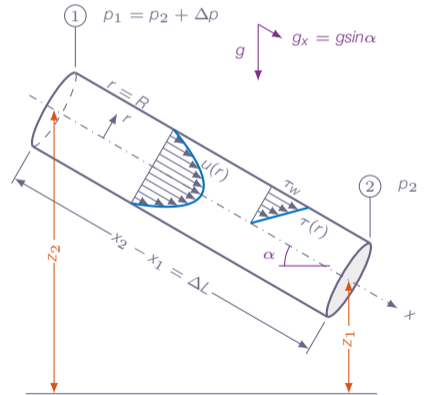
$$h_f = (z_1 - z_2) + \left(\frac{p_1 - p_2}{\rho g} \right) = \Delta z + \frac{\Delta p}{\rho g}$$

Head Loss

Apply the momentum equation along the pipe:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L \sin \alpha - \tau_w(2\pi R)L$$

$$\sum F_x = \dot{m}(V_2 - V_1) = 0$$



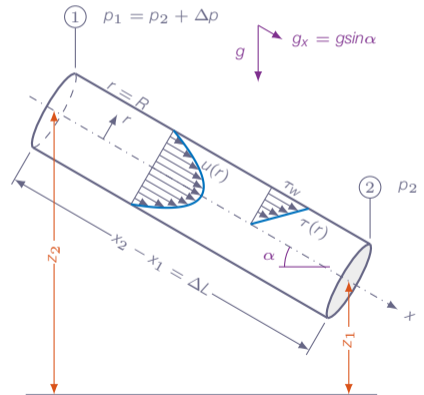
Head Loss

$$\Delta p(\pi R^2) + \rho g(\pi R^2)L \sin \alpha = \tau_w(2\pi R)L$$

$$\frac{\Delta p}{\rho g} + L \sin \alpha = \frac{2\tau_w L}{\rho g R}$$

$$\frac{\Delta p}{\rho g} + \Delta z = \frac{4\tau_w L}{\rho g d}$$

$$h_f = \frac{4\tau_w L}{\rho g d}$$



Friction Factor

$$h_f = f_D \frac{L V^2}{d 2g}$$

where

$$f_D = f(\text{Re}_d, \varepsilon/d, \text{duct shape})$$

is the Darcy friction factor

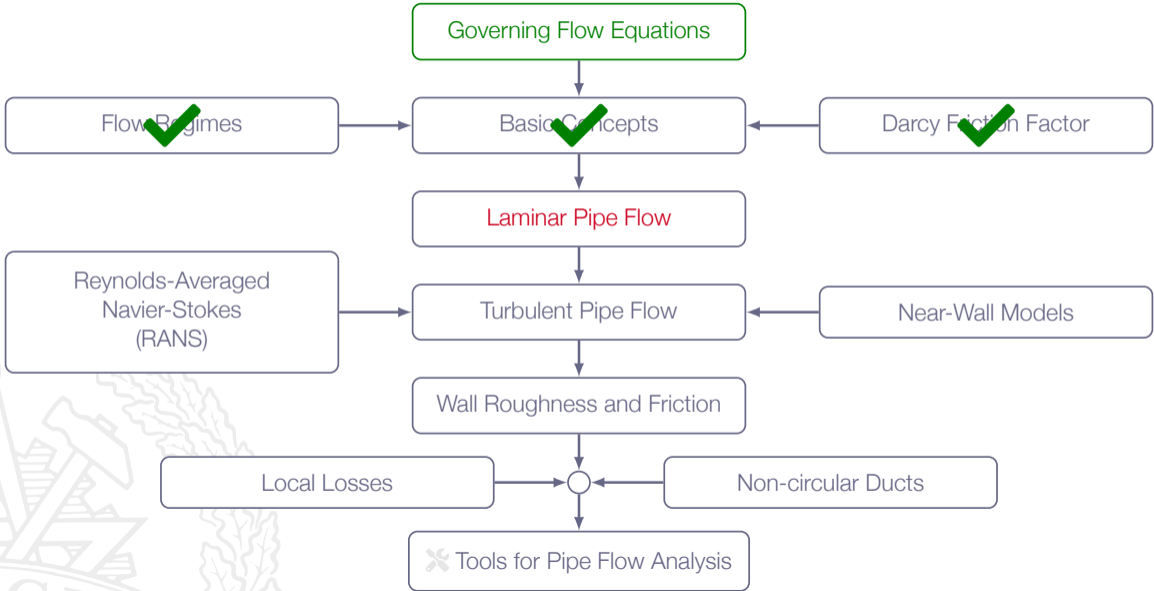
$$\frac{4\tau_w L}{\rho g d} = f_D \frac{L V^2}{d 2g} \Rightarrow f_D = \frac{8\tau_w}{\rho V^2}$$

Note! for non-circular pipes, τ_w is an average value around the duct perimeter



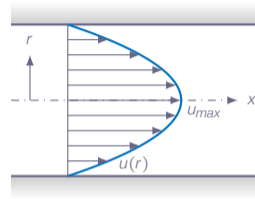
Henry Darcy 1803-1858

Roadmap - Viscous Flow in Ducts



Fully Developed Laminar Pipe Flow

- ▶ Fully developed
- ▶ circular pipe with the diameter D and radius R
- ▶ Pressure driven (Poiseuille flow)



$$u(r) = u_{max} \left(1 - \left(\frac{r}{R} \right)^2 \right) \text{ where } u_{max} = -\frac{dp}{dx} \frac{R^2}{4\mu}$$

$$-\frac{dp}{dx} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right)$$

Fully Developed Laminar Pipe Flow

$$u_{max} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \frac{R^2}{4\mu}$$

$$V = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \frac{R^2}{8\mu}$$

$$\frac{du}{dr} = -2u_{max} \frac{r}{R^2} = -4V \frac{r}{R^2}$$

$$\tau_w = \mu \left. \frac{du}{dr} \right|_{r=R} = \frac{4\mu V}{R}$$



Fully Developed Laminar Pipe Flow

We can now calculate the head loss according to

$$h_f = f_D \frac{L}{D} \frac{V^2}{2g} \quad \text{where } f_D = \frac{8\tau_w}{\rho V^2}$$

$$h_f = \frac{4\tau_w L}{\rho g D} = \frac{16\mu V L}{\rho g D R} = \frac{32\mu V L}{\rho g D^2} = \left\{ V = \frac{4Q}{\pi D^2} \right\} = \frac{128\mu Q L}{\pi \rho g D^4}$$

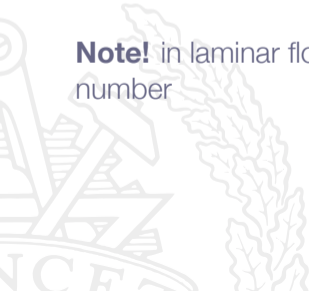
$$Q = \int u dA = VA = V \frac{\pi D^2}{4} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \frac{\pi D^4}{128\mu}$$

Fully Developed Laminar Pipe Flow

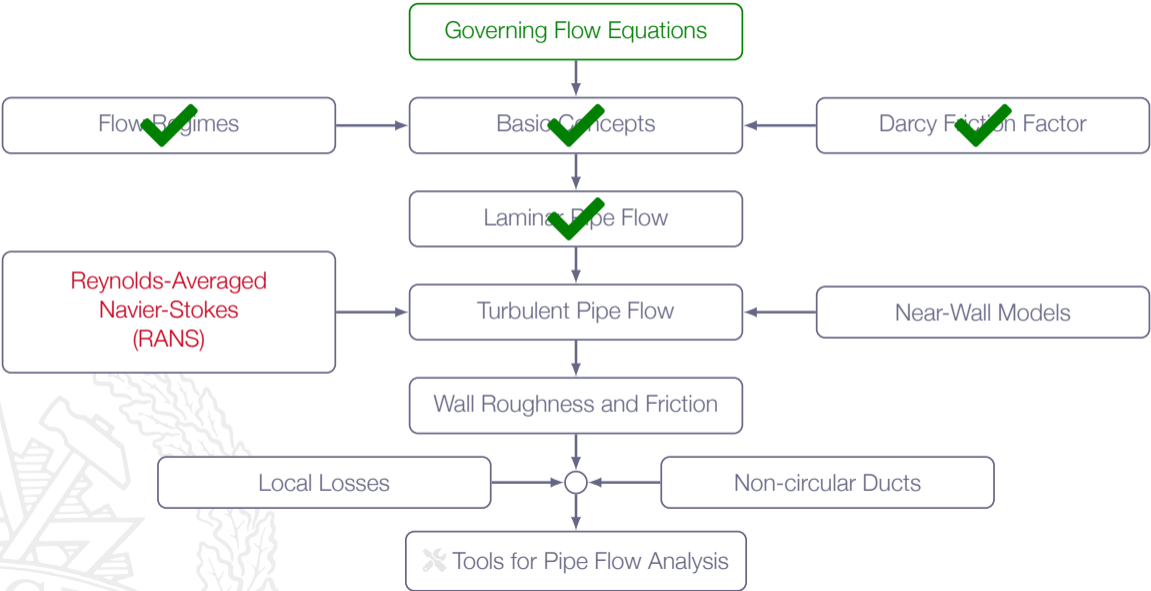
For laminar flow:

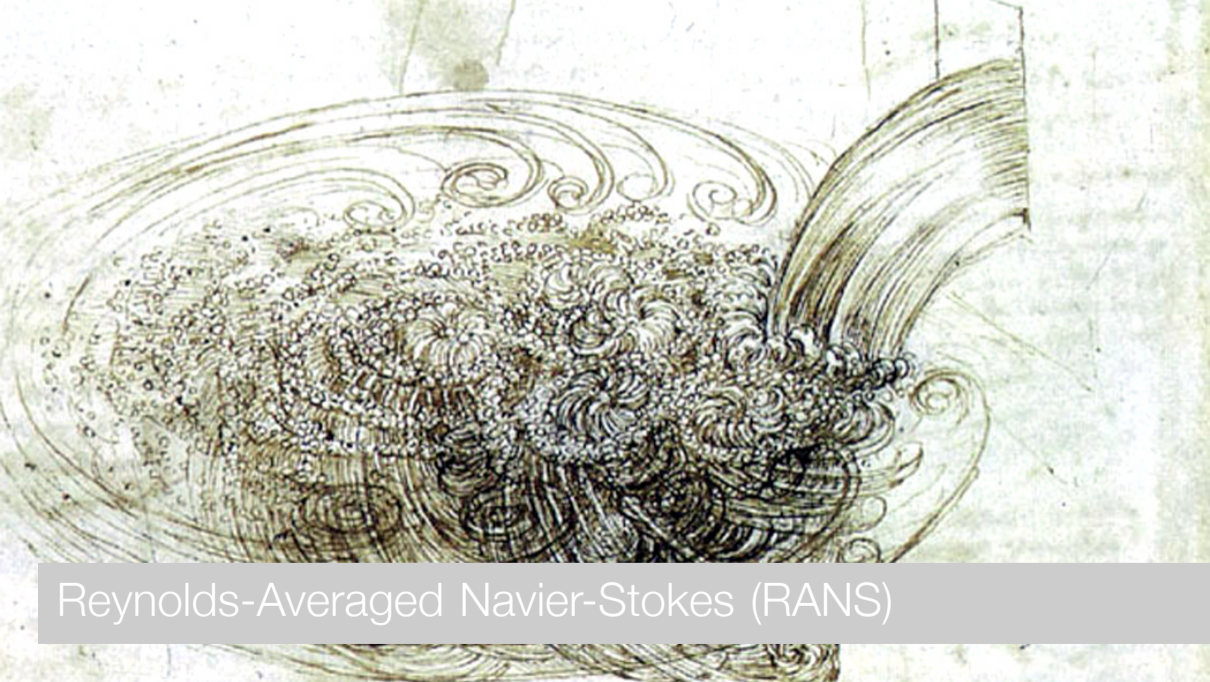
$$f_D = \frac{8\tau_w}{\rho V^2} = \left\{ \tau_w = \frac{8\mu V}{D} \right\} = \frac{64\mu}{\rho V D} = \frac{64}{Re_D}$$

Note! in laminar flow, the friction factor is inversely proportional to the Reynolds number



Roadmap - Viscous Flow in Ducts





Reynolds-Averaged Navier-Stokes (RANS)

Turbulence

Leonardo Da Vinci

*”Observe the motion of the surface of the water, which resembles that of hair, which has **two motions**, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the **principal current**, the other to **random and reverse motion**”*



Governing Equations

We are assuming:

- ▶ constant density and viscosity
- ▶ no thermal interaction

continuity: $\nabla \cdot \mathbf{V} = 0$

momentum: $\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$

Governing Equations

The differential energy equation is not included here but let's have a look at it anyway

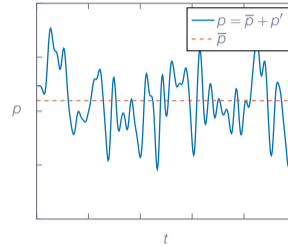
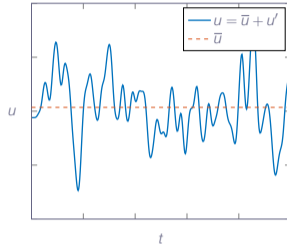
$$\rho \frac{D\hat{u}}{Dt} + \rho \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \phi$$

- ▶ Pressure work:
 - ▶ pressure drives the flow through the duct
- ▶ Viscous work:
 - ▶ no-slip condition
 - ▶ zero velocity at the walls
 - ▶ no work done by wall shear stress

So, where does the energy go?

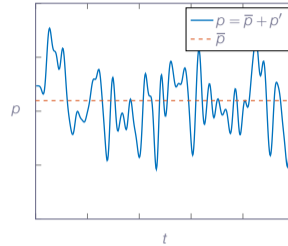
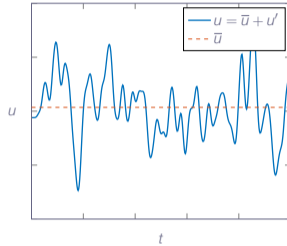
- ▶ pressure work is balanced by viscous dissipation in the interior of the flow

Reynolds' Decomposition



- ▶ Not possible to solve analytically
- ▶ Often, the time-averaged quantities are what we are looking for

Reynolds' Decomposition



$$\bar{u} = \frac{1}{T} \int_0^T u dt$$

$$u' = u - \bar{u}$$

$$\bar{u'} = \frac{1}{T} \int_0^T (u - \bar{u}) dt = \bar{u} - \bar{u} = 0$$

Reynolds' Decomposition

The mean square of the fluctuations are, however, not zero

$$\overline{u'^2} = \frac{1}{T} \int_0^T u'^2 dt \neq 0$$

measure of *turbulence intensity*

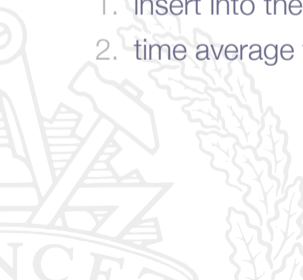
Mean fluctuation products are generally not zero ($\overline{u'v'}$, $\overline{u'p'}$)

Reynolds' Decomposition

Reynolds' idea was to split all properties into mean and fluctuating parts:

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w', \quad p = \bar{p} + p'$$

1. insert into the governing equations
2. time average the equations



Reynolds' Decomposition

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum (x-component):

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Reynolds' Decomposition

Continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

time averaging the equation gives

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

and as a consequence

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Reynolds' Decomposition

Momentum (x-component):

$$\begin{aligned} & \rho \left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} \right) + \\ & \rho \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{u} \frac{\partial u'}{\partial x} + u' \frac{\partial \bar{u}}{\partial x} + u' \frac{\partial u'}{\partial x} \right) + \\ & \rho \left(\bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{v} \frac{\partial u'}{\partial y} + v' \frac{\partial \bar{u}}{\partial y} + v' \frac{\partial u'}{\partial y} \right) + \\ & \rho \left(\bar{w} \frac{\partial \bar{u}}{\partial z} + \bar{w} \frac{\partial u'}{\partial z} + w' \frac{\partial \bar{u}}{\partial z} + w' \frac{\partial u'}{\partial z} \right) = \\ & - \frac{\partial \bar{p}}{\partial x} - \frac{\partial p'}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} \right) \end{aligned}$$

Reynolds' Decomposition

Momentum (x-component):

time averaging the equation gives:

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \overline{u' \frac{\partial u'}{\partial x}} + \overline{v' \frac{\partial u'}{\partial y}} + \overline{w' \frac{\partial u'}{\partial z}} \right) =$$
$$-\frac{\partial \bar{p}}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right)$$

The highlighted terms can be rewritten as:

$$\overline{u' \frac{\partial u'}{\partial x}} + \overline{v' \frac{\partial u'}{\partial y}} + \overline{w' \frac{\partial u'}{\partial z}} = \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} - \underbrace{\overline{u' \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right)}}_{=0}$$

Reynolds' Decomposition

the continuity equation reduces to

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

the axial component of the momentum equation:

$$\rho \frac{D\bar{u}}{Dt} = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x} \left(\mu \frac{\partial \bar{u}}{\partial x} - \overline{\rho u'^2} \right) +$$
$$\frac{\partial}{\partial y} \left(\mu \frac{\partial \bar{u}}{\partial y} - \overline{\rho u'v'} \right) + \frac{\partial}{\partial z} \left(\mu \frac{\partial \bar{u}}{\partial z} - \overline{\rho u'w'} \right)$$

Reynolds' Decomposition

By applying Reynolds' decomposition to our governing equations, we have introduced a number of new unknowns

The number of equations is the same as before, which means problems

Our new problem has a name

The closure problem



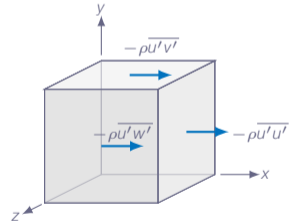
Reynolds' Decomposition

The three correlation terms $-\overline{\rho u'^2}$, $-\overline{\rho u'v'}$, and $-\overline{\rho u'w'}$ are called **Reynolds stresses** or turbulent stresses

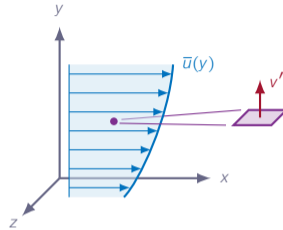
In duct and boundary layer flow, the stress $-\overline{\rho u'v'}$, associated with the direction normal to the wall, is dominant

$$\rho \frac{D\bar{u}}{Dt} \approx -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}$$

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \overline{\rho u'v'} = \tau_{lam} + \tau_{turb}$$



Reynolds' Decomposition



mass flow through surface element: $\dot{m}_y = \rho v' dA$

momentum balance in x-direction: $F_x = \dot{m}_y u = \rho v' (\bar{u} + u') dA$

$$\tau_{dA} = -\frac{F_x}{dA} = -\overline{\rho v' (\bar{u} + u')} = -\overline{\rho v' \bar{u}} - \overline{\rho u' v'} = \{ \overline{v' \bar{u}} = \bar{v}' \bar{u} = 0 \} = -\rho \overline{u' v'}$$

Reynolds' Decomposition

Introducing **turbulent viscosity** μ_t defined such that

$$-\rho \overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

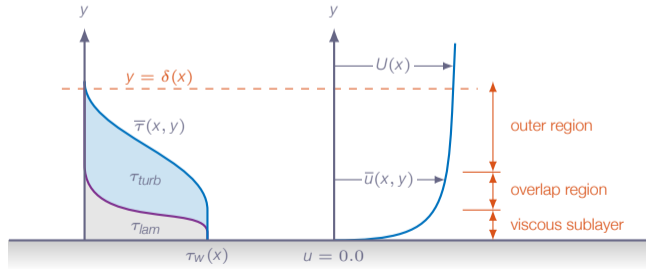
Boussinesq's assumption

With the turbulent viscosity, the total shear stress τ becomes:

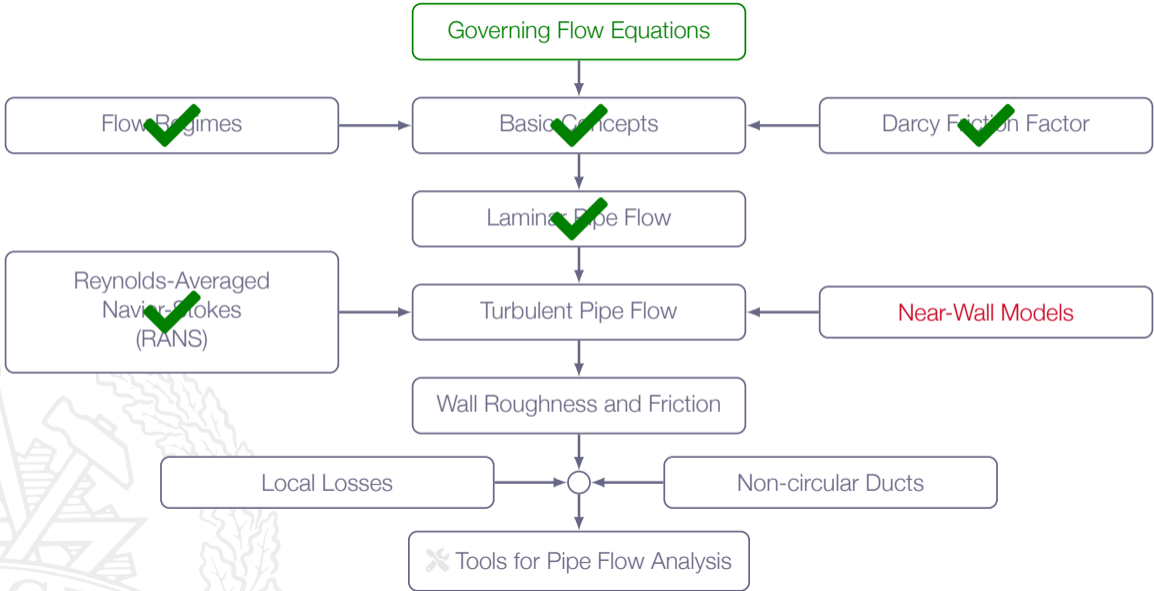
$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y}$$

Reynolds' Decomposition

- ▶ laminar shear (τ_{lam}) dominates in the near-wall region
- ▶ turbulent shear (τ_{turb}) dominates in the outer region
- ▶ both are important in the overlap layer



Roadmap - Viscous Flow in Ducts



Viscous Sublayer

$$\rho \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) = -\frac{d\bar{p}}{dx} + \rho g_x + \frac{\partial \tau}{\partial y}$$

1. $y \rightarrow 0 \Rightarrow \bar{u} \rightarrow 0, \bar{v} \rightarrow 0 \Rightarrow \frac{\partial \tau}{\partial y} = \frac{d\bar{p}}{dx} + \rho g_x$

2. $\tau(y) = \left(\frac{d\bar{p}}{dx} + \rho g_x \right) y + C$ but $y \rightarrow 0 \Rightarrow \tau(0) = C = \tau_w$

3. $y \rightarrow 0 \Rightarrow u' \rightarrow 0, v' \rightarrow 0 \Rightarrow \tau = \tau_w = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = \mu \frac{\partial \bar{u}}{\partial y}$

4. $\bar{u} = \frac{\tau_w}{\mu} y + C = \{ \bar{u}(0) = 0 \Rightarrow C = 0 \} = \frac{\tau_w}{\mu} y$

Viscous Sublayer

Introducing **friction velocity** defined as

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

and thus

$$\bar{u} = \frac{\tau_w}{\mu} y = \frac{\rho u^{*2} y}{\mu} = \frac{u^{*2} y}{\nu}$$

which can be rewritten as:

$$\underbrace{\frac{\bar{u}}{u^*}}_{u^+} = \underbrace{\frac{u^* y}{\nu}}_{y^+} \text{ valid for } y^+ \leq 5 - 10$$

The Log Region

Now, let's move a bit further out from the wall

1. $\tau = \text{const} = \tau_w$ still (*we have not moved that far out from the wall*)
2. outside of the viscous sublayer $\mu_t \gg \mu$ and thus

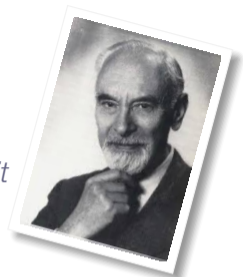
$$\tau = \tau_w = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} \approx -\rho \overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

We need an estimate of μ_t to be able to solve this ...

The Log Region

Prandtl's mixing length concept

"the average distance that a small mass of fluid will travel before it exchanges its momentum with another mass of fluid"

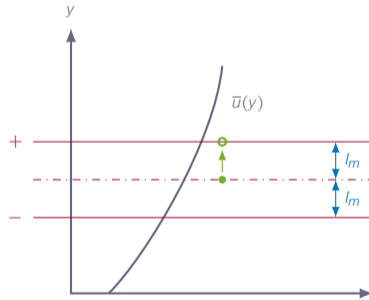


Ludwig Prandtl 1875-1953

$$\bar{u}(y + l_m) = \bar{u}(y) + l_m \frac{\partial \bar{u}}{\partial y}$$

$$\bar{u}(y - l_m) = \bar{u}(y) - l_m \frac{\partial \bar{u}}{\partial y}$$

Prandtl assumed $u' \approx l_m \frac{\partial \bar{u}}{\partial y}$



He further assumed v' to be of the same size as u'

The Log Region

Prandtl's mixing length concept

$$\tau_t = -\rho \overline{u'v'} \approx \rho l_m^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2$$

$$-\rho \overline{u'v'} \approx \mu_t \frac{\partial \bar{u}}{\partial y} \Rightarrow \mu_t \approx \rho l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

$$\nu_t = \frac{\mu_t}{\rho} \approx l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

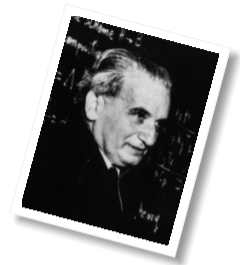


The Log Region

Prandtl's mixing length concept

So, how do we estimate the mixing length l_m

$$l_m(y) = a_0 + a_1 y + a_2 y^2 + \dots$$



Theodore von Kármán 1881-1963

1. $y \rightarrow 0 \Rightarrow l_m \rightarrow 0 \Rightarrow a_0 = 0$
2. small values of y (we are still very close to the wall) $\Rightarrow l_m = a_1 y$

$$l_m = \kappa y$$

where κ is Kármán's constant $\kappa \approx 0.41$

The Log Region

$$\mu_t = \rho \kappa^2 y^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

$$\tau_w = \rho l_m \frac{\partial \bar{u}}{\partial y} = \rho \kappa^2 y^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 = \rho u^{*2}$$

$$\kappa^2 y^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 = u^{*2}$$

$$\frac{1}{u^*} \partial \bar{u} = \frac{1}{\kappa} \frac{\partial y}{y} \Rightarrow \frac{\bar{u}}{u^*} = \frac{1}{\kappa} \ln y + C$$

The Log Region

$$\frac{\bar{u}}{u^*} = \frac{1}{\kappa} \ln y + C$$

or in nondimensional form

$$\frac{\bar{u}}{u^*} = \frac{1}{\kappa} \ln \frac{yu^*}{\nu} + B$$

From experiments we have:

$\kappa \approx 0.41$ and $4.9 < B < 5.5$

flow over a flat plate (external flow): $B \approx 4.9$

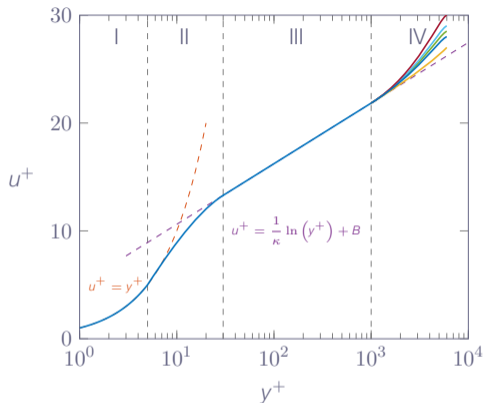
duct flow (internal flow): $B \approx 5.3$ (White: $B \approx 5.0$)

valid for $30 \lesssim y^+ \lesssim 1000$

The Log Region

- ▶ between the viscous sublayer and the log region, none of the models works
- ▶ in the outer region, inertial forces needs to be included

$$\rho \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) \neq 0$$



- I: viscous sublayer
- II: buffer layer
- III: log-law region
- IV: outer layer

Outer region

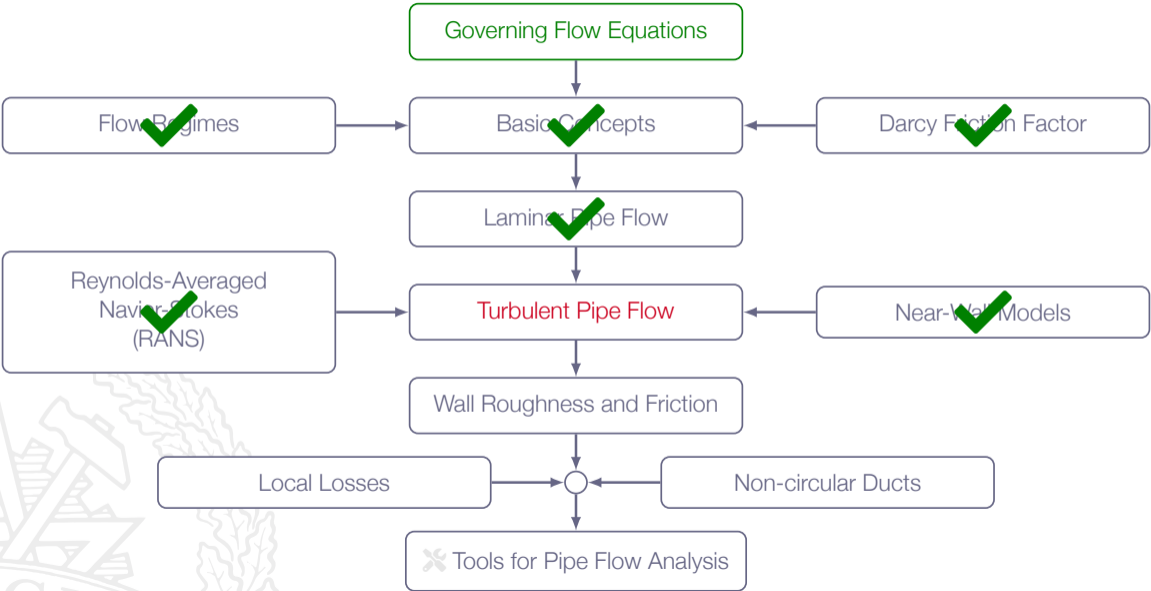
In the outer region it has been found that

$$\frac{U - \bar{u}}{u^*} = f\left(\frac{y}{\delta}\right)$$

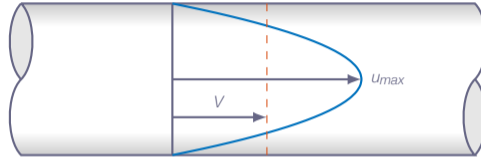
where δ is the thickness of the outer layer and U the velocity at the edge of the outer layer



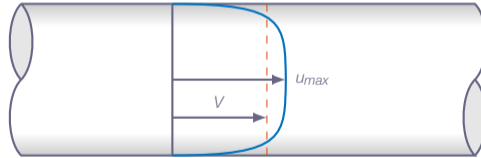
Roadmap - Viscous Flow in Ducts



Turbulent Pipe Flow



Laminar flow



Turbulent flow



Turbulent Pipe Flow

As we did for laminar pipe flow, we will now obtain the friction factor for turbulent pipe flow

$$\tau_w = f_D \frac{\rho V^2}{8} = \rho u^{*2} \Rightarrow f_D = 8 \left(\frac{V}{u^*} \right)^{-2}$$

So, what we need now is an estimate of the average flow velocity in the tube V ...

There are different ways to do this and here is one example:

1. Assume that we can use the log-law all the way across the pipe
2. Integrate to get the average velocity
3. Insert the calculated average velocity into the relation above

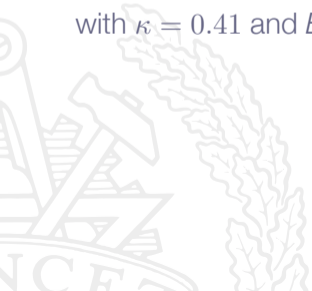
Turbulent Pipe Flow

$$\frac{\bar{u}(r)}{u^*} \approx \frac{1}{\kappa} \ln \frac{(R-r)u^*}{\nu} + B$$

$$V = \frac{Q}{A} = \frac{1}{\pi R^2} \int_0^R \bar{u}(r) 2\pi r dr$$

with $\kappa = 0.41$ and $B = 5.0$ we get

$$\frac{V}{u^*} \approx 2.44 \ln \frac{Ru^*}{\nu} + 1.34$$



Turbulent Pipe Flow

$$\frac{V}{u^*} \approx 2.44 \ln \frac{Ru^*}{\nu} + 1.34$$

The argument of the logarithm can be rewritten as

$$\frac{Ru^*}{\nu} = \frac{VD u^*}{2\nu V} = \frac{1}{2} Re_D \left(\frac{f_D}{8} \right)^{1/2}$$

and thus:

$$\frac{1}{\sqrt{f_D}} \approx 1.99 \log(Re_D \sqrt{f_D}) - 1.02$$

Turbulent Pipe Flow

With $\kappa^{-1} = 2.5$ and $B = 5.5$ we get

$$\frac{1}{\sqrt{f_D}} \approx 2.03 \log(Re_D \sqrt{f_D}) - 0.91$$



Turbulent Pipe Flow

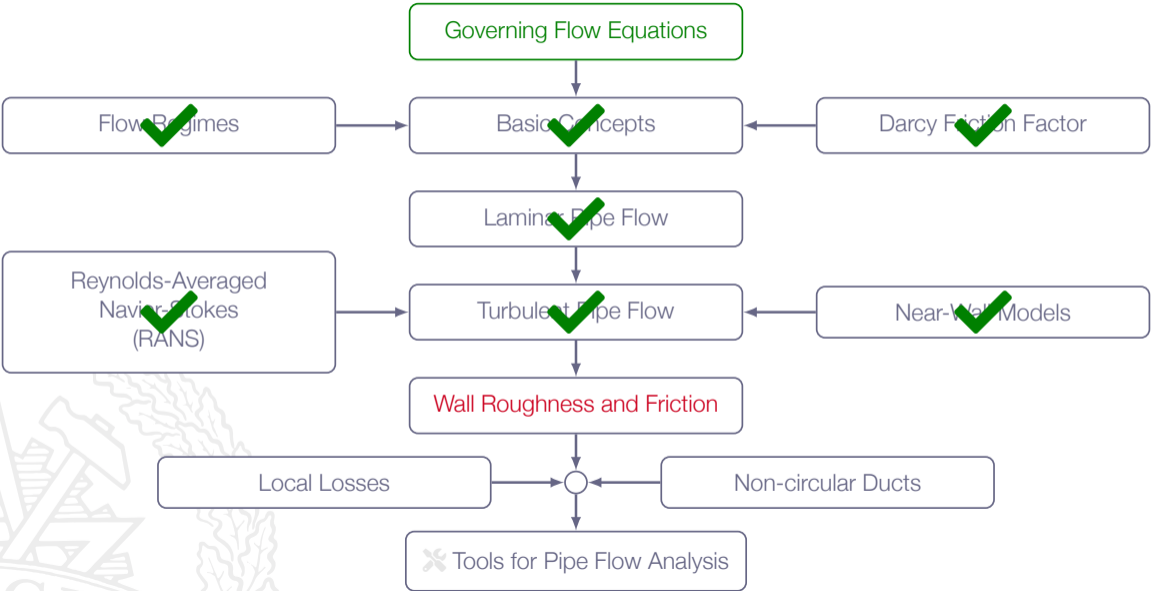
Alternative 2:

If we assume that $\frac{\bar{u}}{u^*} = 8.3 \left(\frac{u^* y}{\nu} \right)^{1/7}$ applies all over the cross section we get

$$f_D = \frac{0.3164}{Re_D^{1/4}}$$



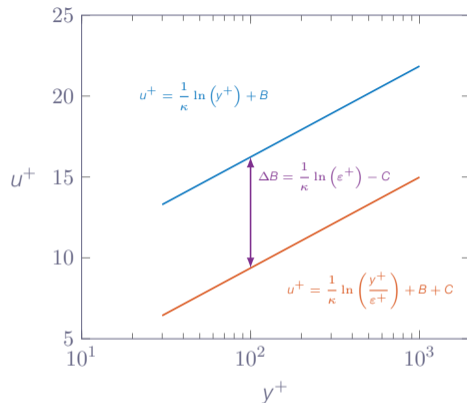
Roadmap - Viscous Flow in Ducts



Wall Roughness

Effects of surface roughness on friction:

- ▶ Negligible for laminar pipe flow
- ▶ Significant for turbulent flow
 - ▶ breaks up the viscous sublayer
 - ▶ modifies the log law (changes the value of the integration constant B)



$$\Delta B \propto (1/\kappa) \ln \epsilon^+ \quad \text{where } \epsilon^+ = \frac{\epsilon U^*}{\nu}$$

Wall Roughness

$$\frac{\epsilon U^*}{\nu} < 5$$

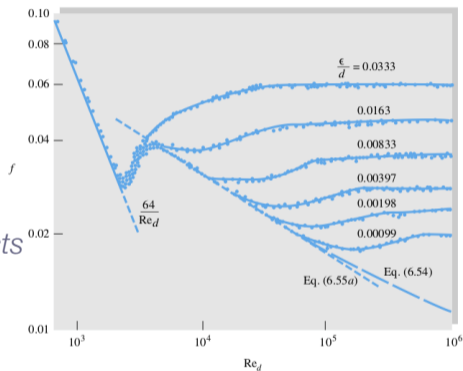
hydraulically smooth
no effects of roughness

$$5 \leq \frac{\epsilon U^*}{\nu} \leq 70$$

transitional
moderate Reynolds number effects

$$\frac{\epsilon U^*}{\nu} > 70$$

fully rough
*sublayer totally broken up
independent of Reynolds number*

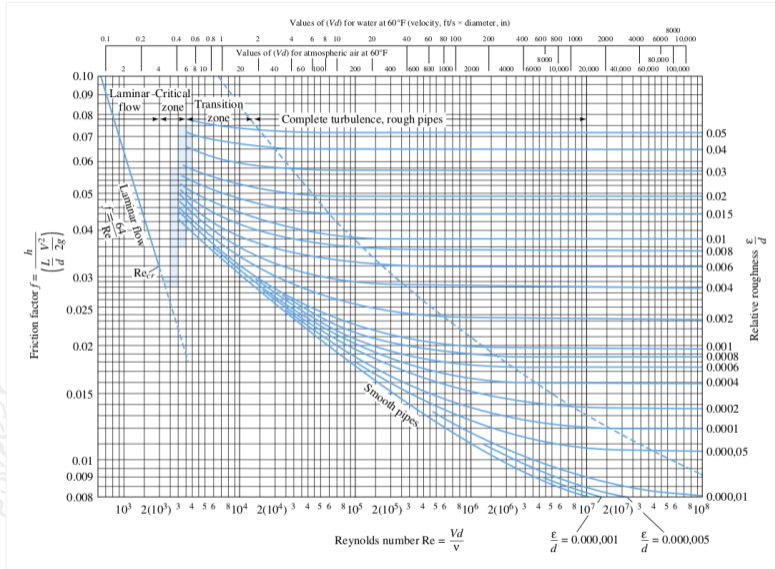


Wall Roughness

$$\frac{1}{\sqrt{f_D}} = -2.0 \log \left(\frac{\epsilon/D}{3.7} + \frac{2.5}{Re_D \sqrt{f_D}} \right)$$



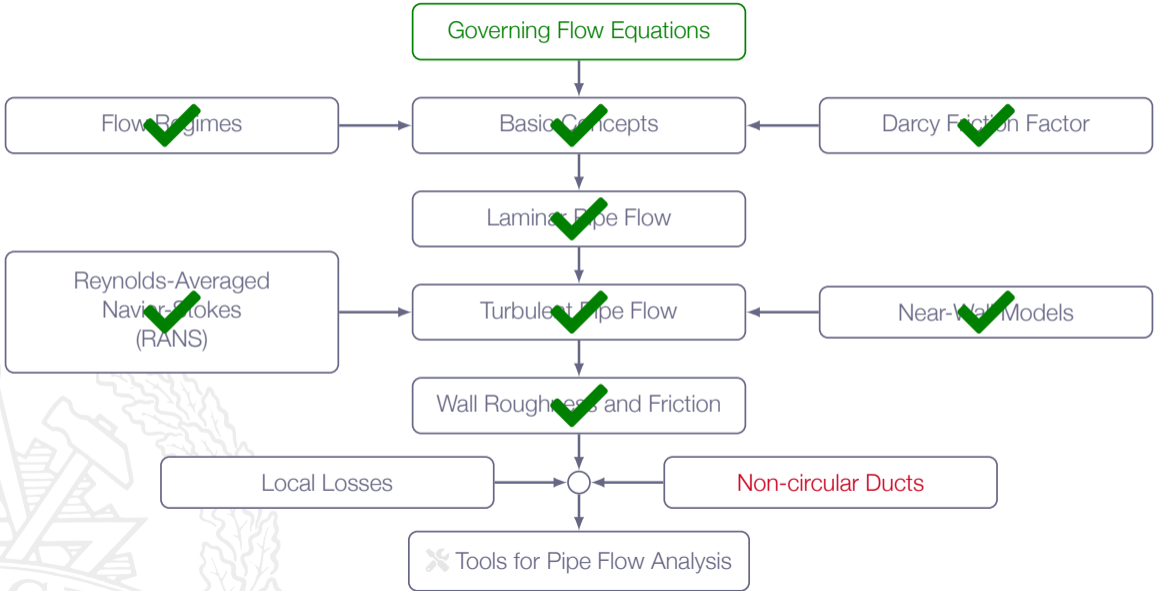
The Moody Chart



Wall Roughness

Material	Condition	ϵ		Uncertainty, %
		ft	mm	
Steel	Sheet metal, new	0.00016	0.05	± 60
	Stainless, new	0.000007	0.002	± 50
	Commercial, new	0.00015	0.046	± 30
	Riveted	0.01	3.0	± 70
	Rusted	0.007	2.0	± 50
Iron	Cast, new	0.00085	0.26	± 50
	Wrought, new	0.00015	0.046	± 20
	Galvanized, new	0.0005	0.15	± 40
	Asphalted cast	0.0004	0.12	± 50
Brass	Drawn, new	0.000007	0.002	± 50
Plastic	Drawn tubing	0.000005	0.0015	± 60
Glass	—	Smooth	Smooth	
Concrete	Smoothed	0.00013	0.04	± 60
	Rough	0.007	2.0	± 50
Rubber	Smoothed	0.000033	0.01	± 60
Wood	Stave	0.0016	0.5	± 40

Roadmap - Viscous Flow in Ducts



Non-circular Ducts

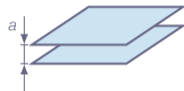
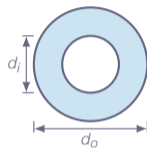
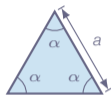
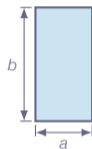
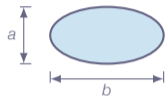
Use the same formulas of the Moody chart but replace the pipe diameter D with the hydraulic diameter D_h

$$D_h = \frac{4A}{\mathcal{P}}$$

where A is the cross section area and \mathcal{P} is the wetter perimeter

$$\Delta p_f = f_D \frac{L}{D_h} \frac{\rho V^2}{2}, \quad Re_{D_h} = \frac{VD_h}{\nu}, \quad \frac{\epsilon}{D_h}$$

Non-circular Ducts



a/b	D_h	C
0.7	$1.17a$	65.0
0.5	$1.30a$	68.0
0.3	$1.44a$	73.0
0.2	$1.50a$	78.0
0.1	$1.55a$	79.0

b/a	D_h	C
1.0	$1.00a$	57.0
1.25	$1.11a$	57.6
2.0	$1.33a$	62.0
3.0	$1.50a$	69.0
4.0	$1.60a$	73.0
5.0	$1.67a$	78.0
8.0	$1.78a$	83.0
10.0	$1.82a$	85.0

D_h	C
$0.58a$	53.0

d_i/d_o	C
$\frac{d_i}{d_o} = 0.10$	89.2
$\frac{d_i}{d_o} = 0.25$	94.0
$0.5 < \frac{d_i}{d_o} < 1.0$	96.0

$$D_h = d_o - d_i$$

D_h	C
$2.0a$	96.0

Non-circular Ducts

Laminar flow:

$$f_D = \frac{C}{Re_{D_h}}$$

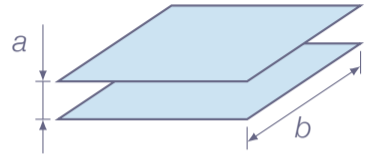
(for circular pipes: $C = 64$ and $D_h = D$)



Non-circular Ducts

Flow between parallel plates

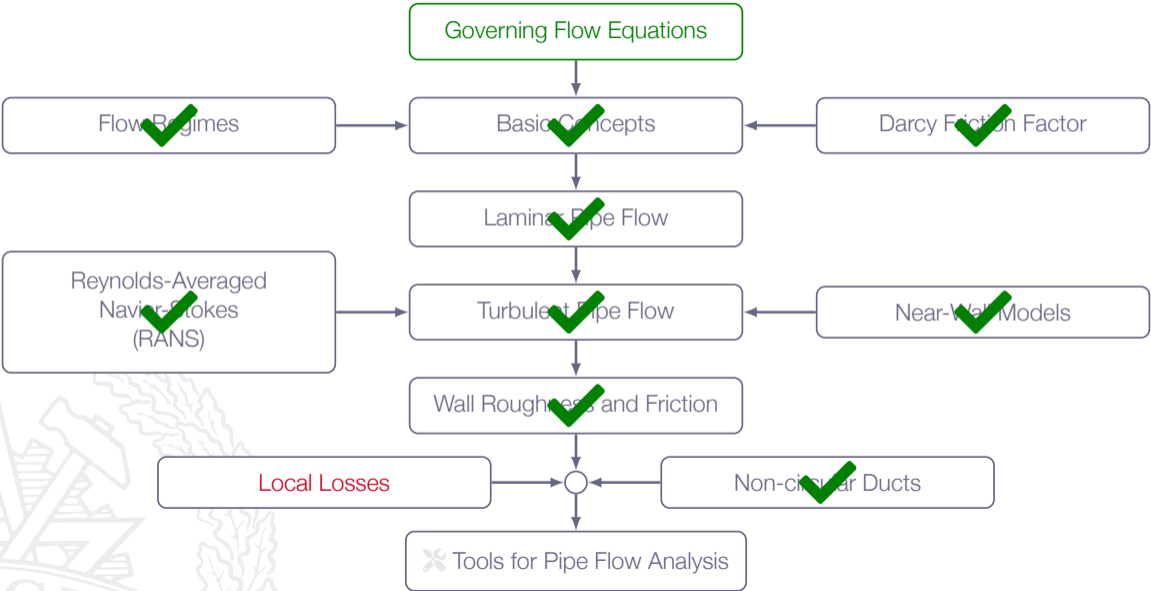
- ▶ vertical distance between plates: a
- ▶ plate width: b



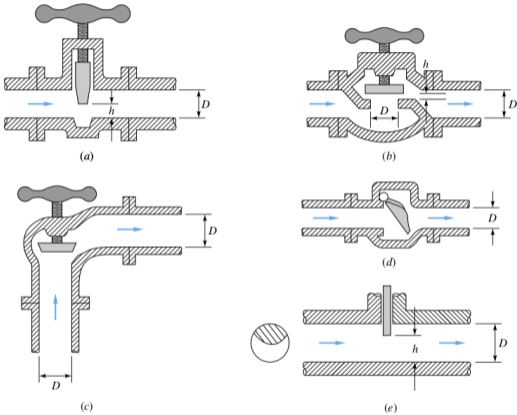
$$D_h = \frac{4A}{\mathcal{P}} = \frac{4ab}{2a + 2b} \Big|_{b \rightarrow \infty} = \frac{4ab}{2b} = 2a$$



Roadmap - Viscous Flow in Ducts



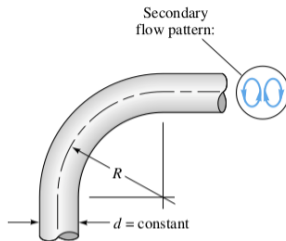
Local Losses



Local Losses

Swirl generated by:

- ▶ Inlets or outlets
- ▶ Sudden area changes
- ▶ Bends
- ▶ Valves
- ▶ Gradual expansions or contractions



$$\Delta p_f = K \frac{\rho V^2}{2}$$

$$\Delta p_{f_{tot}} = \sum_i f_{D_i} \frac{L_i}{D_i} \frac{\rho V_i^2}{2} + \sum_j K_j \frac{\rho V_j^2}{2}$$

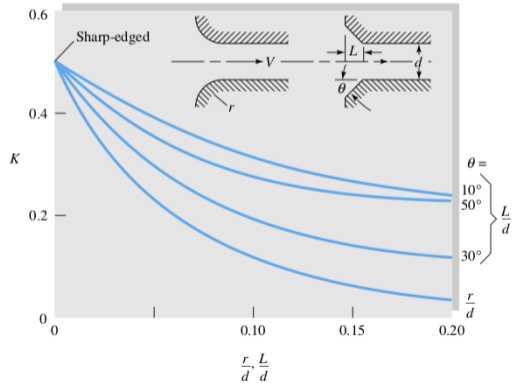
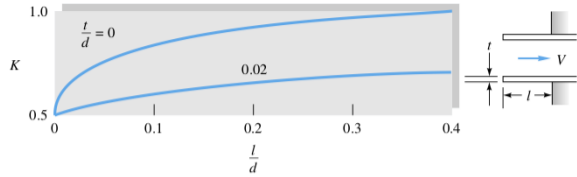
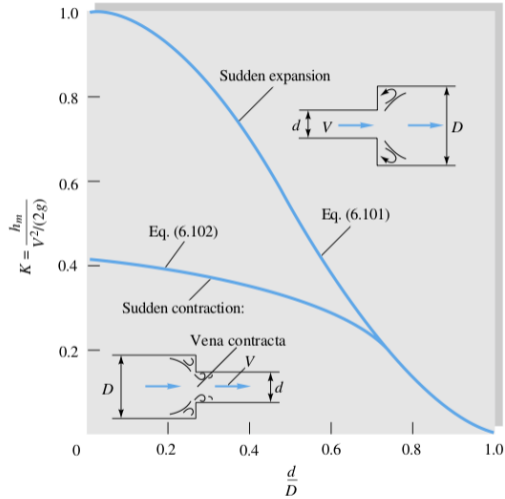
Local Losses

Generated swirl will be damped out by inner friction

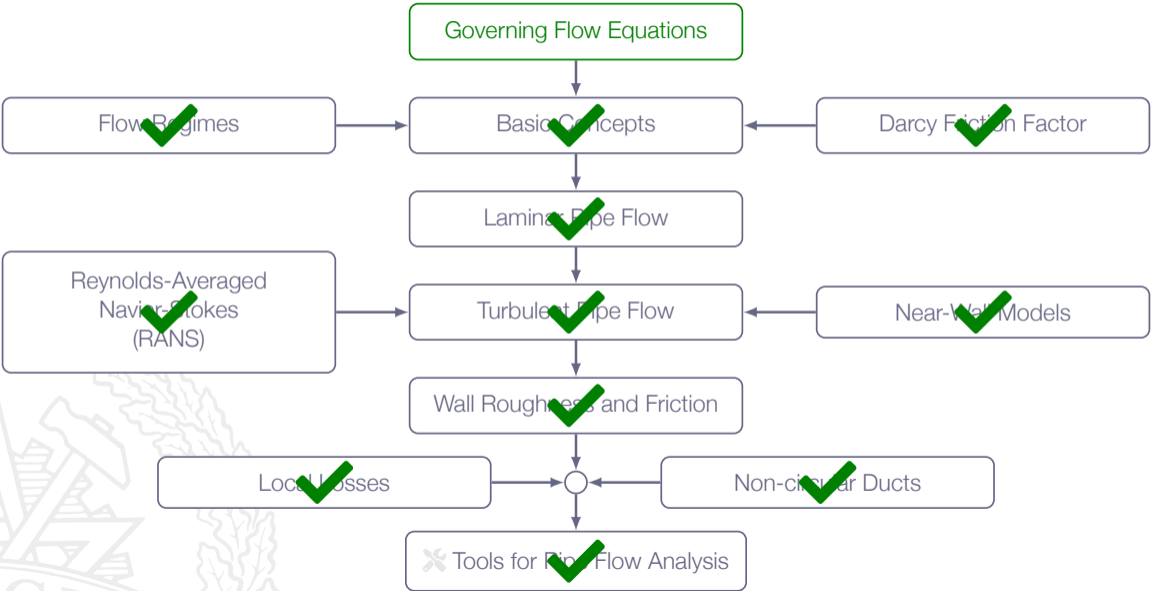
Kinetic energy is converted to internal energy, which results in a pressure loss



Local Losses

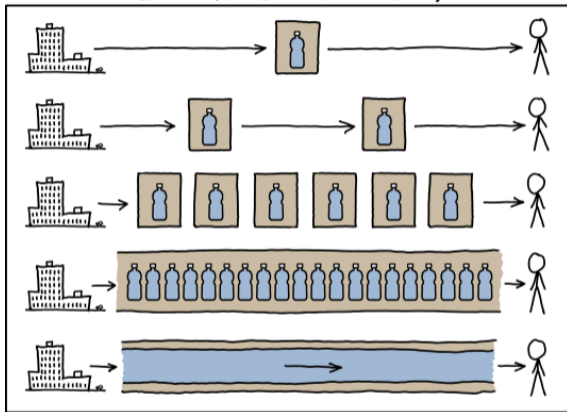


Roadmap - Viscous Flow in Ducts



On-Demand Hyperloop-Style Water Delivery

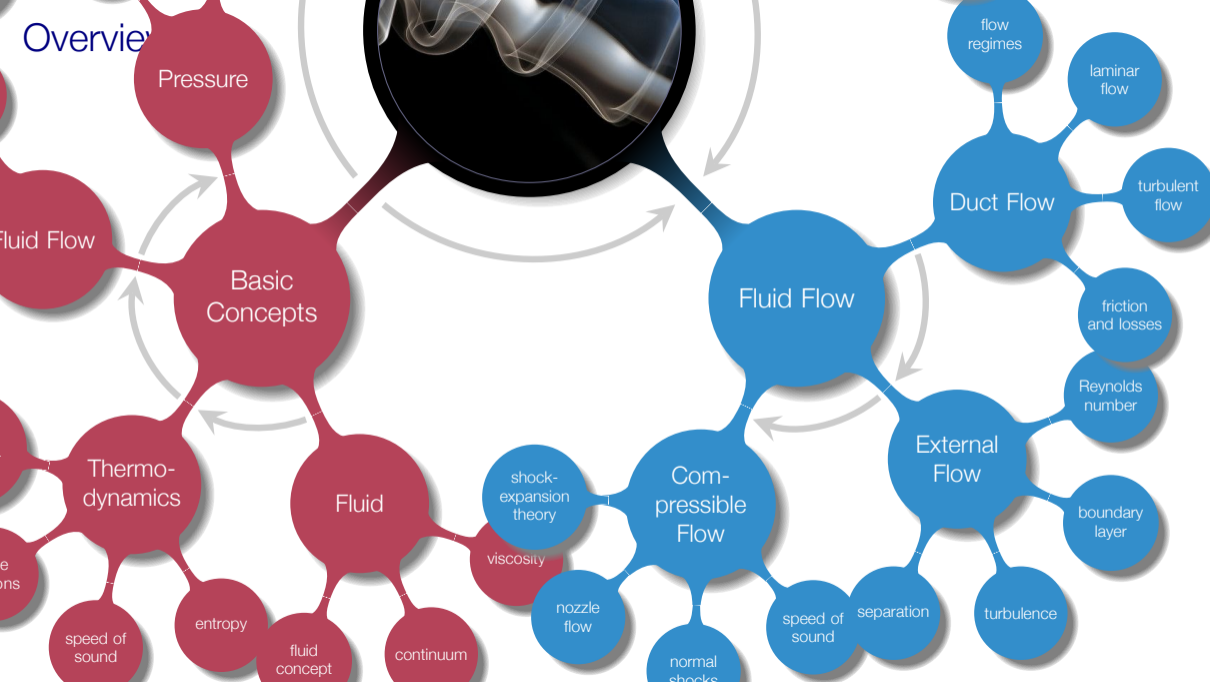
NOW THAT AMAZON IS ADVERTISING
ONE-HOUR DELIVERY OF BOTTLED WATER,



I VOTE WE START CALLING MUNICIPAL PLUMBING
"ON-DEMAND HYPERLOOP-STYLE WATER DELIVERY"
AND SEE IF WE CAN SELL ANYONE ON THE IDEA.



Chapter 7 - Flow Past Immersed Bodies

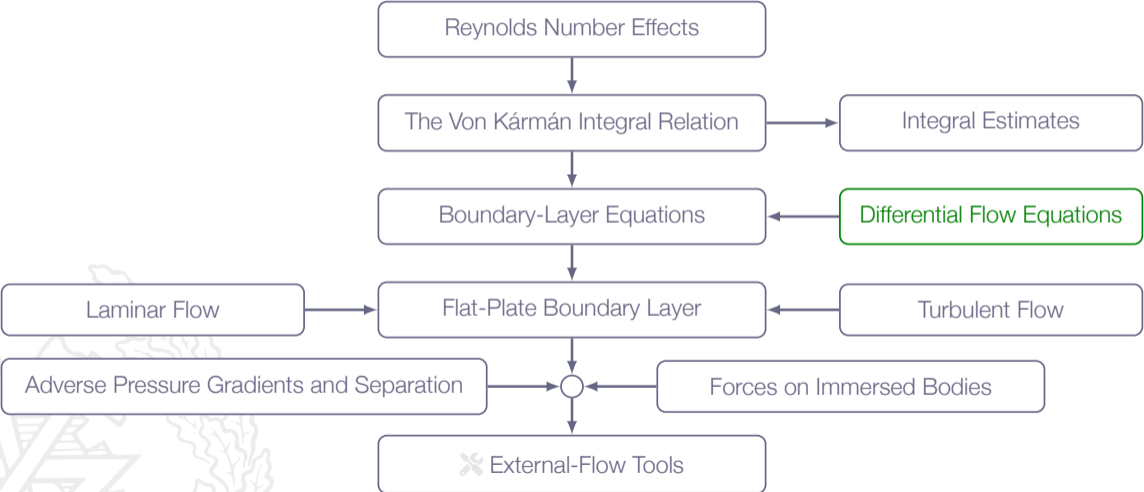


Learning Outcomes

- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 6 **Explain** what a boundary layer is and when/where/why it appears
- 21 **Explain** how the flat plate boundary layer is developed (transition from laminar to turbulent flow)
- 22 **Explain** and use the Blasius equation
- 23 **Define** the Reynolds number for a flat plate boundary layer
- 24 **Explain** what is characteristic for a turbulent flow
- 29 **Explain** flow separation (separated cylinder flow)
- 30 **Explain** how to delay or avoid separation
- 31 **Derive** the boundary layer formulation of the Navier-Stokes equations
- 32 **Understand** and explain displacement thickness and momentum thickness
- 33 **Understand, explain** and **use** the concepts drag, friction drag, pressure drag, and lift

Let's take a deep dive into boundary-layer theory

Roadmap - Flow Past Immersed Bodies

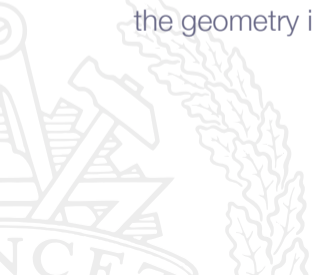


A silver Mercedes-Benz SUV is positioned in a wind tunnel. The car is facing forward, and its license plate reads 'S MB 7177'. The tunnel's interior is dark, and the car is illuminated from the front. Numerous white, curved lines of varying thickness and opacity are superimposed over the scene, representing the flow of air around the vehicle. These lines originate from the right side of the frame and curve around the car, illustrating aerodynamic forces. The background shows the structural elements of the wind tunnel, including a large black cylindrical duct on the left and a metallic wall with a grid pattern on the right.

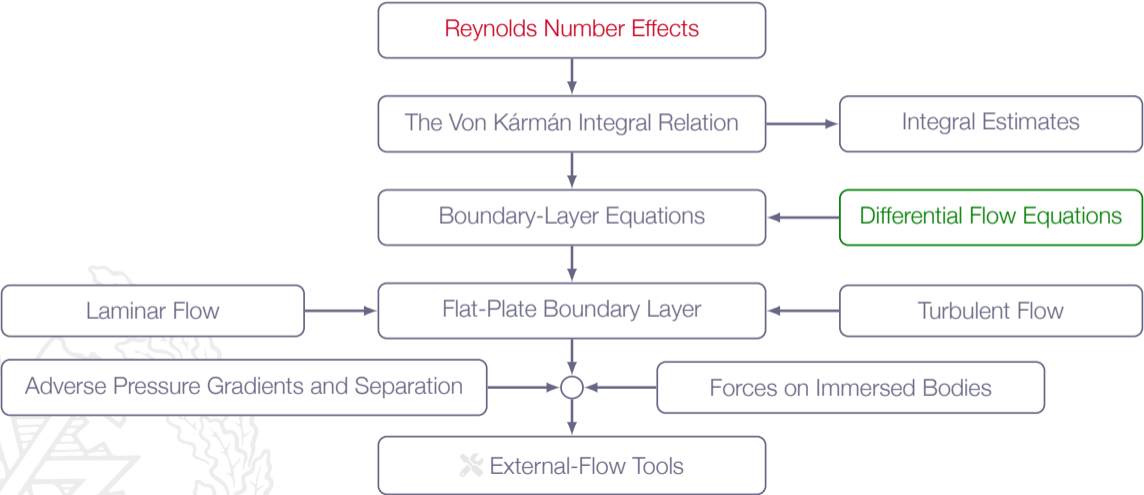
"Understanding the mechanisms behind flow-related forces is a key factor to success in many engineering applications"

External Flow

- ▶ Significant viscous effects near the surface of an immersed body
- ▶ Nearly inviscid far from the body
- ▶ Unconfined - boundary layers are free to grow
- ▶ Most often CFD or experiments are needed to analyze an external flow unless the geometry is very simple

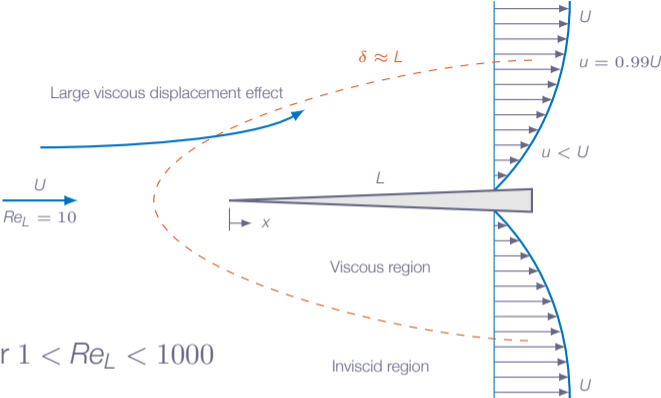


Roadmap - Flow Past Immersed Bodies

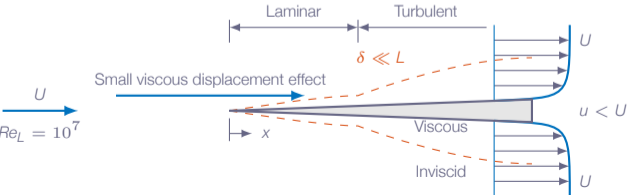


Reynolds Number Effects

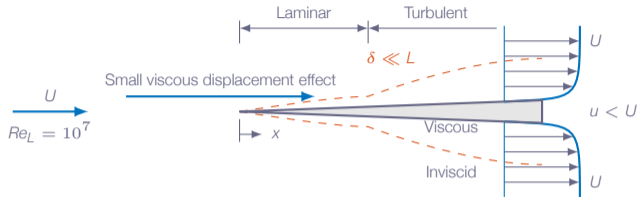
$$Re_L = \frac{UL}{\nu}$$



Note: no simple theory exists for $1 < Re_L < 1000$



Reynolds Number Effects



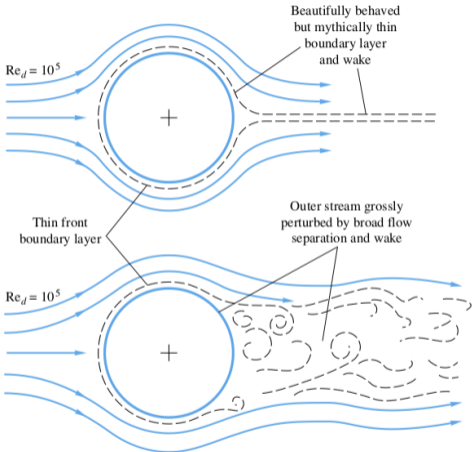
$$\frac{\delta}{x} \approx \begin{cases} \frac{5.0}{Re_x^{1/2}} & \text{laminar} & 10^3 < Re_x < 10^6 \\ \frac{0.16}{Re_x^{1/7}} & \text{turbulent} & 10^6 < Re_x \end{cases}$$

$$Re_L = \frac{UL}{\nu}$$

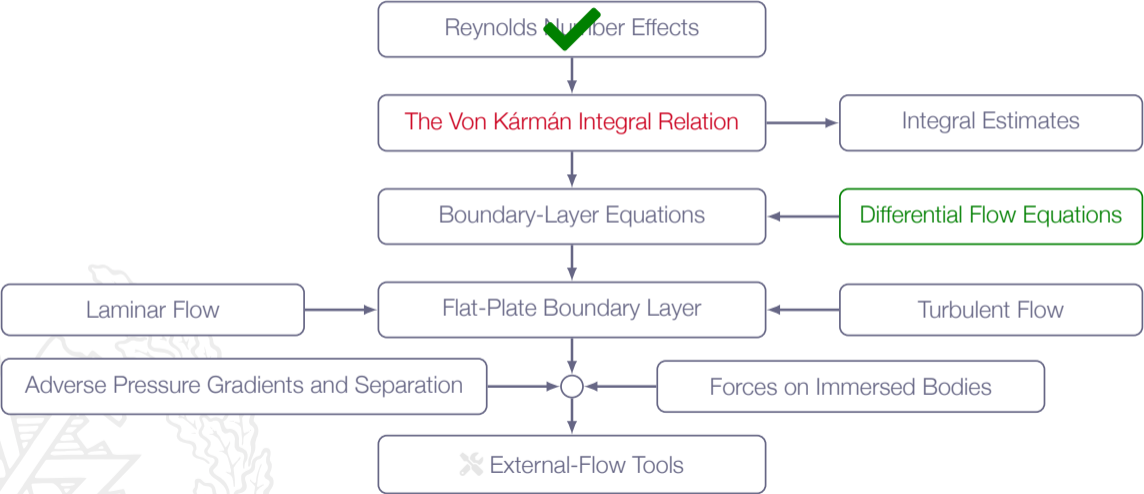
$$Re_x = \frac{Ux}{\nu}$$

Note! Re_L and the local Reynolds number Re_x are not the same

Blunt Objects



Roadmap - Flow Past Immersed Bodies



The Von Kármán Integral Relation

- ▶ Approximate solutions for $\delta(x)$ and $\tau_w(x)$
- ▶ Control volume approach applied to a boundary layer
- ▶ Assuming steady-state incompressible flow

$$\sum \mathbf{F} = \int_{CS} \mathbf{V} \rho (\mathbf{V} \cdot \mathbf{n}) dA$$



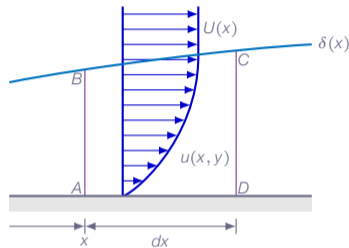
The Von Kármán Integral Relation

Massflow

$$\dot{m}_{AB} = \rho \int_0^{\delta} u dy$$

$$\dot{m}_{CD} = \rho \int_0^{\delta} u dy + \frac{d}{dx} \left[\rho \int_0^{\delta} u dy \right] dx$$

$$\dot{m}_{BC} = \rho \frac{d}{dx} \left[\int_0^{\delta} u dy \right] dx$$



The Von Kármán Integral Relation

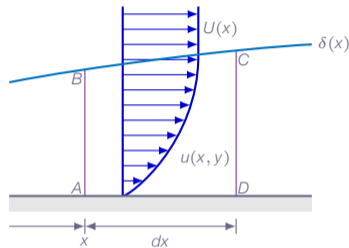
Momentum

$$\dot{I}_{AB} = \rho \int_0^{\delta} u^2 dy$$

$$\dot{I}_{CD} = \rho \int_0^{\delta} u^2 dy + \frac{d}{dx} \left[\rho \int_0^{\delta} u^2 dy \right] dx$$

$$\dot{I}_{BC} = U \dot{m}_{BC} = \rho U \frac{d}{dx} \left[\int_0^{\delta} u dy \right] dx$$

$$\dot{I}_{CD} - \dot{I}_{AB} - \dot{I}_{BC} = \rho \frac{d}{dx} \left[\int_0^{\delta} u^2 dy \right] dx - \rho U \frac{d}{dx} \left[\int_0^{\delta} u dy \right] dx$$



The Von Kármán Integral Relation

Pressure Forces

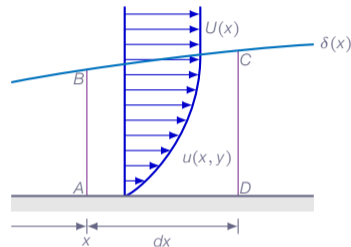
$$AB: p\delta$$

$$CD: -\left(p + \frac{dp}{dx}dx\right)\left(\delta + \frac{d\delta}{dx}dx\right)$$

$$BC: \approx \left(p + \frac{1}{2}\frac{dp}{dx}dx\right)\frac{d\delta}{dx}dx$$

Shear Forces

$$AD: -\tau_w dx$$



The Von Kármán Integral Relation

Forces

$$dF_x = -\tau_w dx + p\delta - \left[p\delta + p \frac{d\delta}{dx} dx + \delta \frac{dp}{dx} dx + \frac{dp}{dx} \frac{d\delta}{dx} dx dx \right] + p \frac{d\delta}{dx} dx + \frac{1}{2} \frac{dp}{dx} \frac{d\delta}{dx} dx dx$$

products of infinitesimal quantities can be regarded to be zero and thus

$$dF_x = -\tau_w dx - \delta \frac{dp}{dx} dx$$

The Von Kármán Integral Relation

Momentum equation

Now we have all components of the momentum equation defined

$$\rho \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] - \rho U \frac{d}{dx} \left[\int_0^\delta u dy \right] = -\tau_w - \delta \frac{dp}{dx}$$

The momentum equation for boundary layers or **Von Kármán's integral relation**

Note! the relation is valid for laminar and turbulent flows (for turbulent flows use time-averaged quantities)

The Von Kármán Integral Relation

Outside of the boundary layer the flow is inviscid \Rightarrow we can use Bernoulli

$$\rho + \frac{1}{2}\rho U^2 = \text{const} \Rightarrow \frac{dp}{dx} + \rho U \frac{dU}{dx} = 0 \Rightarrow -\frac{1}{\rho} \frac{dp}{dx} = U \frac{dU}{dx}$$



The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} - \delta U \frac{dU}{dx} = U \frac{d}{dx} \left[\int_0^\delta u dy \right] - \frac{d}{dx} \left[\int_0^\delta u^2 dy \right]$$

$$\delta U \frac{dU}{dx} = U \frac{dU}{dx} \int_0^\delta dy$$

$$U \frac{d}{dx} \left[\int_0^\delta u dy \right] = \frac{d}{dx} \left[U \int_0^\delta u dy \right] - \frac{dU}{dx} \int_0^\delta u dy$$

$$\frac{\tau_w}{\rho} - U \frac{dU}{dx} \int_0^\delta dy = \frac{d}{dx} \left[U \int_0^\delta u dy \right] - \frac{dU}{dx} \int_0^\delta u dy - \frac{d}{dx} \left[\int_0^\delta u^2 dy \right]$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} - U \frac{dU}{dx} \int_0^\delta dy = \frac{d}{dx} \left[U \int_0^\delta u dy \right] - \frac{dU}{dx} \int_0^\delta u dy - \frac{d}{dx} \left[\int_0^\delta u^2 dy \right]$$

$$U \frac{dU}{dx} \int_0^\delta dy - \frac{dU}{dx} \int_0^\delta u dy = \frac{dU}{dx} \int_0^\delta (U - u) dy$$

$$\frac{d}{dx} \left[U \int_0^\delta u dy \right] - \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] = \frac{d}{dx} \int_0^\delta u(U - u) dy$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx} \int_0^\delta u(U - u)dy + \frac{dU}{dx} \int_0^\delta (U - u)dy$$



The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx} \int_0^\delta u(U - u)dy + \frac{dU}{dx} \int_0^\delta (U - u)dy$$

Now, let's apply the relation to a laminar boundary layer over a flat plate

$$\frac{dU}{dx} = 0 \Rightarrow \frac{\tau_w}{\rho} = \frac{d}{dx} \int_0^\delta u(U - u)dy$$

We need a velocity profile $u = u(y)$ to continue ...

The Von Kármán Integral Relation

Laminar flow \Rightarrow parabolic profile

$$u(y) = A + By + Cy^2$$

The constants A , B , and C are defined using boundary conditions

$$u(0) = 0 \Rightarrow A = 0$$

$$y = \delta \Rightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow B + 2C\delta = 0 \Rightarrow B = -2\delta C$$

$$y = \delta \Rightarrow u = U \Rightarrow B\delta + C\delta^2 = U \Rightarrow -2C\delta^2 + C\delta^2 = U \Rightarrow C = -\frac{U}{\delta^2}$$

$$u(y) = U \left(\frac{2}{\delta}y - \frac{1}{\delta^2}y^2 \right)$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx} \int_0^\delta u(U - u) dy$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \mu \frac{2U}{\delta}$$

$$\int_0^\delta u(U - u) dy = \int_0^\delta U^2 \left(\frac{2}{\delta} y - \frac{1}{\delta^2} y^2 \right) - U^2 \left(\frac{4}{\delta^2} y^2 - \frac{4}{\delta^3} y^3 + \frac{1}{\delta^4} y^4 \right) dy = \frac{2}{15} U^2 \delta$$

$$\frac{\mu 2U}{\rho \delta} = \frac{d}{dx} \left(\frac{2}{15} U^2 \delta \right) \Rightarrow \frac{\nu}{\delta} = \frac{U}{15} \frac{d\delta}{dx} \Rightarrow \delta d\delta = 15 \frac{\nu}{U} dx$$

The Von Kármán Integral Relation

$$\delta d\delta = 15 \frac{\nu}{U} dx$$

$$\frac{\delta^2}{2} = 15 \frac{\nu}{U} x + C = \{x = 0 \Rightarrow \delta = 0 \Rightarrow C = 0\} = 15 \frac{\nu}{U} x$$

$$\delta = \sqrt{\frac{30\nu x}{U}} \Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{Ux}} \approx \frac{5.5}{\sqrt{Re_x}}$$

The Von Kármán Integral Relation

$$\tau_w = \mu \frac{2U}{\delta} = \frac{2\mu U}{\sqrt{\frac{30\nu x}{U}}} = \frac{2}{\sqrt{30}} \frac{\rho U^2}{\sqrt{\frac{Ux}{\nu}}} \approx \frac{0.365}{\sqrt{Re_x}} \rho U^2$$

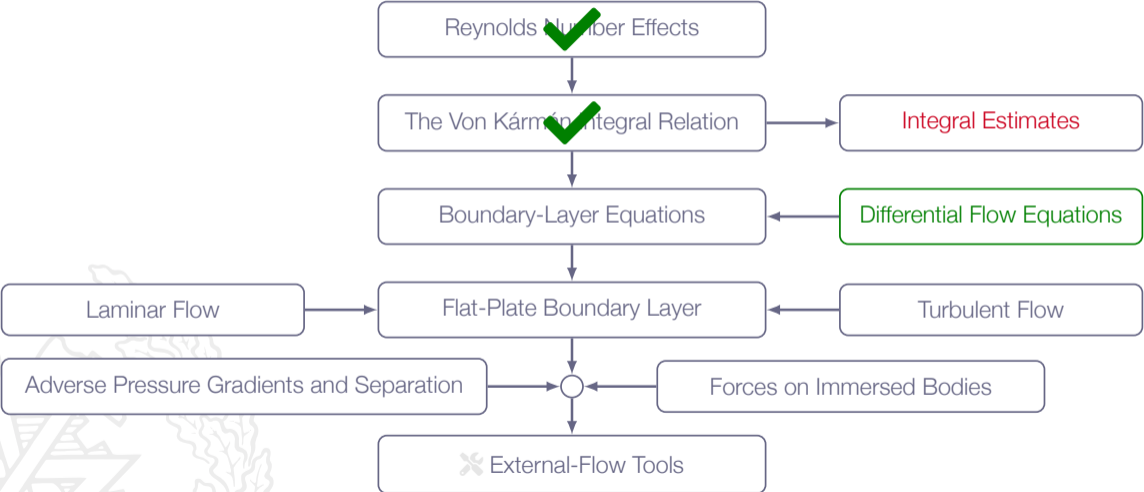
Introducing the **skin friction coefficient** c_f

$$c_f = \frac{2\tau_w}{\rho U^2} \approx \frac{0.73}{\sqrt{Re_x}}$$

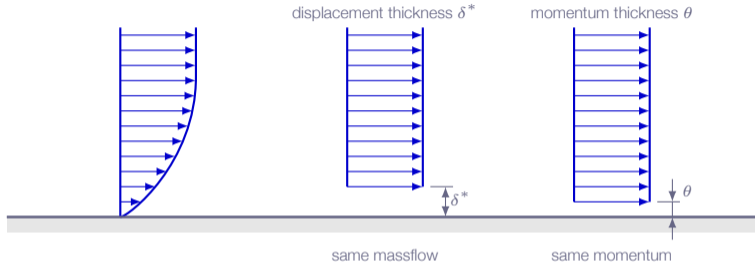
Note! *more accurate solutions for laminar flat plate boundary layers exists:*

$$c_f \approx \frac{0.664}{\sqrt{Re_x}}, \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

Roadmap - Flow Past Immersed Bodies

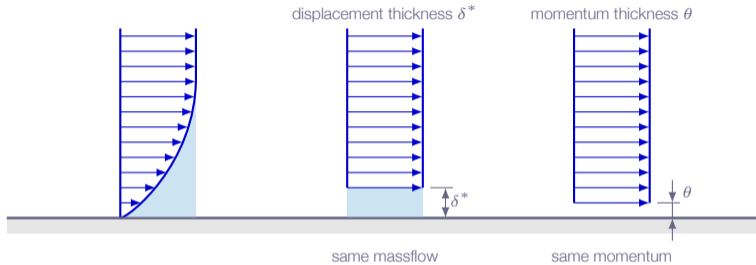


Displacement and Momentum Thickness



"The presence of a boundary layer will result in a small but finite displacement of the flow streamlines"

Displacement Thickness



$$\int_0^{\delta} \rho(U - u) b dy = \rho U b \delta^* \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

Displacement Thickness

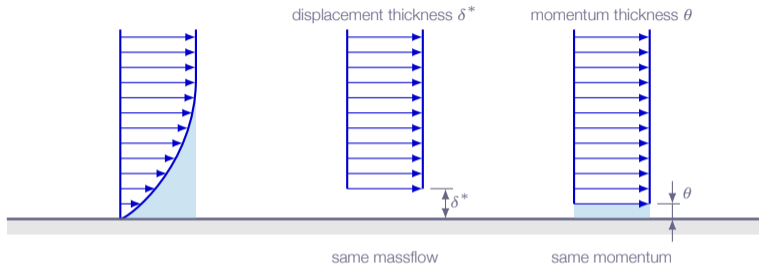
$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

with $u(x, y) \approx U \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$ we get

$$\begin{cases} \delta^* \approx \frac{1}{3}\delta \\ \frac{\delta^*}{x} \approx \frac{1.83}{Re_x^{1/2}} \end{cases}$$

Note! since δ^* is much smaller than x for large values of Re_x , the velocity component in the wall-normal direction will be much smaller than the velocity parallel to the plate

Momentum Thickness



$$\int_0^{\delta} \rho u(U - u) b dy = \rho U^2 b \theta \Rightarrow \theta = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

Momentum Thickness

The drag D for a plate of width b

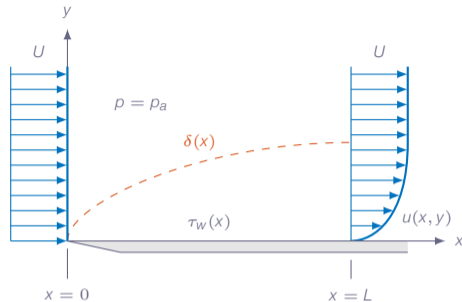
$$D(x) = b \int_0^x \tau_w(x) dx \Rightarrow \frac{dD}{dx} = b\tau_w$$

from before we have

$$\frac{\tau_w}{\rho} = \frac{d}{dx} \int_0^\delta u(U - u) dy = \frac{d}{dx} U^2 \underbrace{\int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy}_\theta = U^2 \frac{d\theta}{dx}$$

and thus

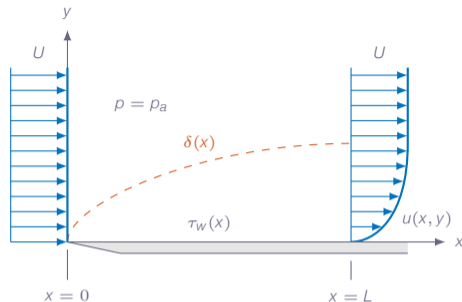
$$\frac{dD}{dx} = b\rho U^2 \frac{d\theta}{dx} \Rightarrow D(x) = \rho b U^2 \theta$$



Momentum Thickness

$$D(x) = \rho b U^2 \theta, \quad \tau_w = \rho U^2 \frac{d\theta}{dx}$$

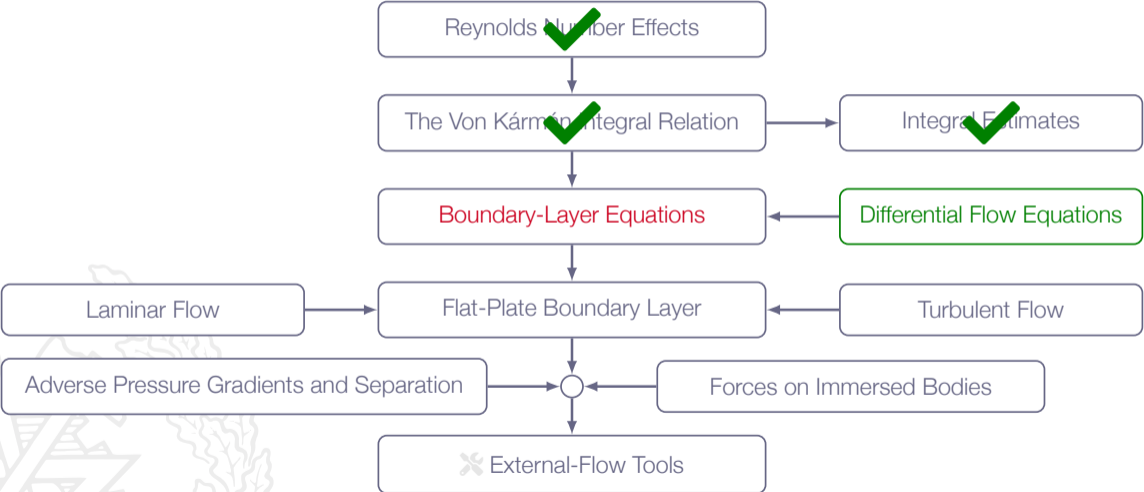
$$\theta = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$



Note!

1. the momentum thickness θ is a measure of the total drag
2. can be used both for laminar and turbulent flows

Roadmap - Flow Past Immersed Bodies



Boundary Layer Equations

- ▶ We will now derive a set of equations suitable for boundary layer analysis
- ▶ Starting point - the non-dimensional equations derived in Chapter 5
- ▶ We will assume two-dimensional, incompressible, steady-state flow
- ▶ We will look at each of the terms and find terms that can be neglected in a thin boundary layer flow



Boundary Layer Equations

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x-momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right)$$

y-momentum:

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re} \left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right)$$

$$x^* = x/L, y^* = y/L$$

$$u^* = u/U, v^* = v/U$$

$$p^* = p/(\rho U^2)$$

$$Re = \frac{UL}{\nu}$$

Boundary Layer Equations

To be able to find the relative sizes of different terms in the equations, we will first have a look at the flow parameters and operators

$$u^* = u/U \sim 1$$

$$x^* = x/L \sim 1$$

$$y^* = y/L \sim \delta^*$$

δ denotes boundary layer thickness and $\delta^* = \delta/L$

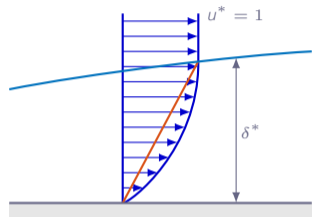
Note! here, δ^* is not the displacement thickness

Boundary Layer Equations

What about derivatives?

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1 - 0}{\delta^*} = \frac{1}{\delta^*}$$

$$\frac{\partial^2 u^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*} \frac{\partial u^*}{\partial y^*} \sim \frac{1/\delta^* - 0}{\delta^*} = \frac{1}{\delta^{*2}}$$



Boundary Layer Equations

continuity:

$$\underbrace{\frac{\partial u^*}{\partial x^*}}_{\sim \frac{1}{l}} + \underbrace{\frac{\partial v^*}{\partial y^*}}_{\sim \frac{?}{\delta^*}} = 0 \Rightarrow v^* \sim \delta^*$$

$\frac{\partial v^*}{\partial y^*}$ must be of the same order of magnitude as $\frac{\partial u^*}{\partial x^*}$ in order to fulfill the continuity equation

Boundary Layer Equations

x-momentum:

$$\underbrace{u^* \frac{\partial u^*}{\partial x^*}}_{\sim 1 \frac{1}{1}} + \underbrace{v^* \frac{\partial u^*}{\partial y^*}}_{\sim \delta^* \frac{1}{\delta^*}} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \left(\underbrace{\frac{\partial^2 u^*}{\partial x^{*2}}}_{\sim \frac{1}{1}} + \underbrace{\frac{\partial^2 u^*}{\partial y^{*2}}}_{\sim \frac{1}{\delta^{*2}}} \right)$$

the boundary layer is assumed to be very thin $\Rightarrow \delta^* \ll 1$ and thus

$$\frac{\partial^2 u^*}{\partial x^{*2}} \ll \frac{\partial^2 u^*}{\partial y^{*2}}$$

assuming the inertial forces to be of the same size as the as the friction forces in the boundary layer we get: $1/Re \sim \delta^{*2}$

Boundary Layer Equations

y-momentum:

$$\underbrace{u^* \frac{\partial v^*}{\partial x^*}}_{\sim 1 \frac{\delta^*}{1}} + \underbrace{v^* \frac{\partial v^*}{\partial y^*}}_{\sim \delta^* \frac{\delta^*}{\delta^*}} = - \frac{\partial p^*}{\partial y^*} + \underbrace{\frac{1}{Re}}_{\sim \delta^{*2}} \left(\underbrace{\frac{\partial^2 v^*}{\partial x^{*2}}}_{\sim \frac{\delta^*}{1}} + \underbrace{\frac{\partial^2 v^*}{\partial y^{*2}}}_{\sim \frac{\delta^*}{\delta^{*2}}} \right)$$

examining the equation we see that all terms are at most of size $\delta^* \Rightarrow \frac{\partial p^*}{\partial y^*} \sim \delta^*$

δ^* is small $\Rightarrow p$ is independent of y

Boundary Layer Equations

With the knowledge gained, we now move back to the dimensional equations

laminar

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

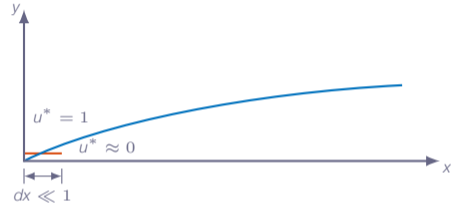
$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

turbulent

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{d\bar{p}}{dx} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y} \overline{u'v'}$$

Boundary Layer Equations



Limitations

1. The boundary layer equations do not apply close to the start of the boundary layer where $\frac{\partial u^*}{\partial x^*} \gg 1$
2. The equations are derived assuming a thin boundary layer

Boundary Layer Equations

The pressure derivative can be replaced with a velocity derivative

Outside of the boundary layer the flow is inviscid \Rightarrow we can use the Bernoulli equation

$$p + \frac{1}{2}\rho U^2 = \text{const} \Rightarrow \frac{dp}{dx} + \rho U \frac{dU}{dx} = 0 \Rightarrow -\frac{1}{\rho} \frac{dp}{dx} = U \frac{dU}{dx}$$

Boundary Layer Equations

laminar boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

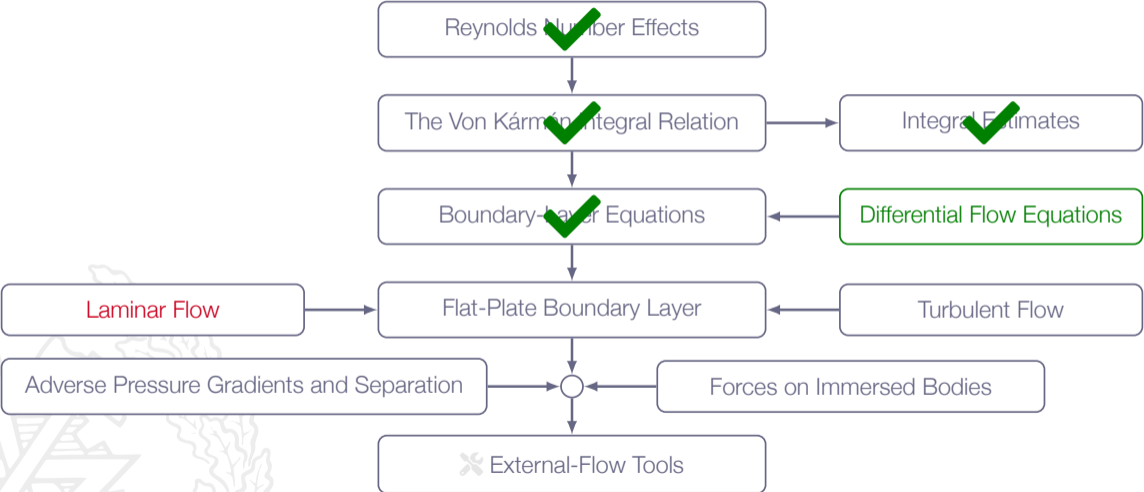
Two equations and two unknowns \Rightarrow possible to solve 😊

Boundary Layer Equations

Note! the boundary layer equations can be used for curved surfaces if the boundary layer thickness δ is small compared to the curvature radius r



Roadmap - Flow Past Immersed Bodies



The Blasius Velocity Profile

For laminar flow the boundary layer equations can be solved exactly for u and v

Blasius presented a solution 1908 where he had used a coordinate transformation and showed that $\frac{u}{U}$ is a function of a single dimensionless variable $\eta = y\sqrt{\frac{U}{\nu x}}$

The coordinate transformation corresponds to a scaling of the y coordinate with the boundary layer thickness δ

$$\frac{y}{\delta} \sim \frac{y}{x/\sqrt{Re_x}} = \frac{y}{x} \sqrt{\frac{Ux}{\nu}} = y\sqrt{\frac{U}{\nu x}}$$

The Blasius Velocity Profile

1. Rewrite the boundary layer equations using the stream function (Chapter 4)
2. Rewrite the equation again $\Psi = f(\eta)\sqrt{\nu Ux}$ where η is the scaled wall-normal coordinate and $f(\eta)$ is a nondimensional streamfunction
3. Lots of math

The Navier-Stokes equations are reduced to an ordinary differential equation (ODE)

$$f''' + \frac{1}{2}ff'' = 0$$

with the boundary conditions

$$\begin{cases} f(0) = f'(0) = 0 \\ f'_{\eta \rightarrow \infty} \rightarrow 1.0 \end{cases}$$

The Blasius Velocity Profile

$$\frac{u}{U} = f'(\eta)$$

Note! $u/U \rightarrow 1$ as $y \rightarrow \infty$ and therefore δ is usually defined as the distance from the wall where $u/U = 0.99$

$$\delta_{99\%} \sqrt{\frac{U}{\nu x}} \approx 5.0 \quad \text{or} \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$



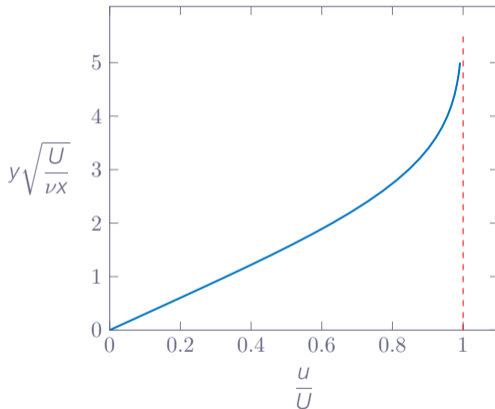
The Blasius Velocity Profile

$$\frac{u}{U} = f'(\eta)$$

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

$$\delta_{99\%}\sqrt{\frac{U}{\nu x}} \approx 5.0$$

Laminar boundary layer (Blasius)



The Blasius Velocity Profile

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \mu U \left[\frac{d}{d\eta} \left(\frac{u}{U} \right) \frac{d\eta}{dy} \right]_{\eta=0}$$

$$\frac{d\eta}{dy} = \sqrt{\frac{U}{\nu x}} \Rightarrow \tau_w = \mu U \sqrt{\frac{U}{\nu x}} \frac{d}{d\eta} \left(\frac{u}{U} \right)_{\eta=0} = \frac{\rho U^2}{\sqrt{Re_x}} \frac{d}{d\eta} \left(\frac{u}{U} \right)_{\eta=0}$$

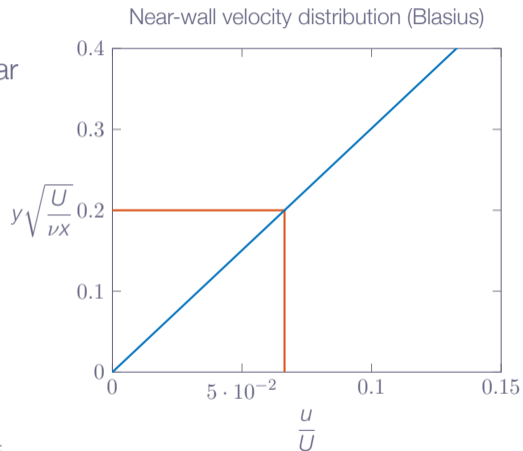
The Blasius Velocity Profile

close to the wall the velocity profile is linear

$$\eta = 0.2 \Rightarrow \frac{u}{U} = 0.0664$$

$$\frac{d}{d\eta} \left(\frac{u}{U} \right)_{\eta=0} = \frac{0.0664}{0.2} = 0.332$$

$$\tau_w = \frac{\rho U^2}{\sqrt{Re_x}} \frac{d}{d\eta} \left(\frac{u}{U} \right)_{\eta=0} = 0.332 \frac{\rho U^2}{\sqrt{Re_x}}$$



Laminar Boundary Layer

$$\tau_w(x) = \frac{0.332\rho^{1/2}\mu^{1/2}U^{3/2}}{x^{1/2}}$$

Note! the wall shear stress drops off with increasing distance due to the boundary layer growth

Recall *for pipe flow, the wall shear stress is independent of x*

Laminar Boundary Layer

wall shear stress:

$$\tau_w(x) = \frac{0.332\rho^{1/2}\mu^{1/2}U^{3/2}}{x^{1/2}}$$

drag force:

$$D(x) = b \int_0^x \tau_w(x) dx = 0.664b\rho^{1/2}\mu^{1/2}U^{3/2}x^{1/2}$$

drag coefficient:

$$C_D = \frac{2D(L)}{\rho U^2 b L} = \frac{1.328}{\sqrt{Re_L}}$$

Laminar Boundary Layer

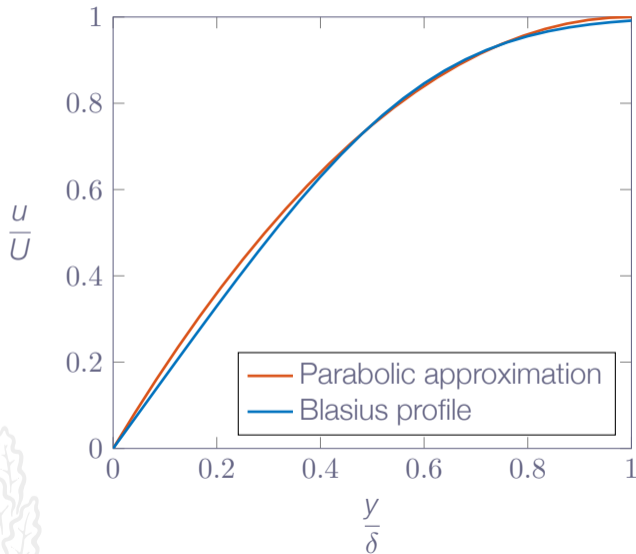
From before we have $D(x) = \rho b \int_0^{\delta(x)} u(U - u)dy$

If we calculate the drag coefficient using $D(x)$ above, we get

$$C_D = \frac{2D(L)}{\rho U^2 b L} = \frac{2}{L} \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy = \frac{2\theta(L)}{L}$$

and thus $\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$

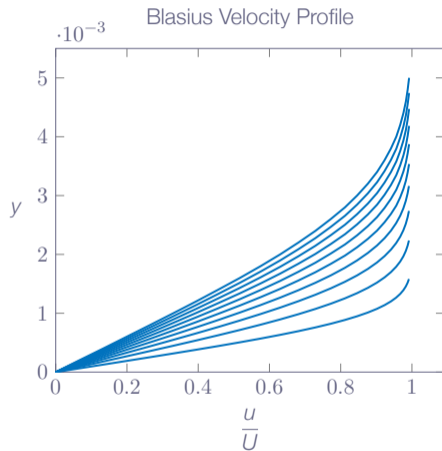
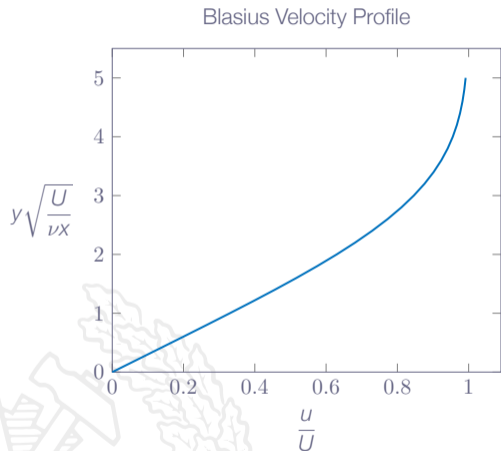
Laminar Boundary Layer



Laminar Boundary Layer

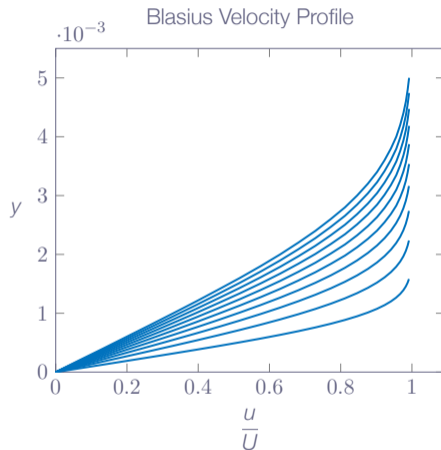
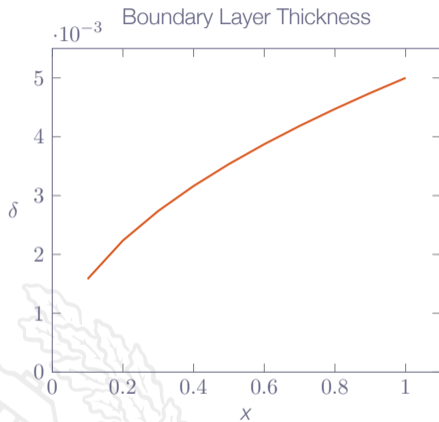
description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	
wall shear stress	τ_w	$0.332 \frac{\rho U^2}{\sqrt{Re_x}}$	
local skin friction coefficient	$C_f = \frac{2\tau_w}{\rho U^2}$	$\frac{0.664}{\sqrt{Re_x}}$	
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	

The Blasius Velocity Profile - Self Similarity



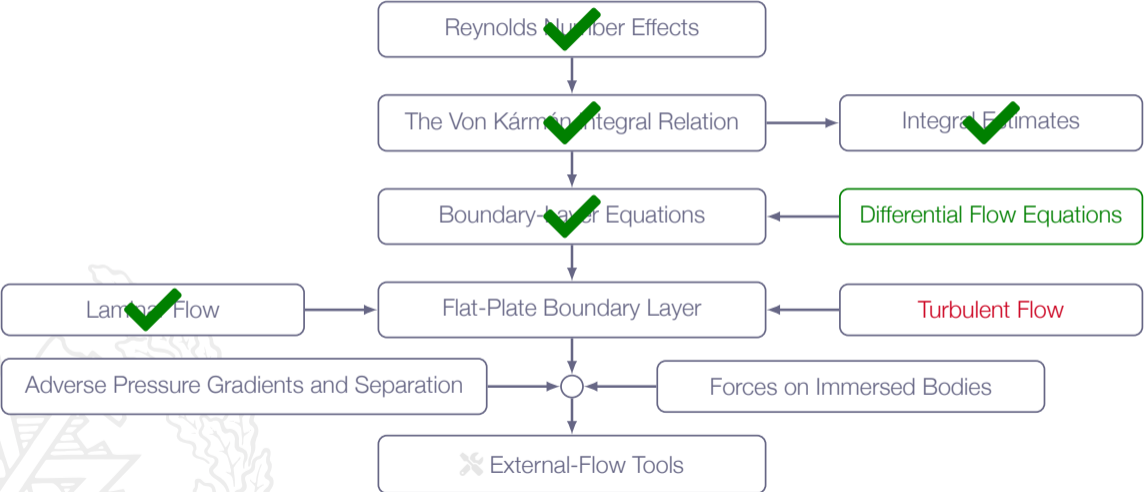
$$y = \eta\sqrt{\frac{\nu x}{U}}, x \in \{0.1 : 0.1 : 1.0\}, U = 1.0, \nu = 10^{-6}$$

The Blasius Velocity Profile - Boundary Layer Growth

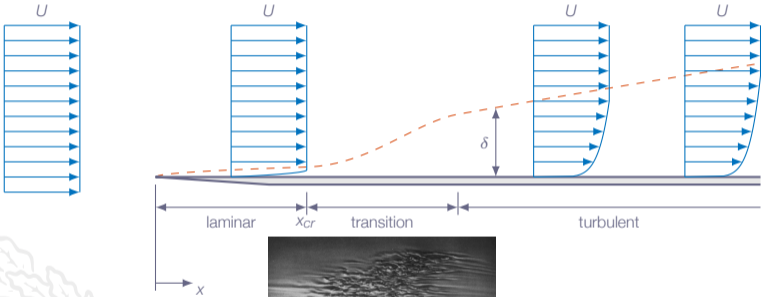


$$y = \eta \sqrt{\frac{\nu X}{U}}, \quad x \in \{0.1 : 0.1 : 1.0\}, \quad U = 1.0, \quad \nu = 10^{-6}$$

Roadmap - Flow Past Immersed Bodies



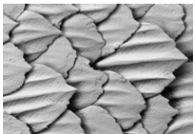
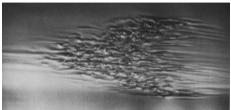
Boundary Layer Transition



--- $u=0.99 U$

$$Re_x = \frac{Ux}{\nu}$$

$$Re_{x_{cr}} = \frac{Ux_{cr}}{\nu} \approx 5.0 \times 10^5$$



Boundary Layer Transition

- ▶ For small Re_x , disturbances in the flow are damped out by viscous forces
- ▶ For somewhat higher Reynolds numbers, friction forces are less important and the flow becomes unstable
- ▶ The transition region is short - can be treated as a point (the transition point)



Boundary Layer Transition

The onset of transition from laminar to turbulent is affected by a number of factors such as:

- ▶ Turbulence in the freestream
- ▶ Surface roughness
- ▶ Pressure gradient

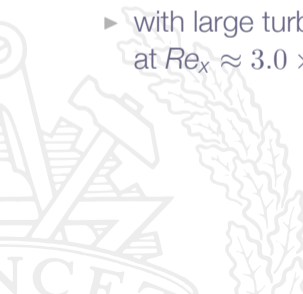
With a smooth surface, no turbulence in the freestream, and zero pressure gradient, the onset of transition can be pushed up to $Re_x \approx 3.0 \times 10^6$

As a rule of thumb, we can assume $Re_{x_{cr}} \approx 5.0 \times 10^5$

Boundary Layer Transition

Freestream turbulence:

- ▶ freestream turbulence reduces the critical Reynolds number
- ▶ with large turbulence intensity in the freestream, the transition can start already at $Re_x \approx 3.0 \times 10^5$ or lower



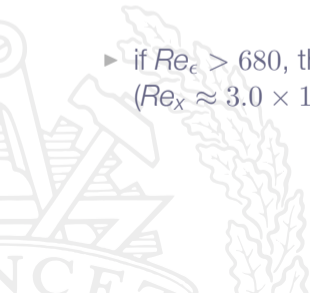
Boundary Layer Transition

Surface roughness:

- ▶ surface roughness does not affect transition significantly if $Re_\epsilon = \frac{U_\epsilon}{\nu} < 680$

Note! rule of thumb

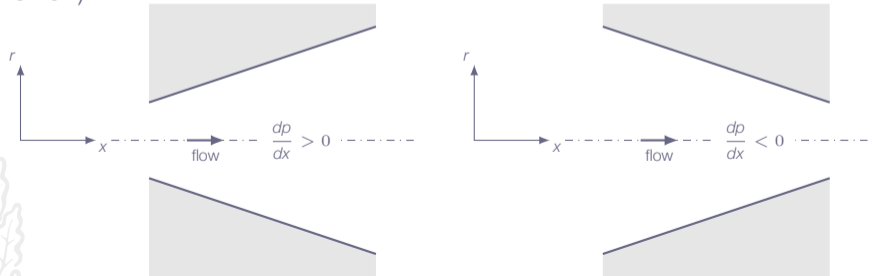
- ▶ if $Re_\epsilon > 680$, the extent of the laminar region can be shortened significantly ($Re_x \approx 3.0 \times 10^5$)



Boundary Layer Transition

Pressure gradient:

- ▶ increasing pressure in the flow direction makes the flow more unstable and thus more prone to separation
- ▶ decreasing pressure in the flow direction has a stabilizing effect on the flow (delayed transition)



Boundary Layer Transition

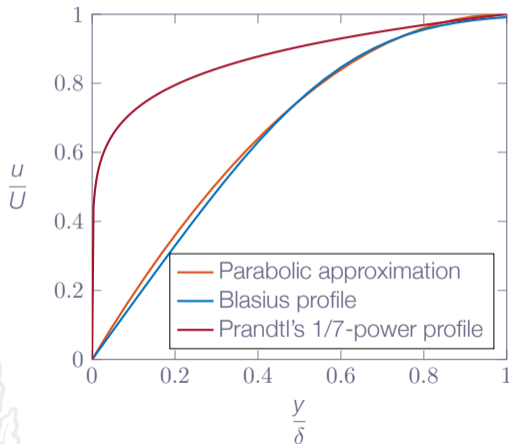
Forced transition:

- ▶ a trip wire or added surface roughness can make the transition to turbulence really fast
- ▶ the critical Reynolds number is not meaningful if the boundary layer is forced to transition



Flat Plate - Turbulent Boundary Layer

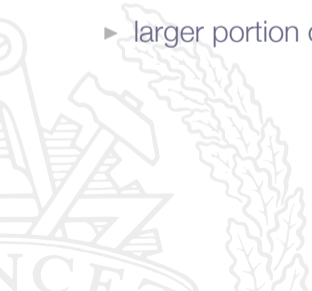
The velocity profile is far from the parabolic profile used for laminar flow



Flat Plate - Turbulent Boundary Layer

A turbulent boundary layer grows faster than a laminar boundary layer

- ▶ the velocity fluctuations (u' , v' , w') leads to increased exchange of momentum
- ▶ increased shear stress compared to the laminar case where we only have forces related to molecular viscosity
- ▶ larger portion of the fluid will be decelerated close to the wall



Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

$$\frac{u}{u^*} \approx \frac{1}{\kappa} \ln \left(\frac{yu^*}{\nu} \right) + B \text{ where } \kappa = 0.41 \text{ and } B = 5.0$$

u^* is the **friction velocity** defined as $u^* = \sqrt{\frac{\tau_w}{\rho}}$

at the edge of the boundary layer $u = U$ and $y = \delta$ and thus

$$\frac{U}{u^*} \approx \frac{1}{\kappa} \ln \left(\frac{\delta u^*}{\nu} \right) + B$$

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

The **skin friction coefficient** c_f is defined as $c_f = \frac{2\tau_w}{\rho U^2} \Rightarrow \tau_w = c_f \frac{1}{2} \rho U^2$

the **friction velocity** can be expressed as $u^* = \sqrt{\frac{\tau_w}{\rho}} = U \sqrt{\frac{c_f}{2}}$

insert in the **log-law** and we get

$$\sqrt{\frac{2}{c_f}} \approx \frac{1}{\kappa} \ln \left(Re_\delta \sqrt{\frac{c_f}{2}} \right) + B$$

rather difficult to work with ...

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Prandtl suggested the following relations:

$$c_f \approx 0.02 Re_\delta^{-1/6}$$

$$\frac{u}{U} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

from before we have the following relation: $\tau_w = \rho U^2 \frac{d\theta}{dx} \Rightarrow c_f = 2 \frac{d\theta}{dx}$

calculate the **momentum thickness** $\theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy = \frac{7}{72} \delta$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Now, combining the two **skin friction coefficient** relations we see that

$$0.02Re_{\delta}^{-1/6} = 2 \frac{d}{dx} \left(\frac{7}{72} \delta \right)$$

$$\text{and thus } Re_{\delta}^{-1/6} \approx 9.72 \frac{d\delta}{dx} = 9.72 \frac{d(Re_{\delta})}{d(Re_x)}$$

$$\text{integration gives } Re_{\delta} \approx 0.16 Re_x^{6/7} \text{ or } \frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$$

Note! the turbulent boundary layer grows significantly faster than the laminar

$$\delta_{turb} \propto x^{6/7} \text{ vs } \delta_{lam} \propto x^{1/2}$$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

$$C_f \approx \frac{0.027}{Re_x^{1/7}}$$

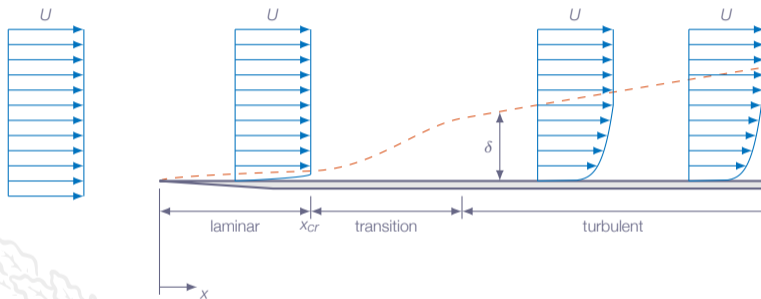
$$\tau_{W_{turb}} \approx \frac{0.0135 \mu^{1/7} \rho^{6/7} U^{13/7}}{x^{1/7}}$$

Note! friction drops slowly with x , increases nearly as ρ and U^2 , and is rather insensitive to viscosity

Flat Plate - Turbulent Boundary Layer

description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	$\frac{0.16}{Re_x^{1/7}}$
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	$\frac{0.02}{Re_x^{1/7}}$
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.016}{Re_x^{1/7}}$
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	1.29
wall shear stress	τ_w	$0.332 \frac{\rho U^2}{\sqrt{Re_x}}$	$0.0135 \frac{\rho U^2}{Re_x^{1/7}}$
local skin friction coefficient	$C_f = \frac{2\tau_w}{\rho U^2}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.027}{Re_x^{1/7}}$
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	$\frac{0.031}{Re_L^{1/7}}$

Flat Plate Boundary Layer



--- $u=0.99 U$

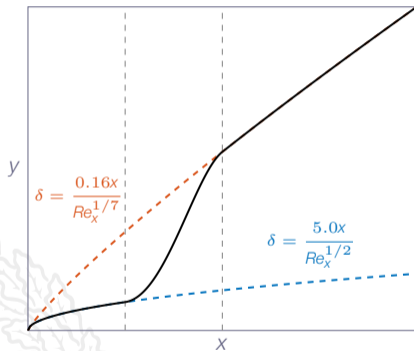
$$Re_x = \frac{Ux}{\nu}$$

$$Re_{x_{cr}} = \frac{Ux_{cr}}{\nu} \approx 5.0 \times 10^5$$

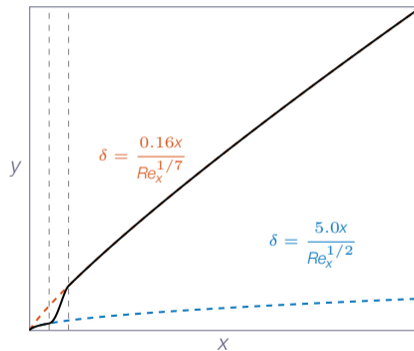
$$D = b \frac{1}{2} \rho U^2 \left[\int_{>0}^{x_{cr}} \frac{0.664}{Re_x^{1/2}} dx + \int_{x_{cr}}^L \frac{0.027}{Re_x^{1/7}} dx \right]$$

Flat Plate Boundary Layer

Boundary layer thickness



Boundary layer thickness



For a long boundary layer the length of the laminar region becomes relatively short in comparison with the length of the turbulent region

Wall Roughness

laminar:

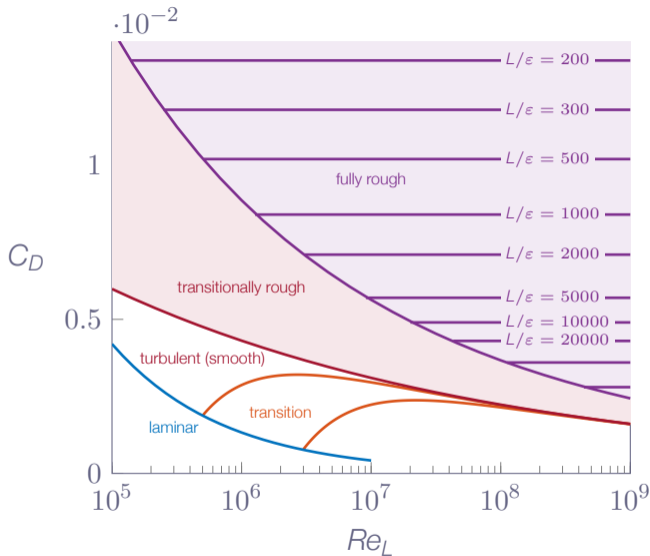
$$C_D = \frac{1.328}{Re_L^{1/2}}$$

turbulent (smooth):

$$C_D = \frac{0.031}{Re_L^{1/7}}$$

turbulent (fully rough):

$$C_D = (1.89 + 1.62 \log(L/\epsilon))^{-2.5}$$



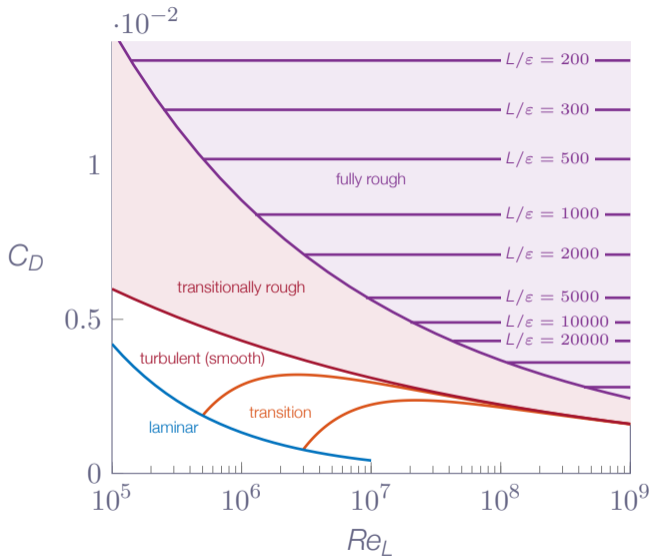
Wall Roughness

transition ($Re_{trans} = 5.0 \times 10^5$):

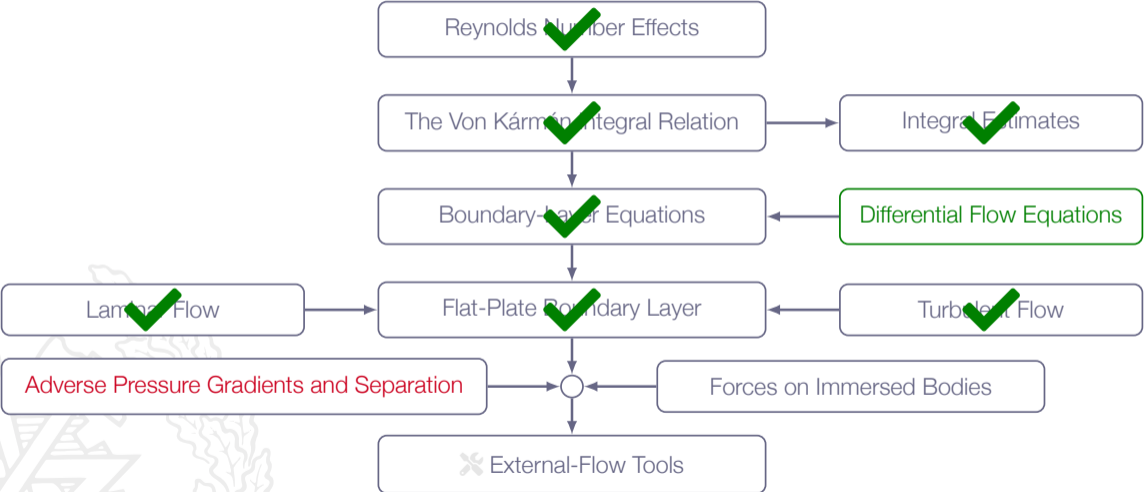
$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L}$$

transition ($Re_{trans} = 3.0 \times 10^6$):

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L}$$

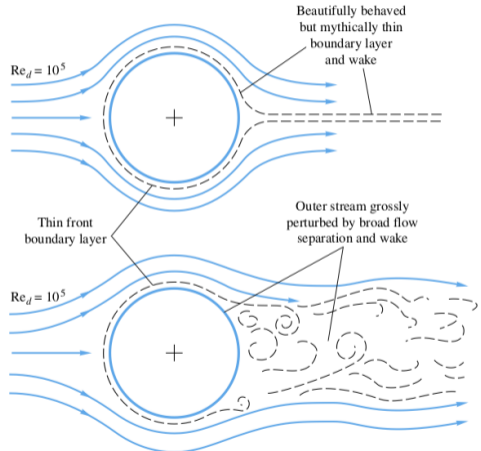


Roadmap - Flow Past Immersed Bodies



Pressure Gradient

- ▶ Adverse pressure gradient
 - ▶ pressure increases in the flow direction
 - ▶ may lead to separation
- ▶ Favorable pressure gradient
 - ▶ pressure decreases in the flow direction
 - ▶ the flow will not separate
- ▶ Separation mechanism
 - ▶ loss of momentum near the wall
 - ▶ adverse pressure gradient
 - ▶ decelerated fluid will force flow to separate from the body



Pressure Gradient

Boundary layer formulation of the momentum equation:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y}$$

with $u = v = 0$ close at the wall, we get

$$\left. \frac{\partial \tau}{\partial y} \right|_{wall} = \mu \left. \frac{\partial^2 u}{\partial y^2} \right|_{wall} = -\rho U \frac{dU}{dx} = \frac{dp}{dx} \Rightarrow \left. \frac{\partial^2 u}{\partial y^2} \right|_{wall} = \frac{1}{\mu} \frac{dp}{dx}$$

Note! applies both for laminar and turbulent flow

Pressure Gradient

$$\left. \frac{\partial^2 u}{\partial y^2} \right|_{wall} = \frac{1}{\mu} \frac{dp}{dx}$$

Adverse pressure gradient ($\frac{dp}{dx} > 0$):

$$\frac{\partial^2 u}{\partial y^2} > 0 \text{ at the wall}$$

$$\frac{\partial^2 u}{\partial y^2} < 0 \text{ at the outer layer } y = \delta$$

thus $\frac{\partial^2 u}{\partial y^2} = 0$ somewhere in the boundary layer

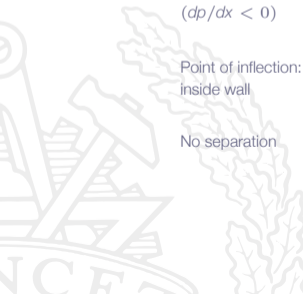
Pressure Gradient



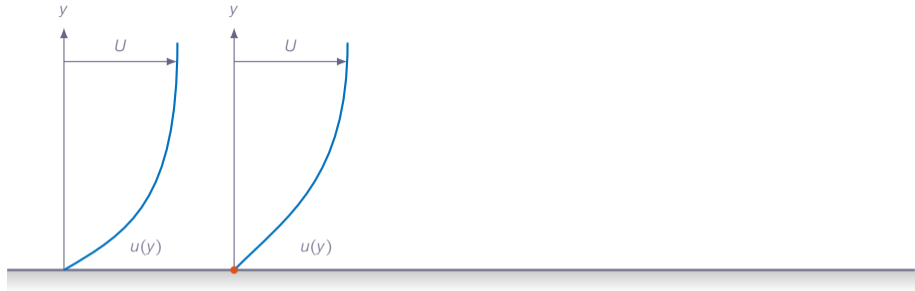
Favorable gradient
($dp/dx < 0$)

Point of inflection:
inside wall

No separation



Pressure Gradient



Favorable gradient
($dp/dx < 0$)

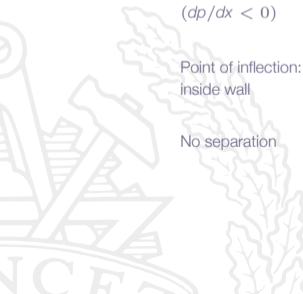
Zero gradient
($dp/dx = 0$)

Point of inflection:
inside wall

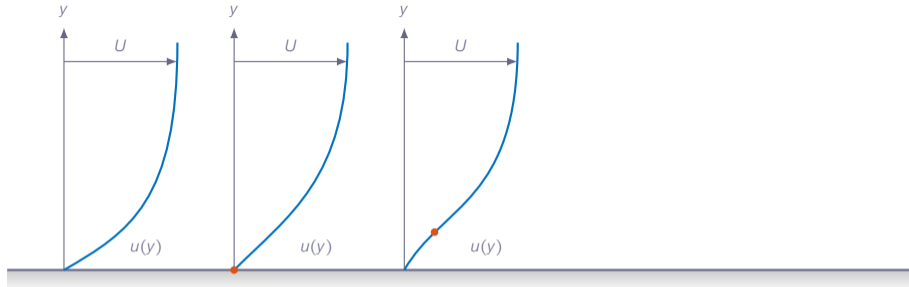
Point of inflection:
at the wall

No separation

No separation



Pressure Gradient



Favorable gradient
($dp/dx < 0$)

Zero gradient
($dp/dx = 0$)

Weak adverse
gradient ($dp/dx > 0$)

Point of inflection:
inside wall

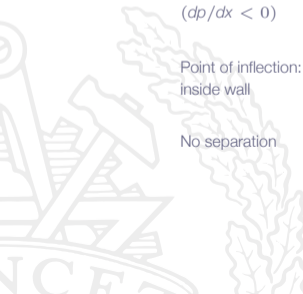
Point of inflection:
at the wall

Point of inflection:
in the flow

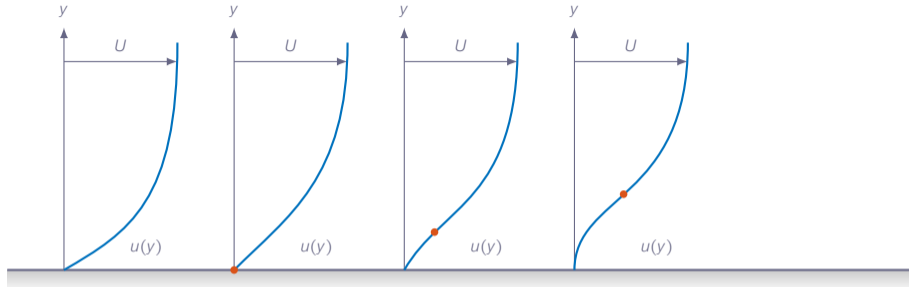
No separation

No separation

No separation



Pressure Gradient



Favorable gradient
($dp/dx < 0$)

Zero gradient
($dp/dx = 0$)

Weak adverse
gradient ($dp/dx > 0$)

Critical adverse
gradient ($dp/dx > 0$)

Point of inflection:
inside wall

Point of inflection:
at the wall

Point of inflection:
in the flow

Point of inflection:
in the flow

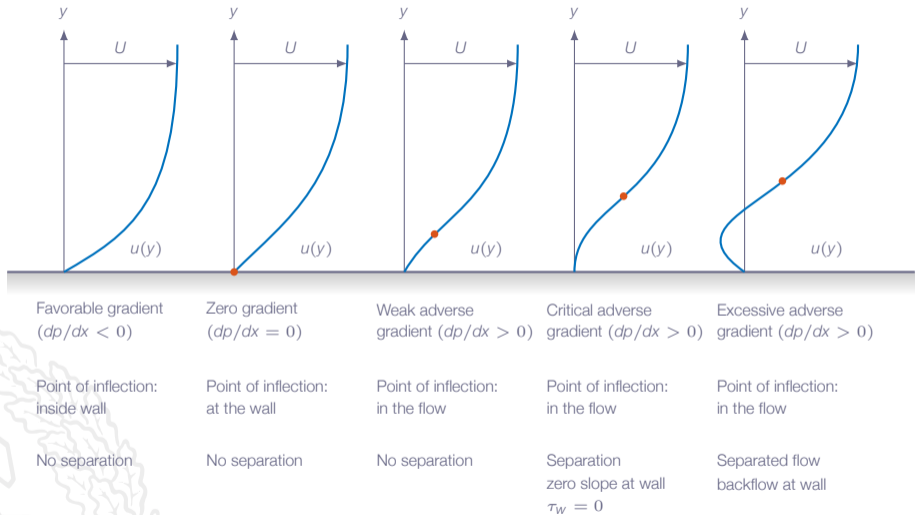
No separation

No separation

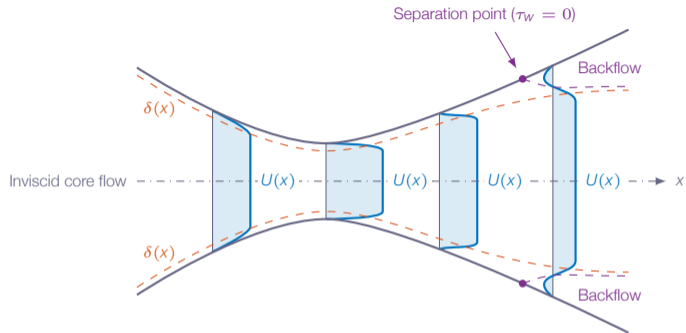
No separation

Separation
zero slope at wall
 $\tau_w = 0$

Pressure Gradient



Pressure Gradient



Nozzle

decreasing area

favorable pressure gradient

$$dp/dx < 0$$

$$dU/dx > 0$$

Throat

minimum area

zero pressure gradient

$$dp/dx = 0$$

$$dU/dx = 0$$

Diffuser

increasing area

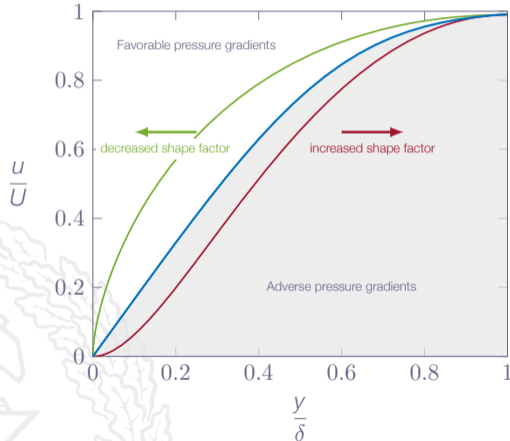
adverse pressure gradient

$$dp/dx > 0$$

$$dU/dx < 0$$

Shape Factor

$$\text{Shape factor: } H = \frac{\delta^*}{\theta}$$



Laminar flow:

No pressure gradient: $H \approx 2.6$

Separation: $H \approx 3.5$

Turbulent flow:

No pressure gradient: $H \approx 1.3$

Separation: $H \approx 2.4$

Avoid or Delay Separation

Decrease magnitude of adverse pressure gradient

- ▶ Guide vanes
- ▶ Streamlining



Avoid or Delay Separation

Decrease wall friction

- ▶ Polish surfaces
- ▶ Fluid injection



Avoid or Delay Separation

Remove decelerated fluid

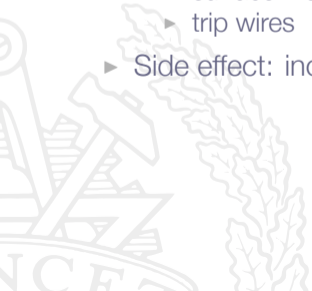
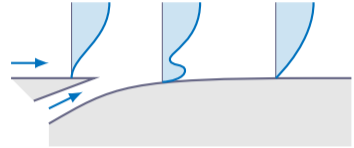
- ▶ Boundary layer suction



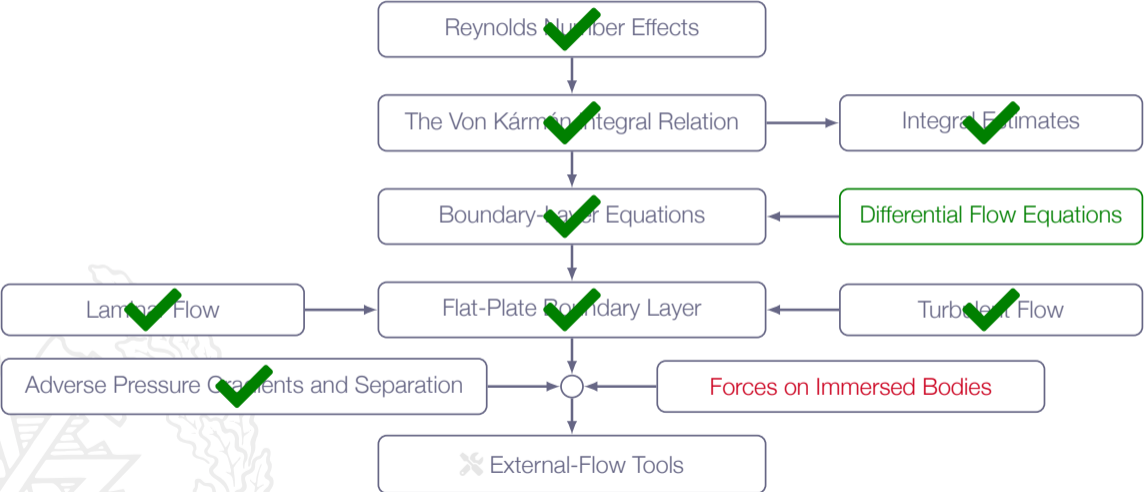
Avoid or Delay Separation

Increase near-wall momentum

- ▶ Force transition to turbulence
 - ▶ surface roughness
 - ▶ surface irregularities (dimples on the surface of a golf ball)
 - ▶ trip wires
- ▶ Side effect: increased friction



Roadmap - Flow Past Immersed Bodies

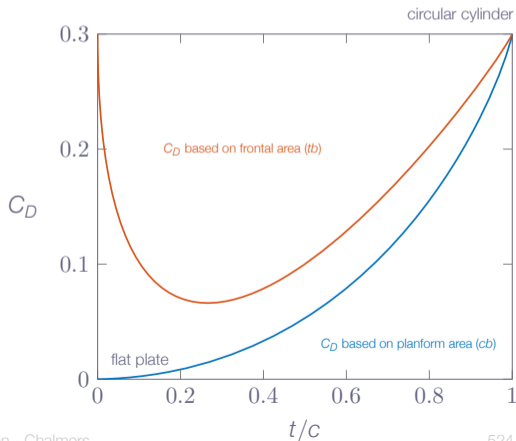
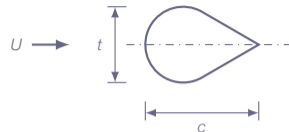


Drag of Immersed Bodies

$$C_D = \frac{\text{drag}}{\frac{1}{2}\rho U^2 A} = f\left(\frac{UL}{\nu}\right)$$

Characteristic area A:

1. Frontal area
blunt objects: *cylinders, cars*
2. Planform area
wide flat bodies: *wings, hydrofoils*
3. Wetted area
ships



Drag of Immersed Bodies

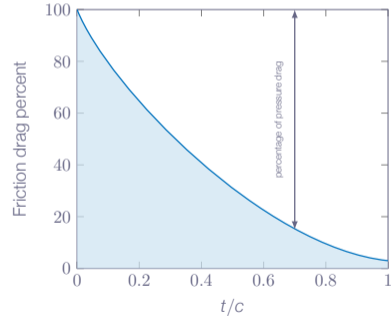
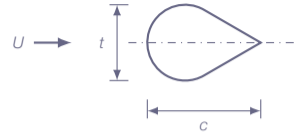
- ▶ No reliable theory for drag prediction (with the exception of flat plates)
- ▶ The separation point can be predicted with some accuracy but not the wake flow
- ▶ CFD or experiments needed



Drag of Immersed Bodies

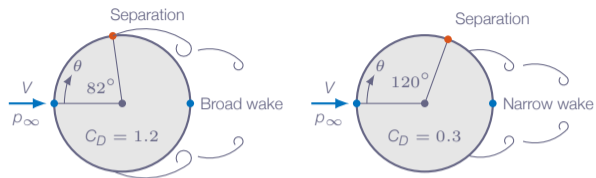
$$C_D = C_{D_{pressure}} + C_{D_{friction}}$$

- ▶ Pressure drag
 - ▶ difference between the high front stagnation pressure and the low wake pressure on the backside of the body
 - ▶ often larger than the friction drag
- ▶ The relative importance of friction and pressure drag depends on
 - ▶ shape
 - ▶ surface roughness

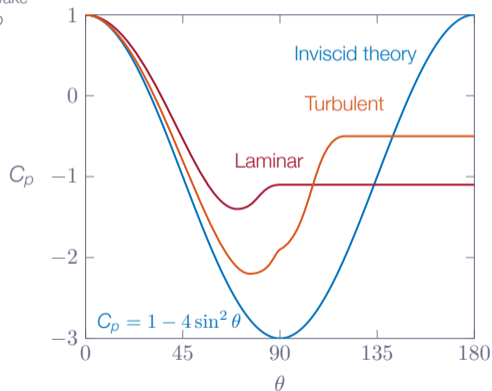


Note! for a cylinder, friction drag can be as low as a few percent of the total drag

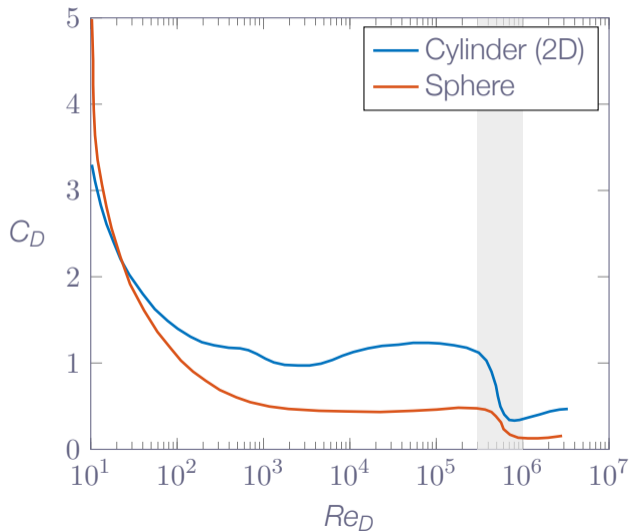
Cylinder Surface Pressure



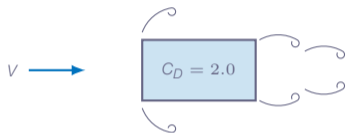
$$C_p = \frac{p - p_\infty}{\rho V^2 / 2}$$



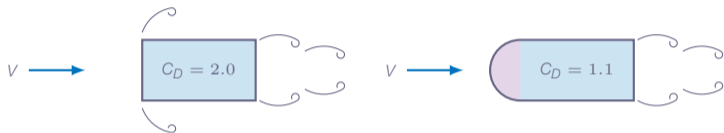
Cylinder Drag



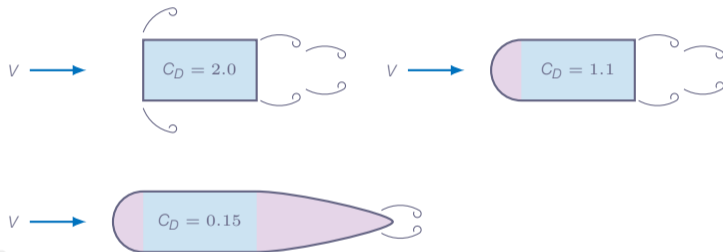
Streamlining



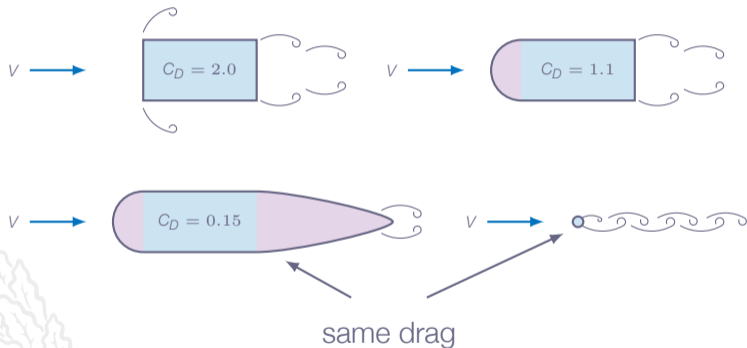
Streamlining



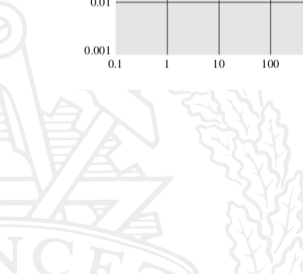
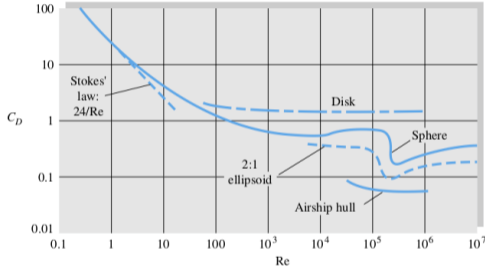
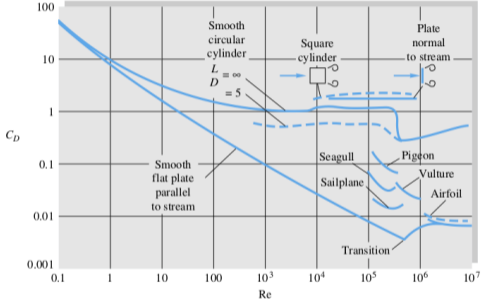
Streamlining



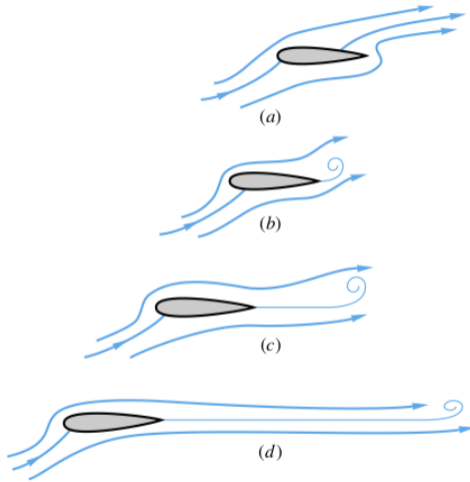
Streamlining



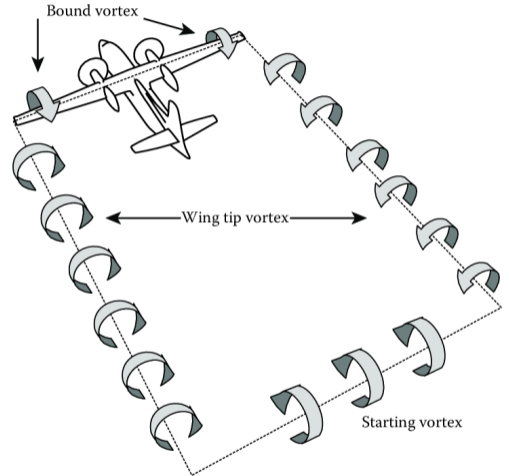
Drag vs Reynolds Number for Immersed Bodies



Wing Lift and Drag



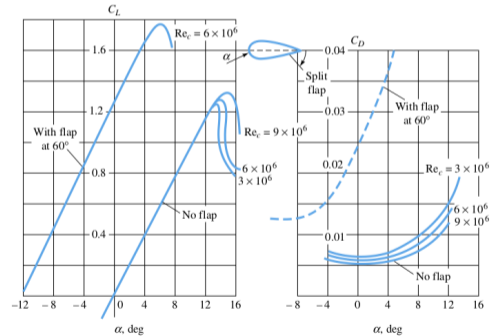
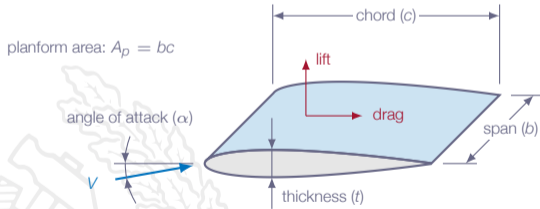
Wing Lift and Drag



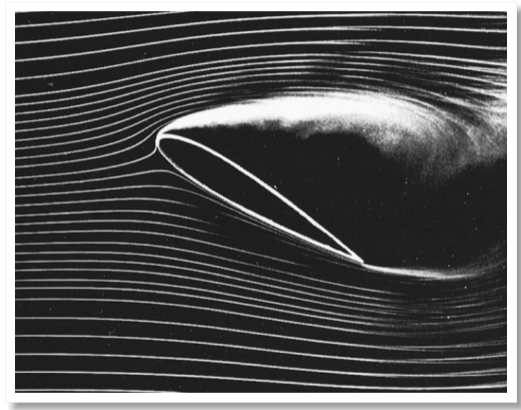
Wing Lift and Drag

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A_p}$$

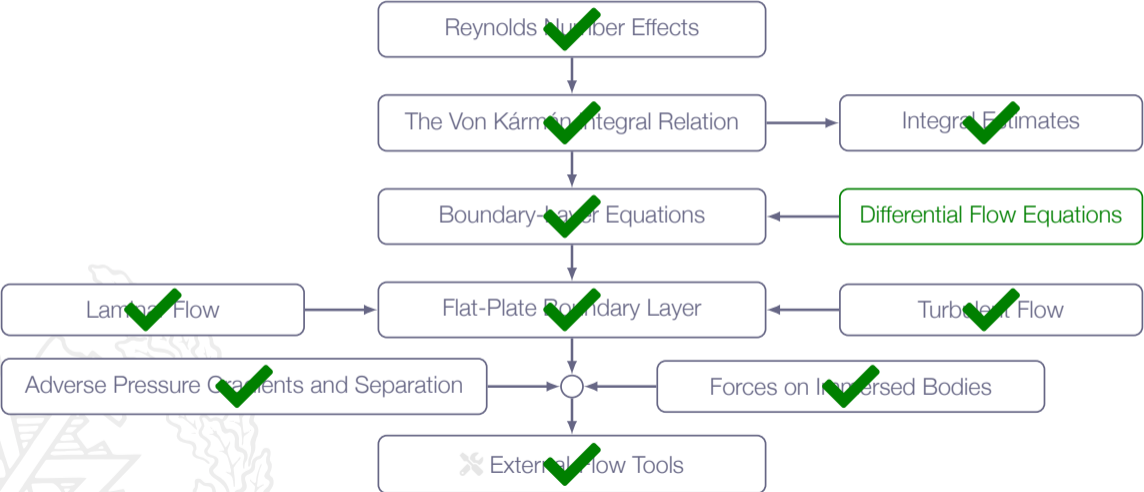
$$C_L = \frac{F_L}{\frac{1}{2}\rho U^2 A_p}$$



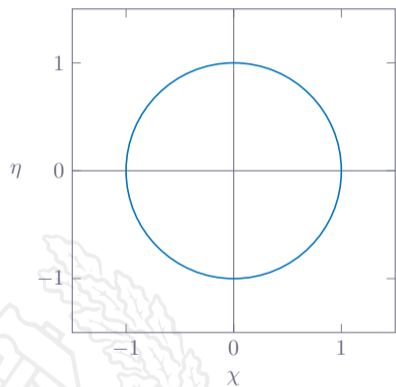
Wing Lift and Drag



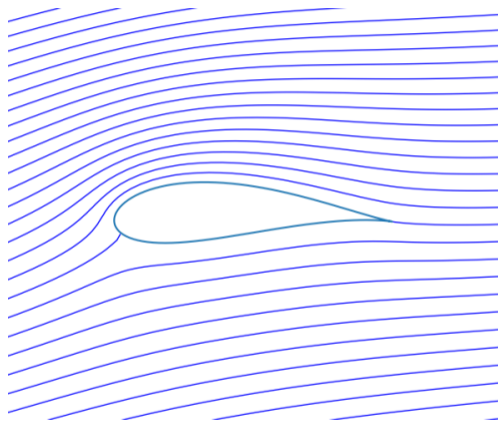
Roadmap - Flow Past Immersed Bodies



Joukowski Transform

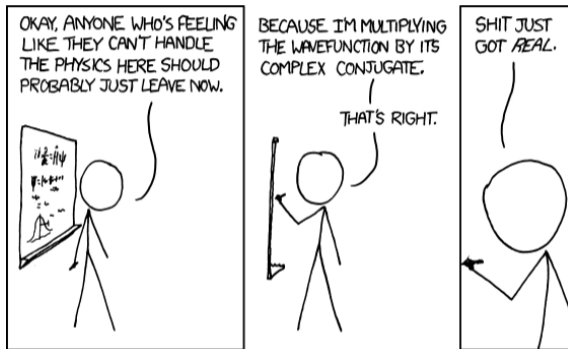


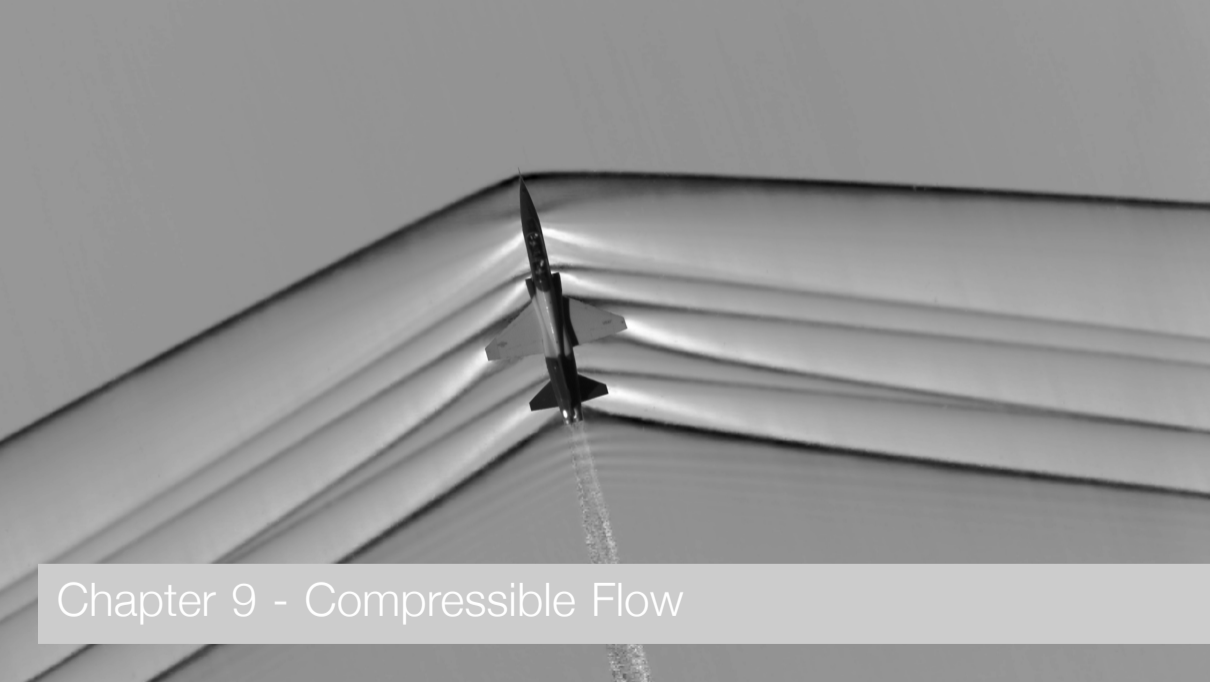
$$\zeta = \chi + i\eta$$



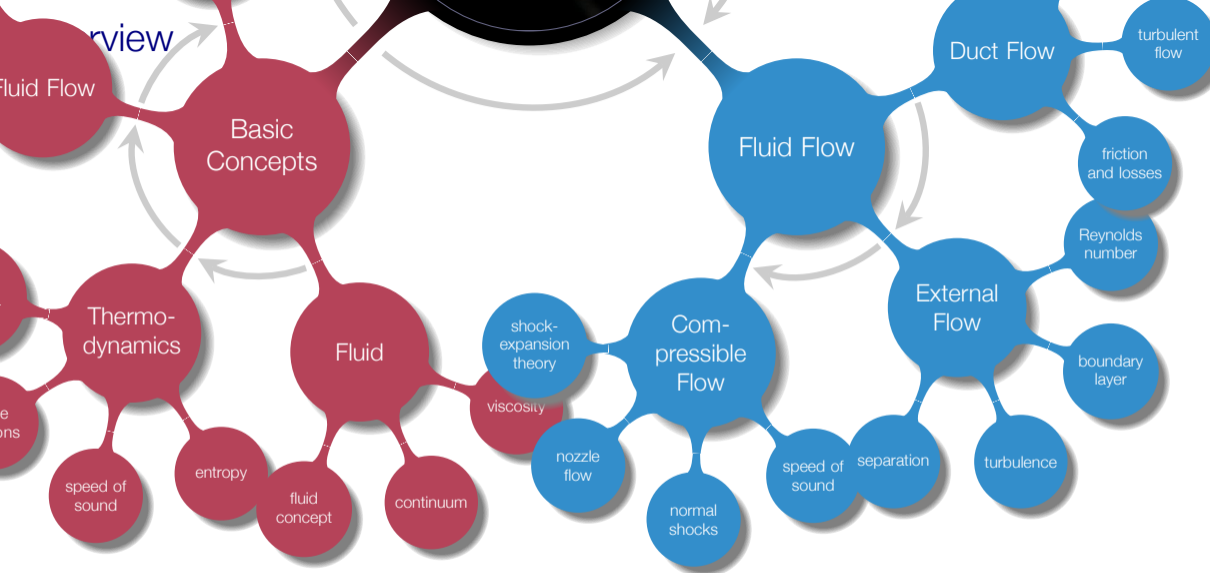
$$z = \zeta + \frac{1}{\zeta} = x + iy$$

Complex Conjugate





Chapter 9 - Compressible Flow

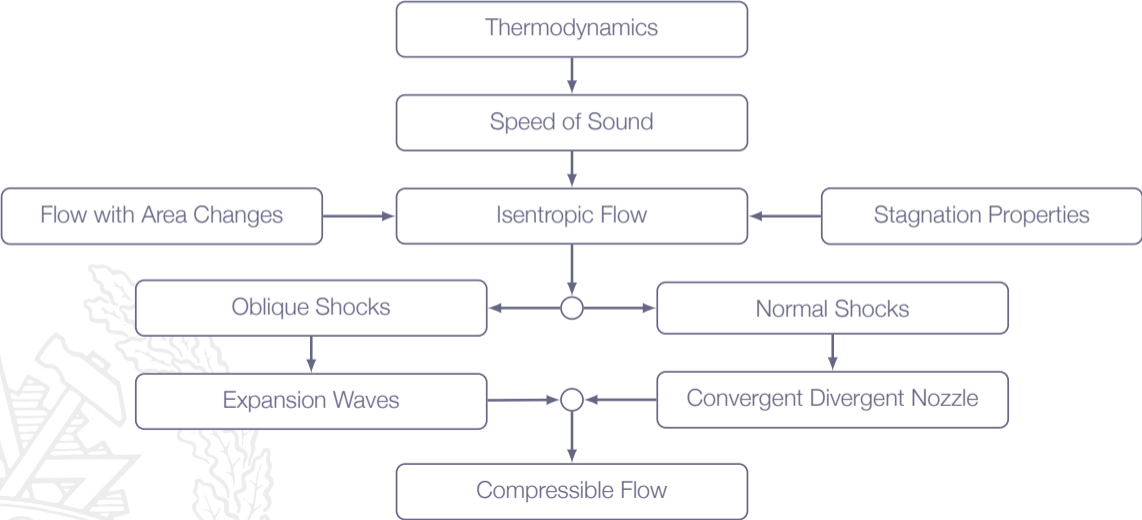


Learning Outcomes

- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 37 **Understand and explain** basic concepts of compressible flows (the gas law, speed of sound, Mach number, isentropic flow with changing area, normal shocks, oblique shocks, Prandtl-Meyer expansion)

Let's go supersonic ...

Roadmap - Compressible Flow



Motivation

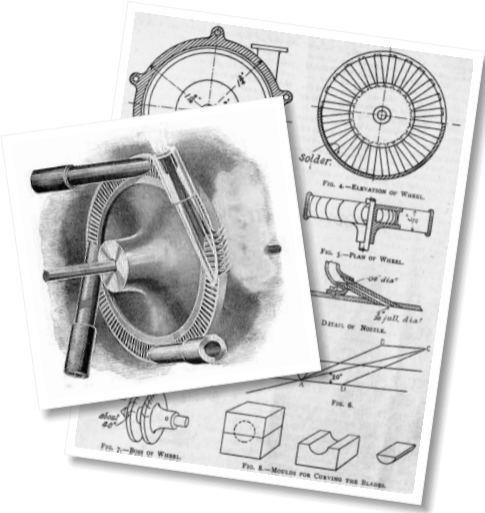
- ▶ Compressible flow:
 - ▶ flows where variations in density are significant
 - ▶ most often high-speed gas flows (gas dynamics)
 - ▶ fluids moving at speeds comparable to the speed of sound
 - ▶ not common in liquids (would require very high pressures)



Historical Milestones



First supersonic flight - Charles Yeager 1947

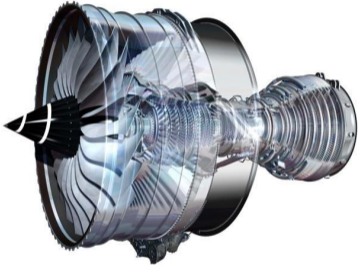


Steam turbine with convergent-divergent nozzles - Carl Gustav de Laval 1893

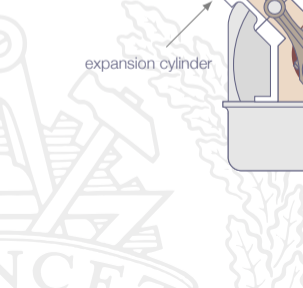
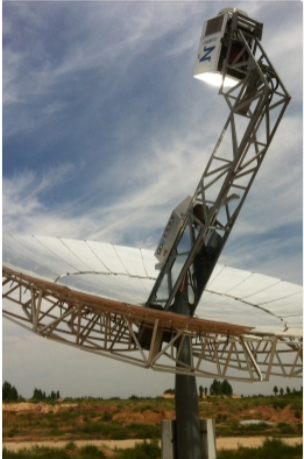
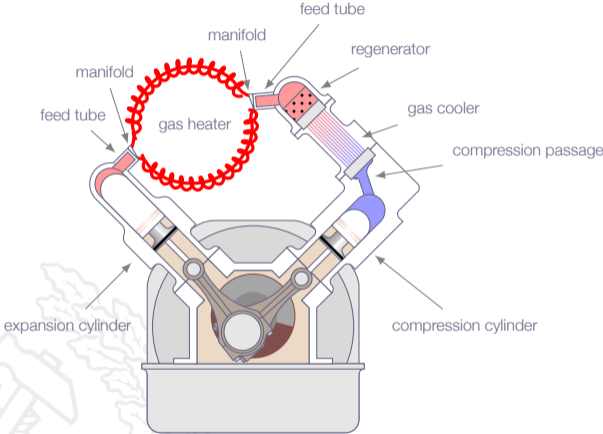
Compressible Flow Applications



Compressible Flow Applications



Compressible Flow Applications



Governing Equations

- ▶ With significant density changes follows substantial changes in pressure and temperature
- ▶ The energy equation must be included
- ▶ Four equations:
 1. Continuity
 2. Momentum
 3. Energy
 4. Equation of state
- ▶ Unknowns: ρ , p , T , and \mathbf{V}
- ▶ The four equations must be solved simultaneously

Mach Number Regimes

Incompressible flow

- ▶ insignificant density changes

subsonic flow

- ▶ local and global Mach number less than unity

transonic flow

- ▶ subsonic flow with regions of supersonic flow (local Mach number can be higher than one)
- ▶ supersonic flow with regions of subsonic flow (local Mach number can be less than one)

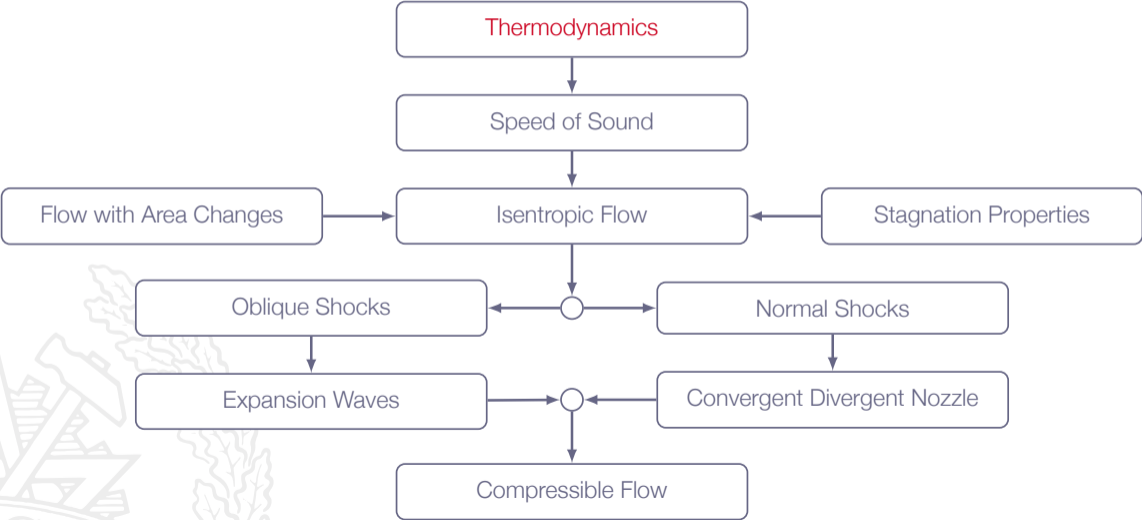
supersonic flow

- ▶ local and global Mach number higher than one

hypersonic flow

- ▶ Mach number higher than 5.0

Roadmap - Compressible Flow



Ratio of Specific Heats

- ▶ The ratio of specific heats is important in compressible flow

$$\gamma = \frac{C_p}{C_v}$$

- ▶ γ is a fluid property
- ▶ For moderate temperatures γ is a constant
- ▶ For higher temperatures γ varies with temperature
- ▶ For air, $\gamma = 1.4$

Equation of State

In the following ideal gas law and constant specific heats will be assumed:

$$p = \rho RT$$

$$R = C_p - C_v = \text{const}$$

$$\gamma = \frac{C_p}{C_v} = \text{const}$$

Auxiliary relations:

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

Internal Energy and Enthalpy

Constant specific heats:

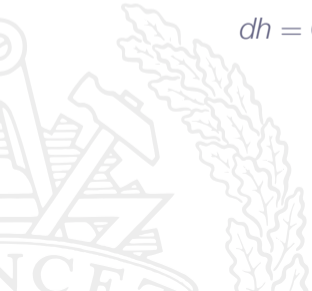
$$d\hat{u} = C_v dT$$

$$dh = C_p dT$$

Variable specific heats:

$$\hat{u} = \int C_v dT$$

$$h = \int C_p dT$$



Isentropic Relations

compute entropy change from the first and second law of thermodynamics

$$Tds = dh - \frac{dp}{\rho}$$

for calorically perfect gases, $dh = C_p dT$

$$\int_1^2 ds = \int_1^2 C_p \frac{dT}{T} - R \int_1^2 \frac{dp}{\rho}$$

for constant specific heats

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

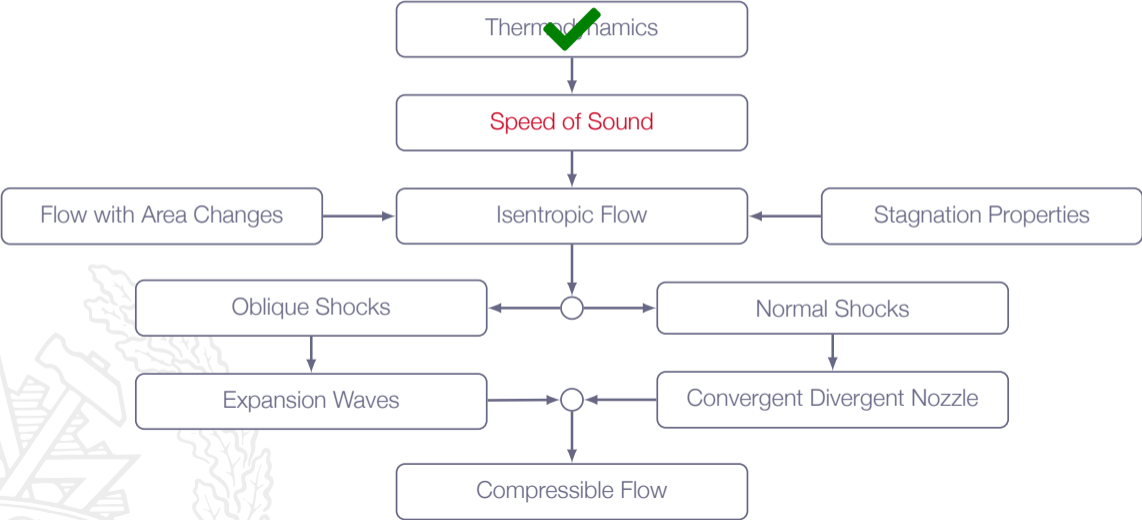
Isentropic Relations

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

for isentropic flow ($s_2 = s_1$) we get

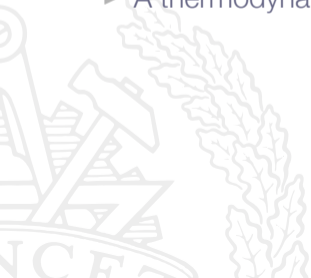
$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1} \right)^{\gamma/(\gamma-1)} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma}$$

Roadmap - Compressible Flow

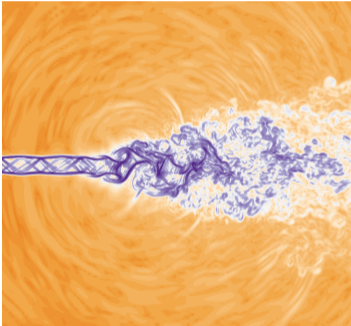
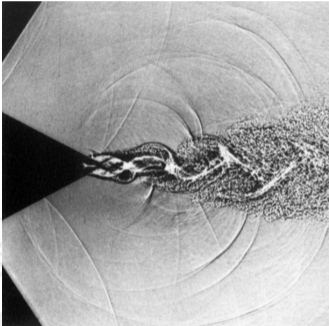


Speed of Sound

- ▶ The rate of propagation of a pressure pulse of infinitesimal strength through a fluid at rest
- ▶ Related to the molecular activity of the fluid
- ▶ A thermodynamic property



Speed of Sound



Speed of Sound

frame of reference fixed to fluid

$$\begin{array}{c} \rho \\ \rho \\ T \\ V = 0 \end{array} \quad \begin{array}{c} \leftarrow C \\ \parallel \\ \rho + \Delta\rho \\ \rho + \Delta\rho \\ T + \Delta T \\ V = \Delta V \\ \leftarrow \end{array}$$

frame of reference following the wave

$$\begin{array}{c} \rho \\ \rho \\ T \\ V = C \\ \rightarrow \end{array} \quad \begin{array}{c} \parallel \\ \rho + \Delta\rho \\ \rho + \Delta\rho \\ T + \Delta T \\ V = C - \Delta V \\ \rightarrow \end{array}$$

Speed of Sound

frame of reference following the wave

continuity:

$$\begin{array}{ccc} \rho & & \rho + \Delta\rho \\ p & & p + \Delta p \\ T & & T + \Delta T \\ V = C & & V = C - \Delta V \end{array}$$

$$\rho AC = (\rho + \Delta\rho)A(C - \Delta V)$$

$$\Delta V = C \frac{\Delta\rho}{\rho + \Delta\rho}$$

Note! there are no gradients in the flow so viscous effects are confined to the interior of the wave

Speed of Sound

frame of reference following the wave

momentum:

$$\begin{array}{ccc} \rho & & \rho + \Delta\rho \\ p & & p + \Delta p \\ T & & T + \Delta T \\ v = c & & v = c - \Delta v \end{array}$$

$$pA - (p + \Delta p)A = (\rho AC)(C - \Delta V - C) \Rightarrow \Delta p = \rho C \Delta V$$

with ΔV from the continuity equation we get

$$C^2 = \frac{\Delta p}{\Delta \rho} \left(1 + \frac{\Delta \rho}{\rho} \right)$$

Note! the larger $\Delta \rho / \rho$, the higher the propagation velocity

Speed of Sound

In the limit of infinitesimal strength $\Delta\rho \rightarrow 0$ and thus

$$C^2 = a^2 = \frac{\partial p}{\partial \rho}$$

- ▶ There is no added heat and thus the process adiabatic
- ▶ For weak waves the process can also be assumed to be reversible

$$a^2 = \left. \frac{\partial p}{\partial \rho} \right|_s$$



Speed of Sound

$$a^2 = \left. \frac{\partial p}{\partial \rho} \right|_s$$

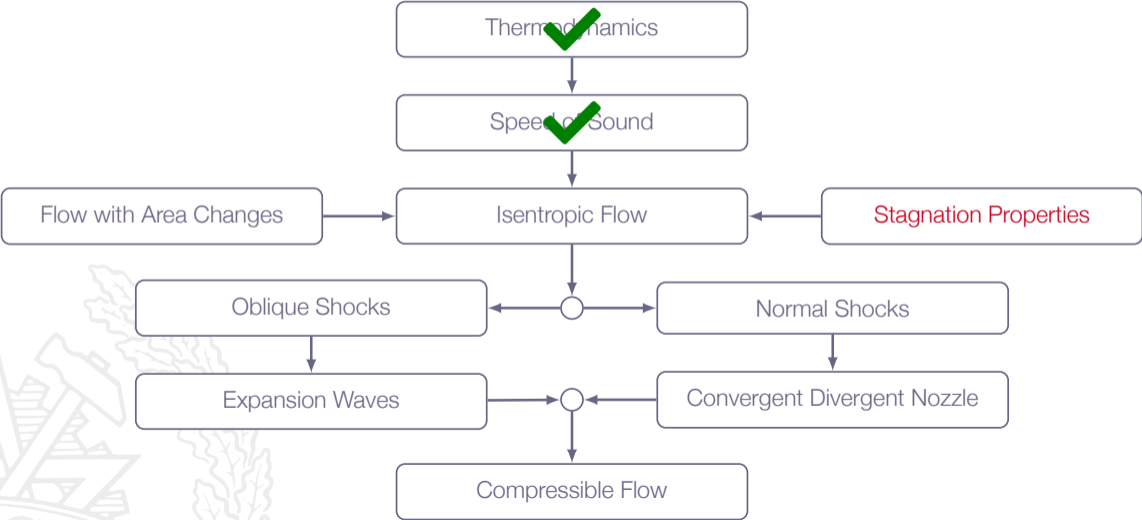
The isentropic relation gives

$$p = \rho^\gamma \Rightarrow \frac{\partial p}{\partial \rho} = \gamma \rho^{\gamma-1} = \gamma \frac{p}{\rho} = \gamma RT$$

and thus

$$a = \sqrt{\gamma RT}$$

Roadmap - Compressible Flow



Stagnation Enthalpy

Consider high-speed gas flow past an insulated wall

$$h_1 + \frac{1}{2}V_1^2 + gz_1 = h_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_\nu$$

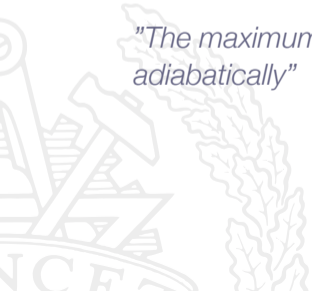
- ▶ differences in potential energy extremely small
- ▶ outside the boundary layer, heat transfer and viscous work are zero

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 = \text{const}$$

Stagnation Enthalpy

$$h + \frac{1}{2}V^2 = h_o$$

"The maximum enthalpy that the fluid would achieve if brought to rest adiabatically"



Stagnation Temperature

For a calorically perfect gas $h = C_p T$

$$h + \frac{1}{2}V^2 = h_o$$

$$C_p T + \frac{1}{2}V^2 = C_p T_o$$

Where T_o is the stagnation temperature



Mach Number Relations

$$C_p T + \frac{1}{2} V^2 = C_p T_o \Rightarrow 1 + \frac{V^2}{2C_p T} = \frac{T_o}{T}$$

$$C_p T = \frac{\gamma R}{\gamma - 1} T = \frac{\gamma R T}{\gamma - 1} = \frac{a^2}{\gamma - 1}$$

$$\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2} \right) M^2$$

Mach Number Relations

Since $a \propto T^{1/2}$ we get

$$\frac{a_0}{a} = \left(\frac{T_0}{T} \right)^{1/2} = \left[1 + \left(\frac{\gamma - 1}{2} \right) M^2 \right]^{1/2}$$



Mach Number Relations

If the flow is adiabatic and reversible (isentropic), we may use the isentropic relations

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\gamma/(\gamma-1)}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{1/(\gamma-1)} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{1/(\gamma-1)}$$



Stagnation Properties

- ▶ p_o and ρ_o - the pressure and density that the flow would achieve if brought to rest isentropically
- ▶ All stagnation properties are constants in an isentropic flow
- ▶ h_o , T_o , and a_o are constants in an adiabatic flow but not necessarily p_o and ρ_o
- ▶ p_o and ρ_o will vary throughout an adiabatic flow as the entropy changes due to friction or shocks

Critical Properties

Another useful set of reference variables is the critical properties (sonic conditions)

$$\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2 = \{M = 1.0\} = 1 + \left(\frac{\gamma - 1}{2}\right) = \left(\frac{2 + \gamma - 1}{2}\right) = \left(\frac{\gamma + 1}{2}\right)$$



Critical Properties

$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma + 1} \right)$$

$$\frac{a^*}{a_o} = \left(\frac{2}{\gamma + 1} \right)^{1/2}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1} \right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1} \right)^{1/(\gamma-1)}$$

Critical Properties

Air $\gamma = 1.4$

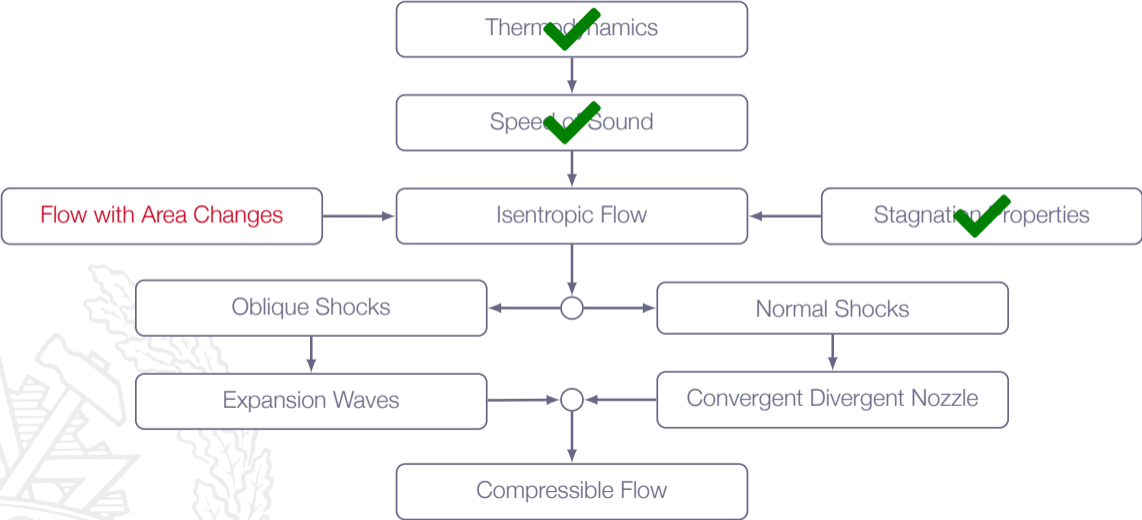
$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma + 1} \right) = 0.8333$$

$$\frac{a^*}{a_o} = \left(\frac{2}{\gamma + 1} \right)^{1/2} = 0.9129$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1} \right)^{\gamma/(\gamma-1)} = 0.5283$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma + 1} \right)^{1/(\gamma-1)} = 0.6339$$

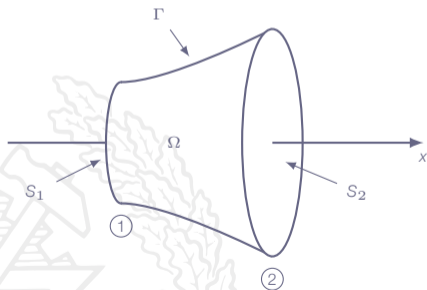
Roadmap - Compressible Flow



Isentropic Quasi-1D Flow

Quasi-1D:

- ▶ Flow properties varies in one direction only (x)
- ▶ The flow area is a smooth function $A = A(x)$
- ▶ Steady-state, inviscid and isentropic flow



The Area-Velocity Relation

Continuity:

$$\rho(x)V(x)A(x) = \text{const} \Rightarrow d(\rho VA) = 0 \Rightarrow AVd\rho + \rho AdV + \rho VdA = 0$$

divide by ρVA gives

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$



The Area-Velocity Relation

From the definition of stagnation enthalpy and isentropic flow we get

$$h_o = h + \frac{1}{2}V^2 = \text{const} \Rightarrow dh + VdV = 0$$

The second law of thermodynamics and isentropic flow

$$Tds = 0 = dh - \frac{dp}{\rho} \Rightarrow dh = \frac{dp}{\rho}$$

and thus

$$\frac{dp}{\rho} + VdV = 0$$

The Area-Velocity Relation

$$\frac{dp}{\rho} + VdV = 0$$

From the definition of the speed of sound

$$dp = a^2 d\rho \Rightarrow a^2 \frac{d\rho}{\rho} + VdV = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{1}{a^2} VdV$$



The Area-Velocity Relation

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = \frac{dV}{V} - \frac{1}{a^2}VdV + \frac{dA}{A} = 0$$

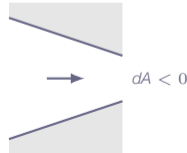
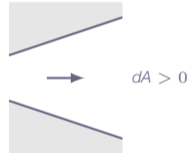
$$\frac{dV}{V} \left(\frac{V^2}{a^2} - 1 \right) = \frac{dA}{A}$$

$$\frac{dV}{V} = \frac{1}{M^2 - 1} \frac{dA}{A} = -\frac{dp}{\rho V^2}$$



The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$



Subsonic $M < 1$ **Supersonic** $M > 1$

subsonic diffuser

$$dV < 0$$

$$dp > 0$$

supersonic nozzle

$$dV > 0$$

$$dp < 0$$

subsonic nozzle

$$dV > 0$$

$$dp < 0$$

supersonic diffuser

$$dV < 0$$

$$dp > 0$$

The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

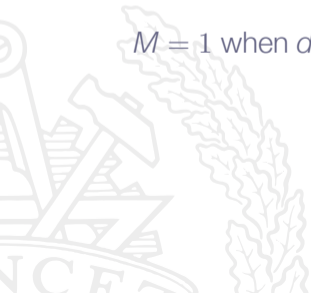


The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

$M = 1$ when $dA = 0$



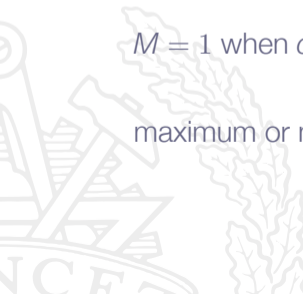
The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$

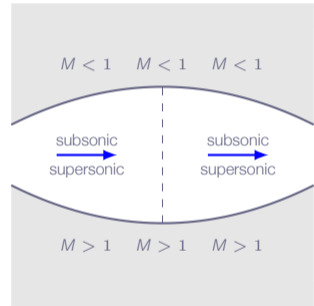
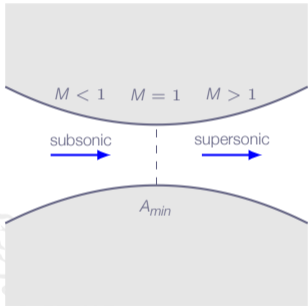
What happens when $M = 1$?

$M = 1$ when $dA = 0$

maximum or minimum area



The Area-Velocity Relation



The Area-Mach-Number Relation

$$\rho AV = \rho^* A^* V^* \Rightarrow \frac{A}{A^*} = \frac{\rho^* V^*}{\rho V}$$

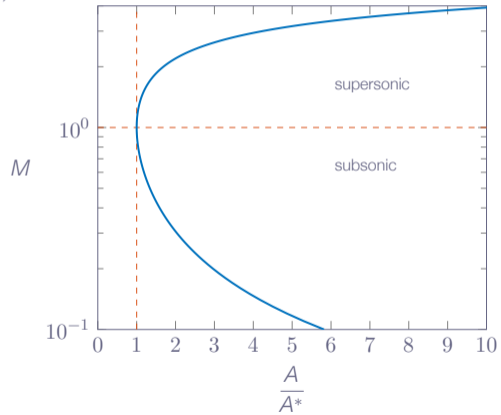
$$\frac{\rho^*}{\rho} = \frac{\rho^* \rho_0}{\rho_0 \rho} = \left[\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{1/(\gamma - 1)}$$

$$\frac{V^*}{V} = \frac{(\gamma RT^*)^{1/2}}{V} = \frac{(\gamma RT)^{1/2}}{V} \left(\frac{T^*}{T} \right)^{1/2} \left(\frac{T_0}{T} \right)^{1/2} = \frac{1}{M} \left[\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{1/2}$$

$$\left(\frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{(\gamma + 1)/(\gamma - 1)}$$

The Area-Mach-Number Relation

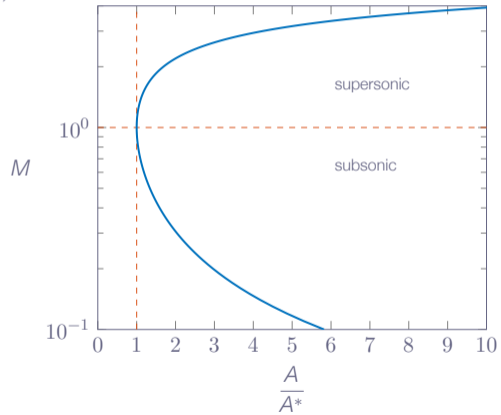
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{(\gamma+1)/(\gamma-1)}$$



The Area-Mach-Number Relation

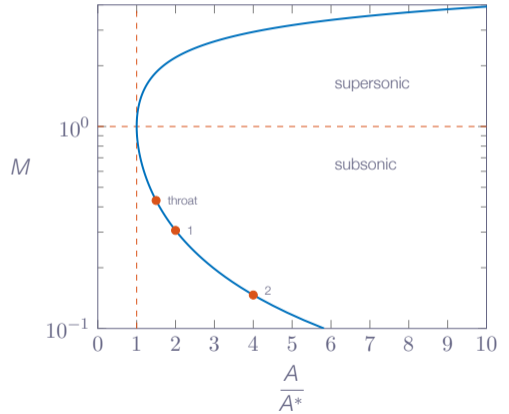
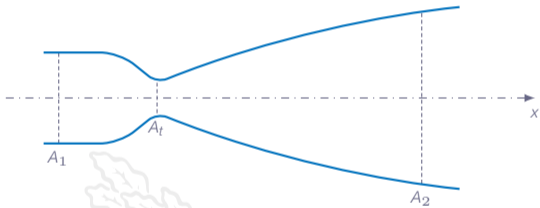
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{(\gamma+1)/(\gamma-1)}$$

Note! $\frac{A}{A^*} = \frac{\rho^* V^*}{\rho V}$



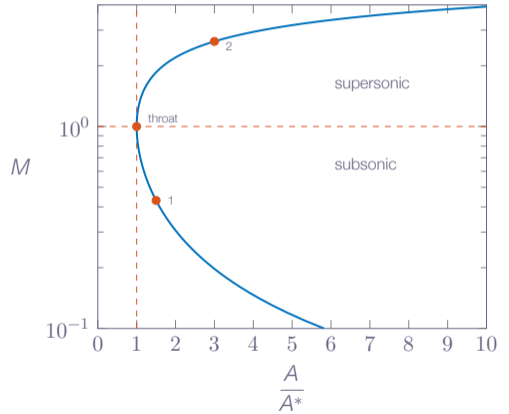
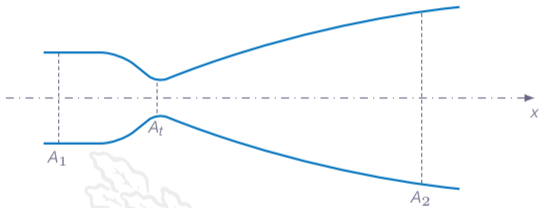
The Area-Mach-Number Relation

Sub-critical (non-choked) nozzle flow



The Area-Mach-Number Relation

Critical (choked) nozzle flow

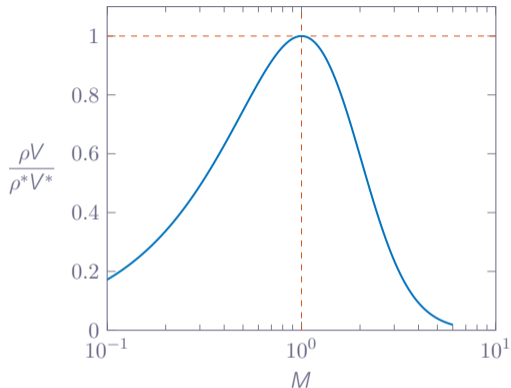


Choking

$$\rho VA = \rho^* A^* V^* \Rightarrow \frac{A^*}{A} = \frac{\rho V}{\rho^* V^*}$$

From the area-Mach-number relation

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if } M < 1 \\ 1 & \text{if } M = 1 \\ < 1 & \text{if } M > 1 \end{cases}$$



The maximum possible massflow through a duct is achieved when its throat reaches sonic conditions

Choking

$$\dot{m}_{max} = \rho^* A^* V^*$$

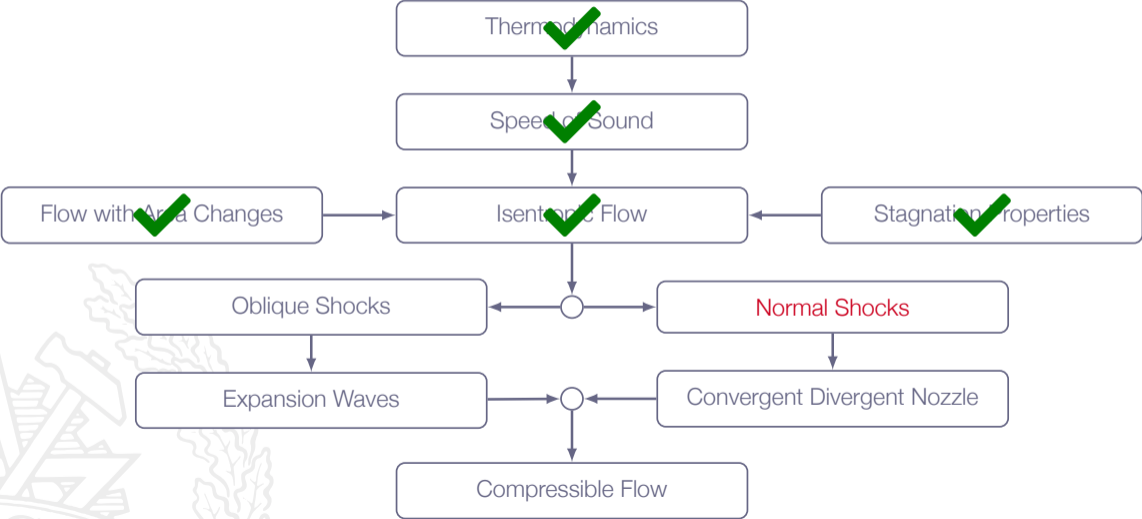
$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1} \right)^{1/(\gamma-1)}$$

$$V^* = \sqrt{\gamma R T^*}$$

$$\frac{T^*}{T_o} = \frac{2}{\gamma + 1}$$

$$\dot{m}_{max} = \frac{A^* p_o}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1} \right)^{\frac{(\gamma+1)}{(\gamma-1)}}}$$

Roadmap - Compressible Flow



Shock Waves

"Shock waves are nearly discontinuous changes in a supersonic flow"

- ▶ higher downstream pressure
- ▶ sudden changes in flow direction
- ▶ blockage by a downstream body
- ▶ explosion



Normal Shocks

Continuity:

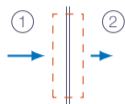
$$\rho_1 u_1 = \rho_2 u_2$$

Momentum:

$$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2$$

Energy:

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 = h_o$$



The Rankine-Hugoniot relation:

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1) \left(\frac{1}{\rho_2} + \frac{1}{\rho_1} \right)$$

Normal Shocks

$$h_2 - h_1 = \frac{1}{2}(\rho_2 - \rho_1) \left(\frac{1}{\rho_2} + \frac{1}{\rho_1} \right)$$

Note! The Rankine-Hugoniot relation only includes thermodynamic properties and gives a relation between the flow state upstream of the shock and the flow downstream of the shock

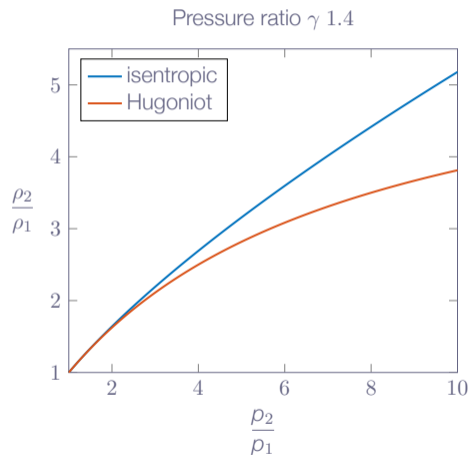
Normal Shocks

The Rankine-Hugoniot relation

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right) \left(\frac{\rho_2}{\rho_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{\rho_2}{\rho_1}\right)}$$

The isentropic relation

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$



Normal Shocks

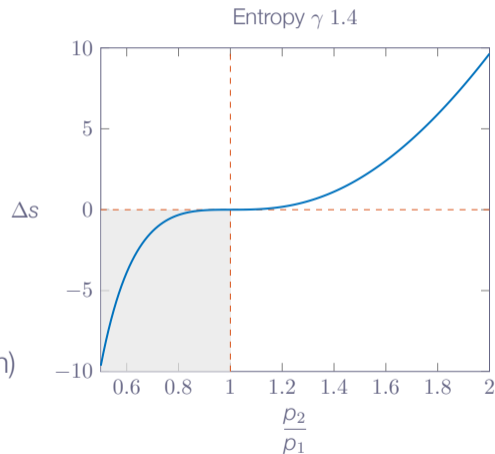
The second law of thermodynamics

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

can be rewritten as

$$s_2 - s_1 = C_v \ln \left[\frac{\rho_2}{\rho_1} \left(\frac{\rho_1}{\rho_2} \right)^\gamma \right]$$

(ρ_2/ρ_1 from the Rankine-Hugoniot relation)



Note! a reduction of entropy is a violation of the second law of thermodynamics

Normal Shocks

For a perfect gas, it is possible to obtain relations for normal shocks that only include upstream variables

$$\text{Momentum equation: } \rho_1 + \rho_1 u_1^2 = \rho_2 + \rho_2 u_2^2$$

$$\rho_2 - \rho_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_2\} = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

divide by ρ_1

$$\frac{\rho_2}{\rho_1} = 1 + \frac{\rho_1 u_1^2}{\rho_1} \left(1 - \frac{u_2}{u_1}\right)$$

$$u_1^2 = M_1^2 a_1^2 = M_1^2 \gamma R T_1 = \gamma M_1^2 \frac{\rho_1}{\rho_1} \Rightarrow \frac{\rho_2}{\rho_1} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Normal Shocks

$$\frac{\rho_2}{\rho_1} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1} \right)$$

Using the energy equation its possible obtain a relation for $\frac{u_2}{u_1}$

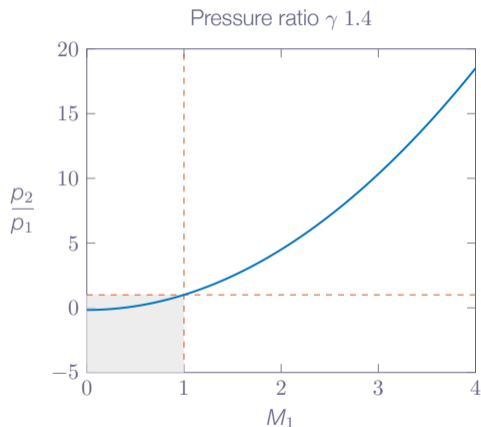
$$\frac{u_2}{u_1} = \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}$$

and thus

$$\frac{\rho_2}{\rho_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1)$$

Normal Shocks

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1)$$



Note! from before we know that p_2/p_1 must be greater than 1.0, which means that M_1 must be greater than 1.0

Normal Shocks

Momentum equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$$M = \frac{u}{a} \Rightarrow p_1 + \rho_1 M_1^2 a_1^2 = p_2 + \rho_2 M_2^2 a_2^2$$

$$a = \sqrt{\gamma RT} = \sqrt{\frac{\gamma p}{\rho}} \Rightarrow p_1 + \rho_1 M_1^2 \frac{\gamma p_1}{\rho_1} = p_2 + \rho_2 M_2^2 \frac{\gamma p_2}{\rho_2}$$

$$p_1 (1 + \gamma M_1^2) = p_2 (1 + \gamma M_2^2)$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Normal Shocks

Two ways to calculate the pressure ratio over the shock

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1)$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Setting the relations equal gives a relation for the downstream Mach number

$$M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)}$$

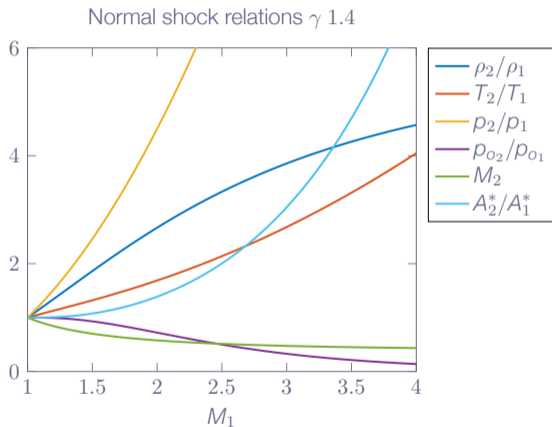
Note! for $\gamma > 1$ and $M_1 > 1$, the downstream Mach number must be less than 1.0, i.e we will always have subsonic flow behind a normal shock

Normal Shocks

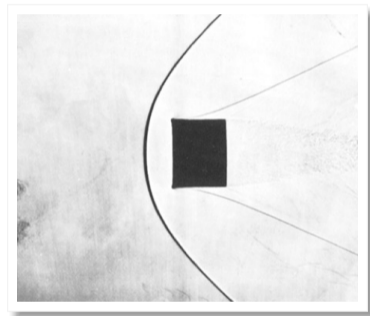
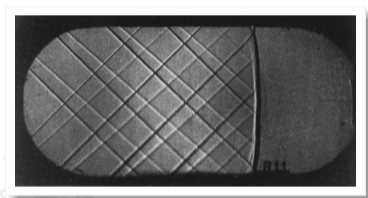
1. Supersonic flow upstream of normal shock
2. Subsonic flow downstream of normal shock
3. Entropy increases over the shock and consequently total pressure decreases
4. Sonic throat area increases
5. Very weak shock waves are nearly isentropic



Normal Shocks



Normal Shocks



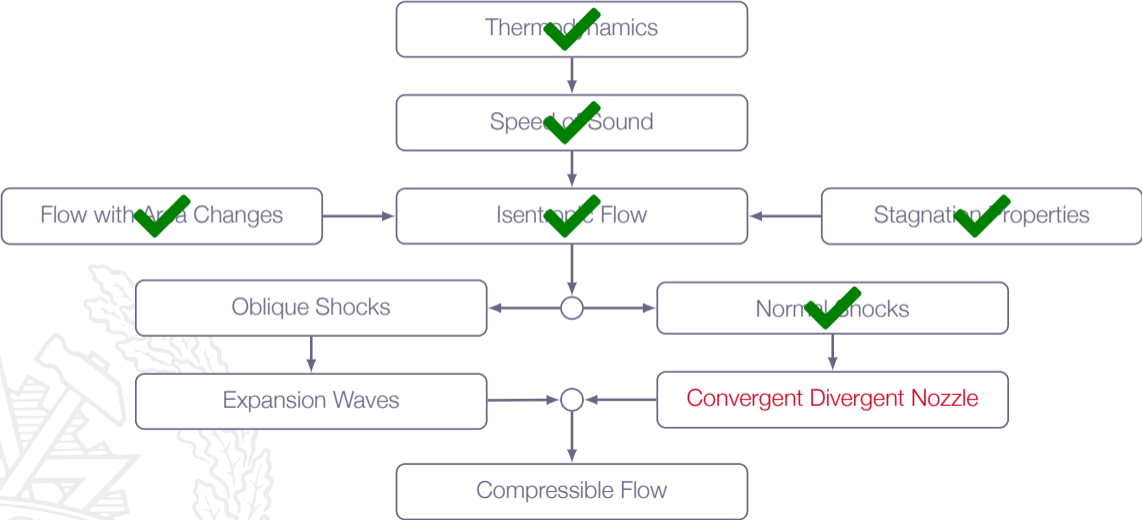
Moving Normal Shocks

Change frame of reference

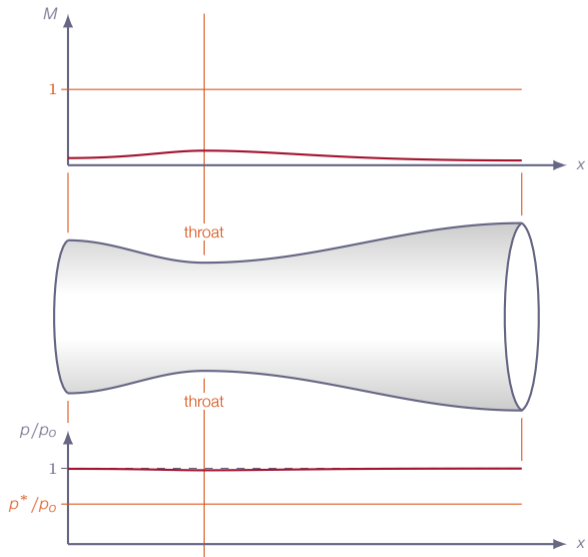
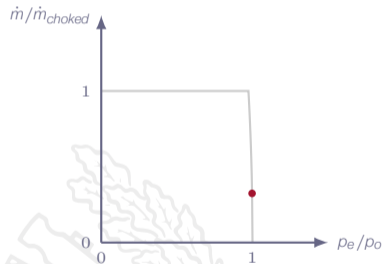
- ▶ coordinate system moving with the shock
- ▶ thermodynamic properties does not change



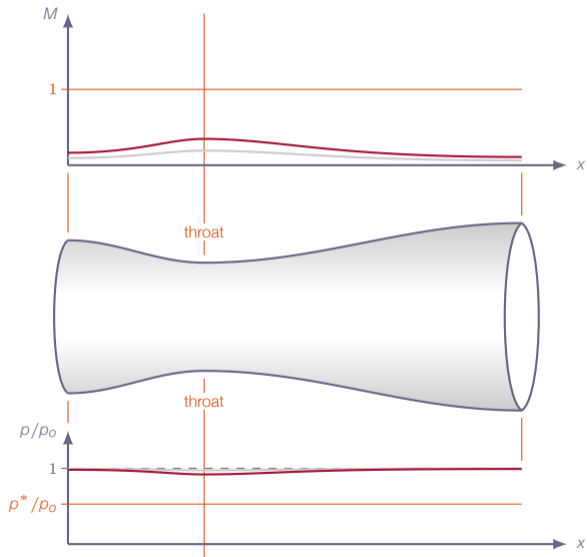
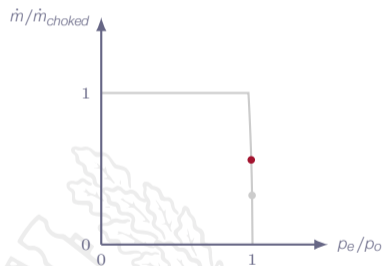
Roadmap - Compressible Flow



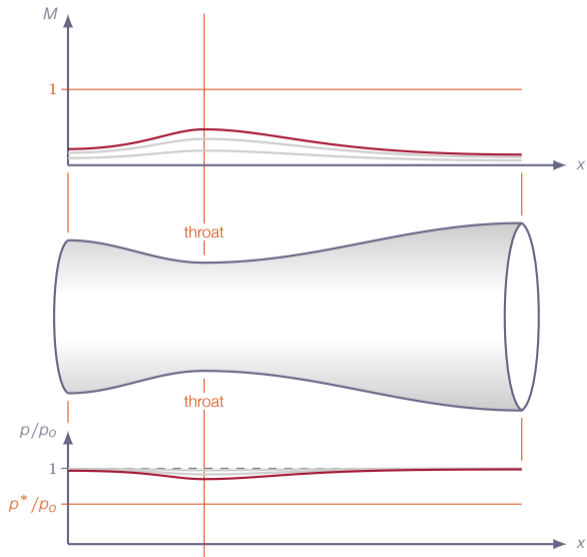
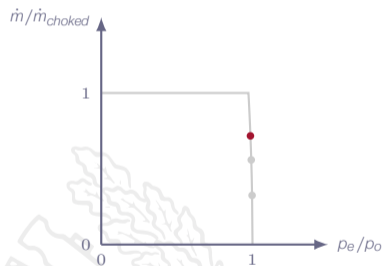
Convergent-Divergent Nozzle



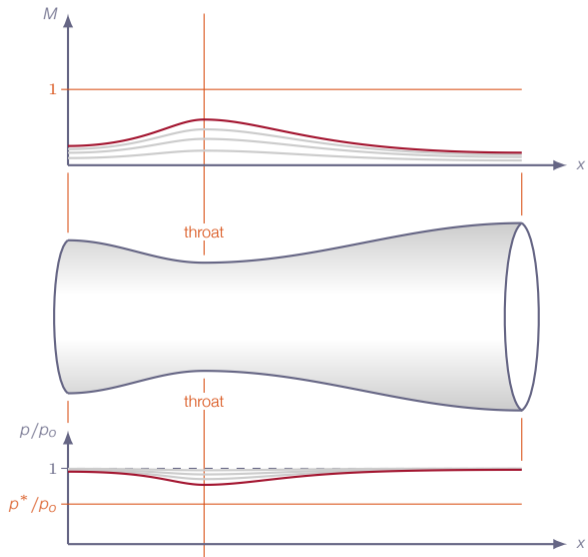
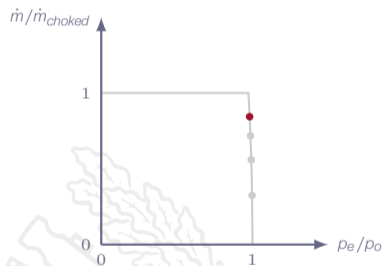
Convergent-Divergent Nozzle



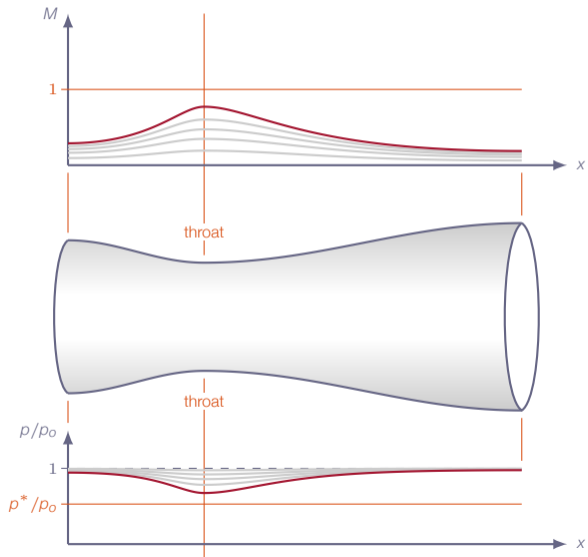
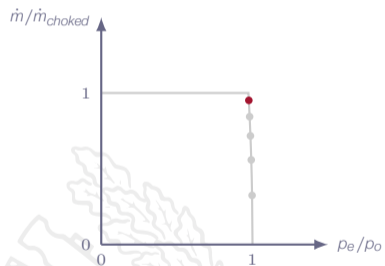
Convergent-Divergent Nozzle



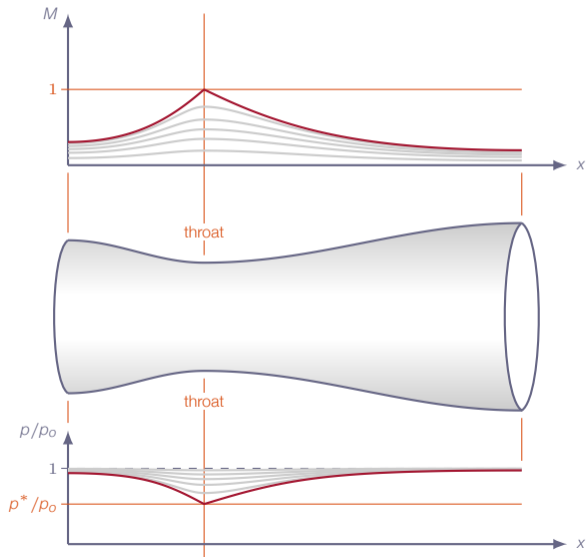
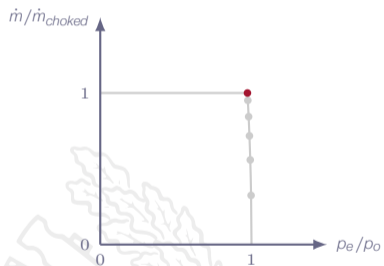
Convergent-Divergent Nozzle



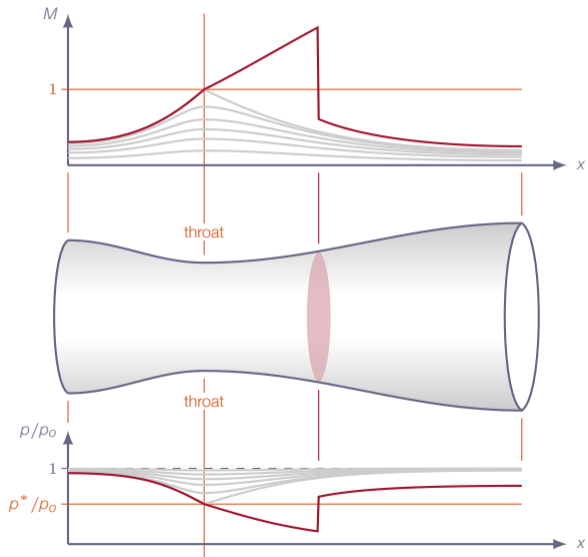
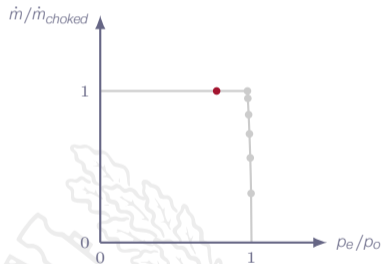
Convergent-Divergent Nozzle



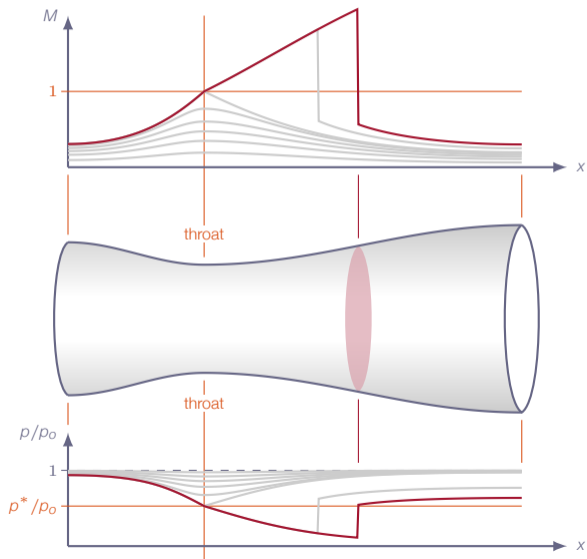
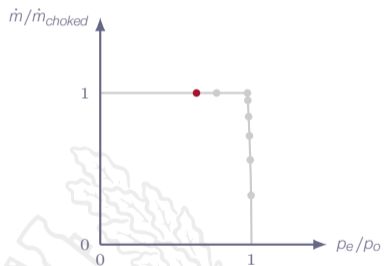
Convergent-Divergent Nozzle



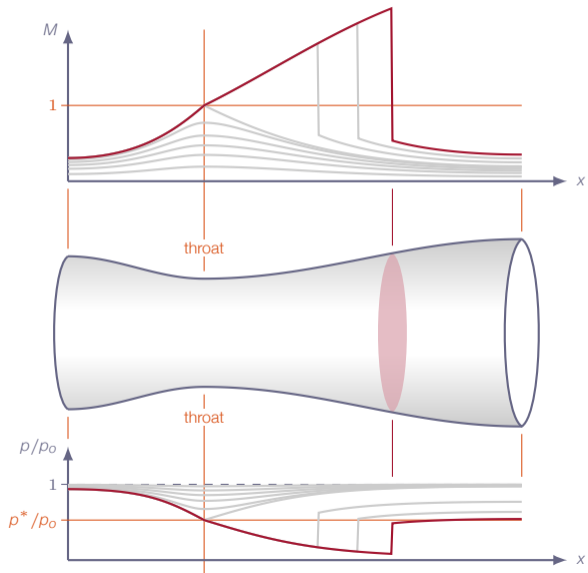
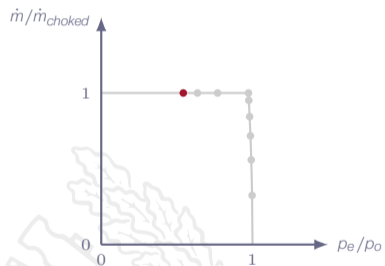
Convergent-Divergent Nozzle



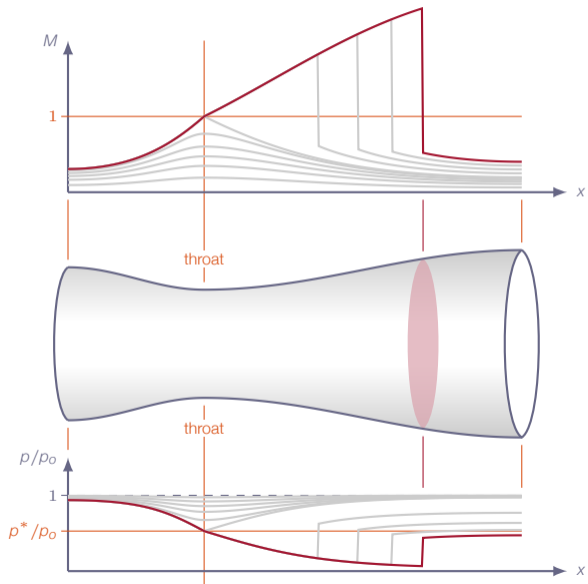
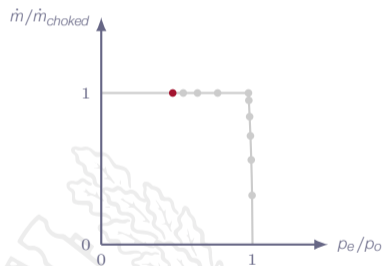
Convergent-Divergent Nozzle



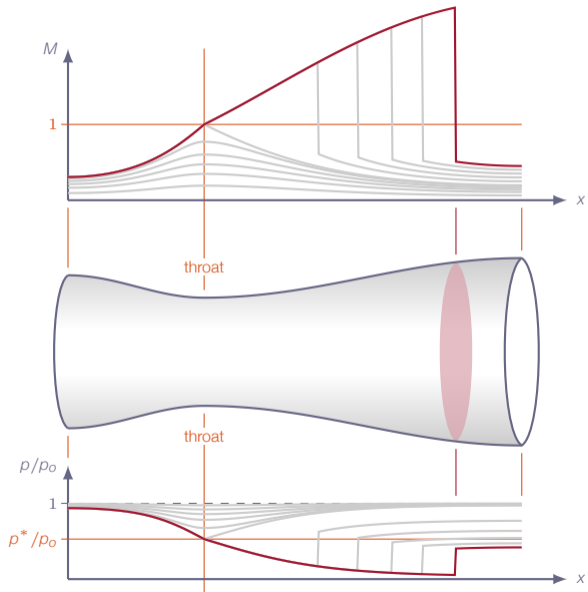
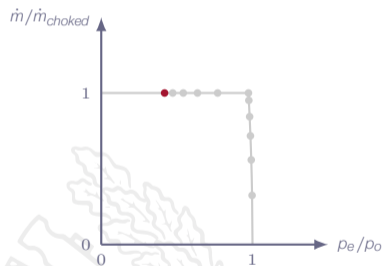
Convergent-Divergent Nozzle



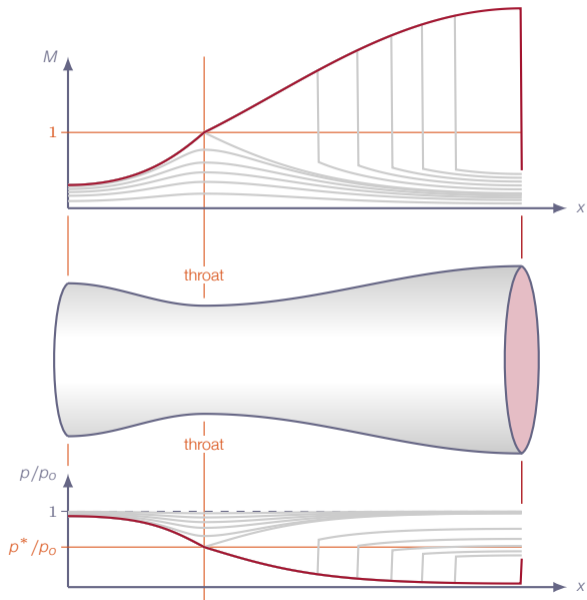
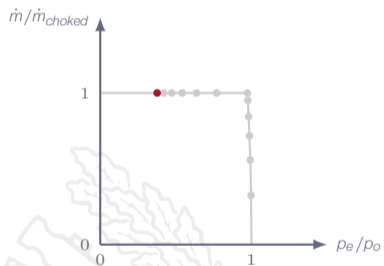
Convergent-Divergent Nozzle



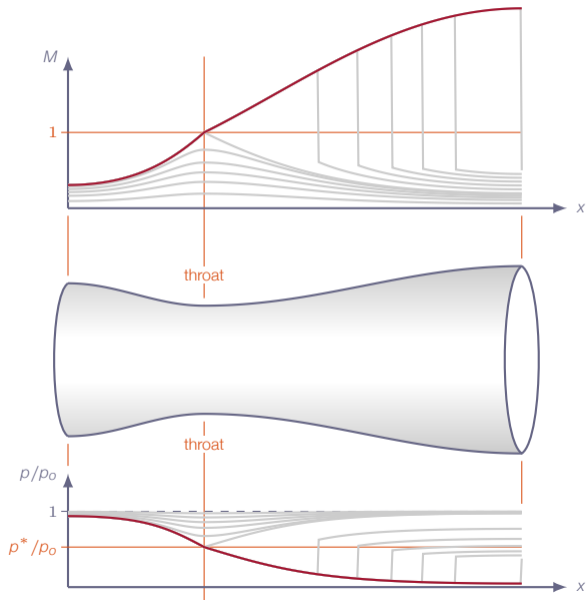
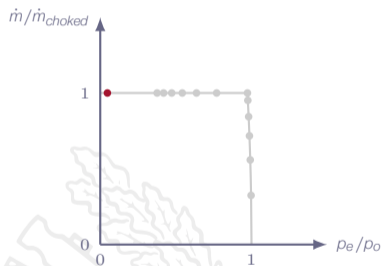
Convergent-Divergent Nozzle



Convergent-Divergent Nozzle



Convergent-Divergent Nozzle



Convergent-Divergent Nozzle



normal shock

$$\rho_o/\rho_e = (\rho_o/\rho_e)_{ne}$$

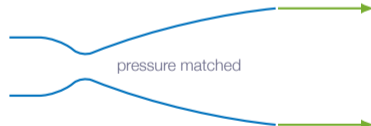
normal shock at nozzle exit



oblique shock

$$(\rho_o/\rho_e)_{ne} < \rho_o/\rho_e < (\rho_o/\rho_e)_{sc}$$

overexpanded nozzle flow



pressure matched

$$\rho_o/\rho_e = (\rho_o/\rho_e)_{sc}$$

pressure matched nozzle flow



expansion fan

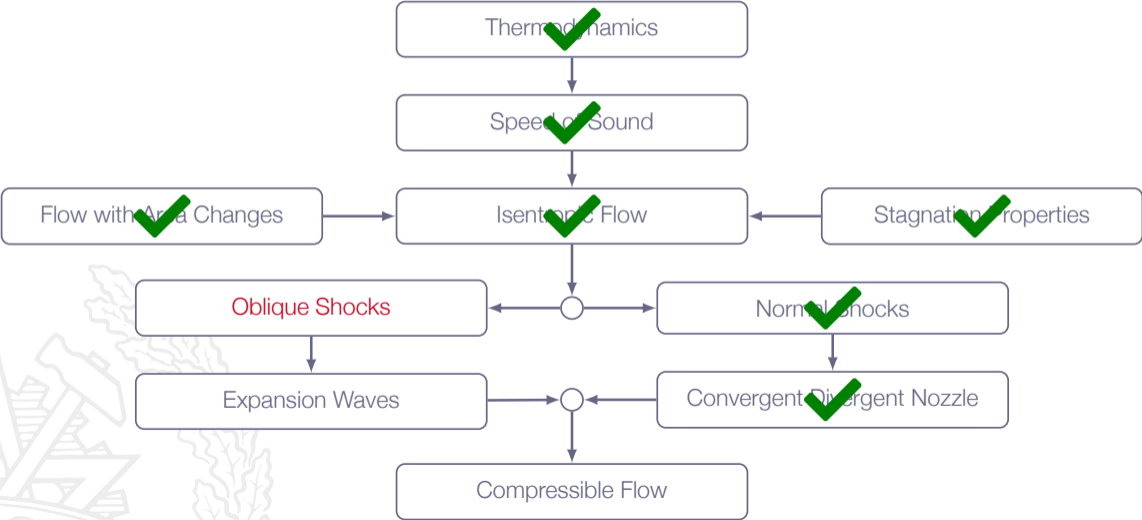
$$\rho_o/\rho_e > (\rho_o/\rho_e)_{sc}$$

underexpanded nozzle flow

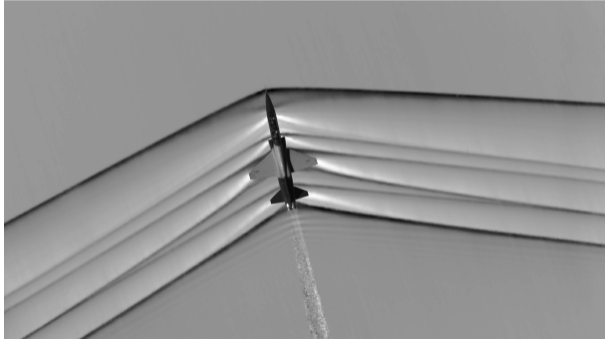
Convergent-Divergent Nozzle



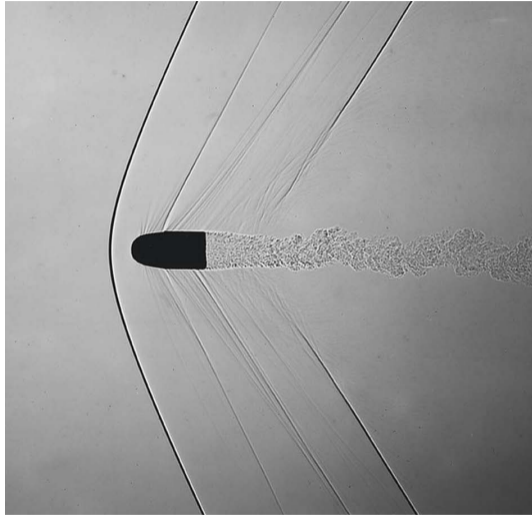
Roadmap - Compressible Flow



Oblique Shocks

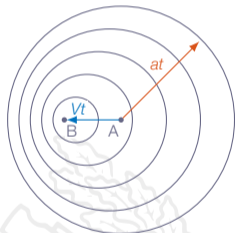


Oblique Shocks

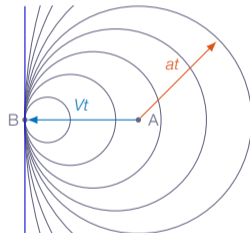


Mach Wave

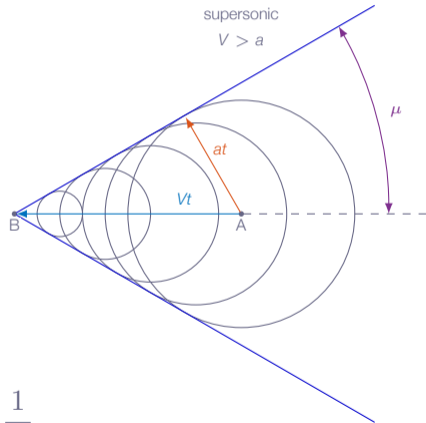
subsonic
 $V < a$



sonic
 $V = a$



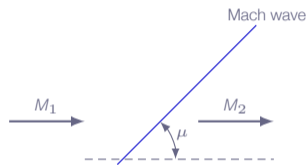
supersonic
 $V > a$



$$\sin \mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Mach Wave

A Mach wave is an infinitely weak oblique shock



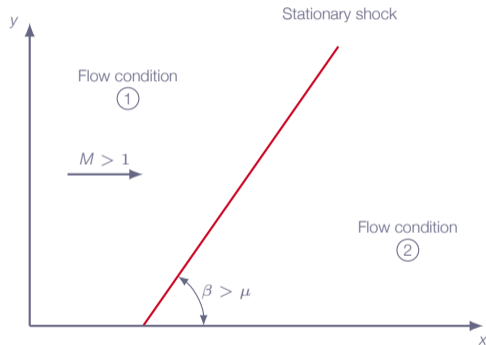
No substantial changes of flow properties over a single Mach wave

$M_1 > 1.0$ and $M_1 \approx M_2$

Isentropic

Oblique Shocks and Mach Waves

Two-dimensional steady-state flow



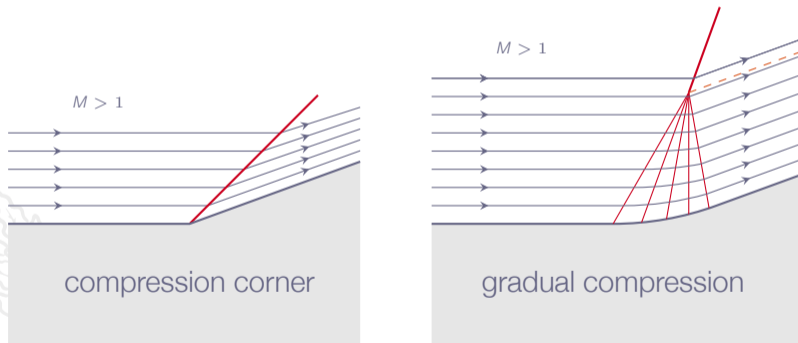
Significant changes of flow properties from 1 to 2

$M_1 > 1.0$, $\beta > \mu$, and $M_1 \neq M_2$

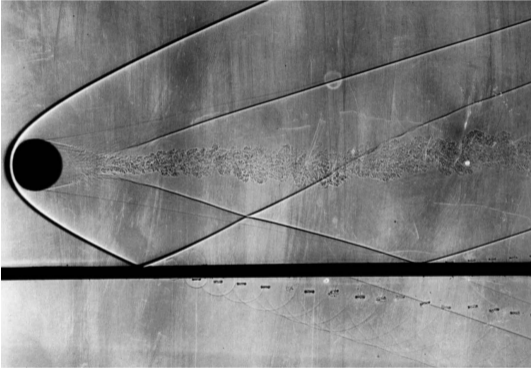
Not isentropic

Oblique Shocks and Mach Waves

When does an oblique shock appear in a flow?

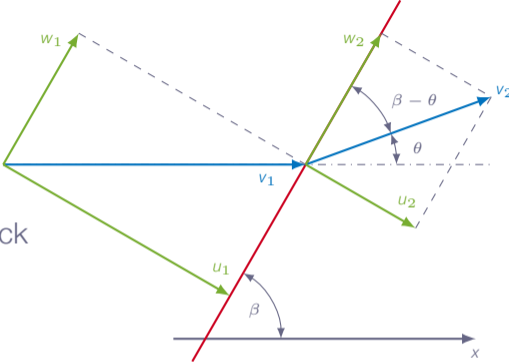


Mach Wave and Mach Waves

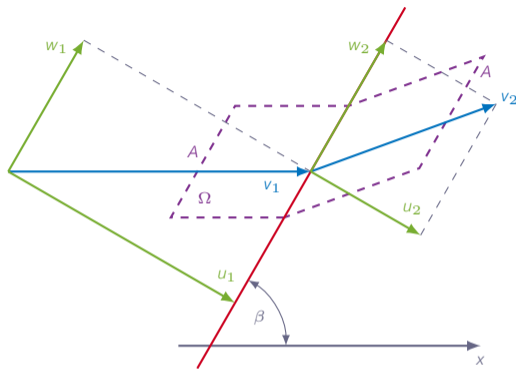


Oblique Shocks

Stationary oblique shock

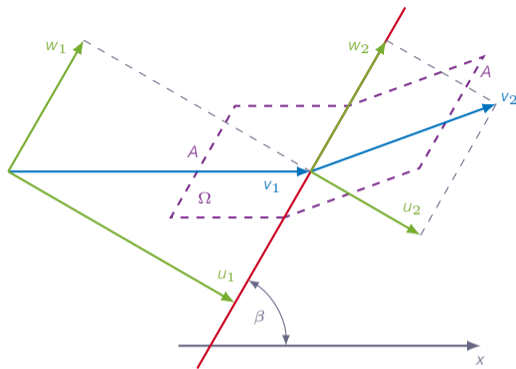


Oblique Shock Relations



- ▶ Two-dimensional steady-state flow
- ▶ Control volume aligned with flow stream lines

Oblique Shock Relations



Velocity notations:

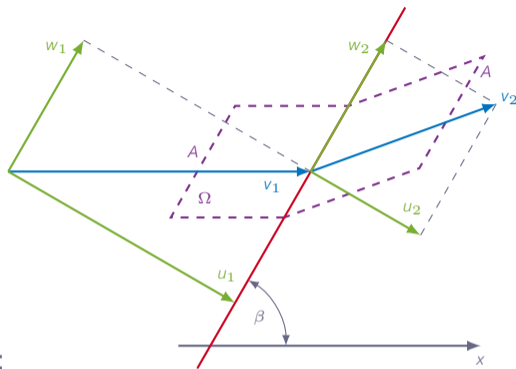
$$M_{n1} = \frac{u_1}{a_1} = M_1 \sin(\beta)$$

$$M_{n2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta)$$

$$M_1 = \frac{v_1}{a_1}$$

$$M_2 = \frac{v_2}{a_2}$$

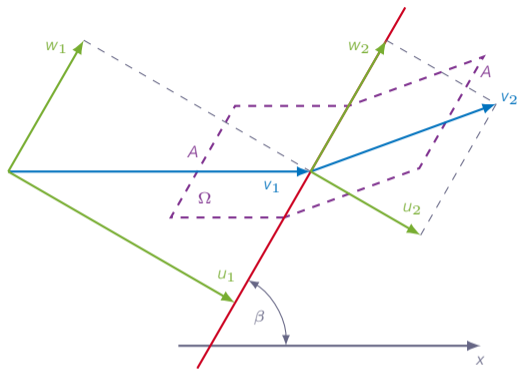
Oblique Shock Relations



Conservation of mass:

$$\rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2$$

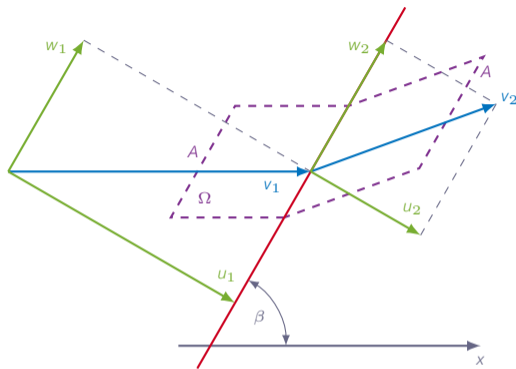
Oblique Shock Relations



Conservation of momentum (shock-normal direction):

$$(\rho_1 u_1^2 + p_1)A + (\rho_2 u_2^2 + p_2)A = 0 \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

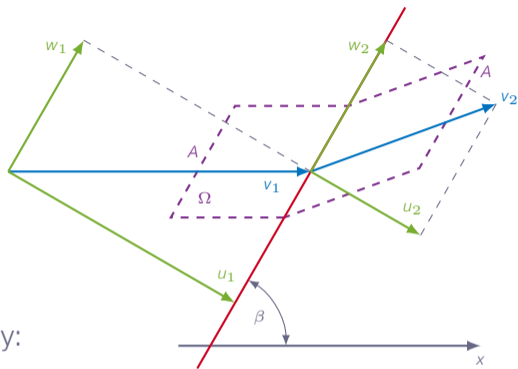
Oblique Shock Relations



Conservation of momentum (shock-tangential direction):

$$\rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow w_1 = w_2$$

Oblique Shock Relations



Conservation of energy:

$$\rho_1 u_1 \left[h_1 + \frac{1}{2} (u_1^2 + w_1^2) \right] A + \rho_2 u_2 \left[h_2 + \frac{1}{2} (u_2^2 + w_2^2) \right] A = 0 \Rightarrow h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$$

Oblique Shock Relations

We can use the same equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

Ratios such as ρ_2/ρ_1 , p_2/p_1 , and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?



Oblique Shock Relations

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The answer is no, but why?



Oblique Shock Relations

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The answer is no, but why?

P_{o1} , T_{o1} , etc are calculated using M_1 not M_{n1} (the tangential velocity is included)



Oblique Shock Relations

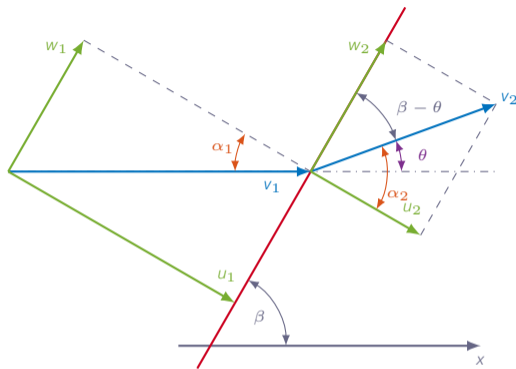
What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

The answer is no, but why?

P_{o1} , T_{o1} , etc are calculated using M_1 not M_{n1} (the tangential velocity is included)

OBS! Do not use ratios involving total quantities, e.g. ρ_{o2}/ρ_{o1} , T_{o2}/T_{o1} , obtained from formulas or tables for normal shock

Deflection Angle (for the interested)



$$\theta = \alpha_2 - \alpha_1 = \tan^{-1} \left(\frac{W}{U_2} \right) - \tan^{-1} \left(\frac{W}{U_1} \right)$$

$$\frac{\partial \theta}{\partial W} = \frac{U_2}{W^2 + U_2^2} - \frac{U_1}{W^2 + U_1^2}$$

Deflection Angle (for the interested)

$$\frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2} = 0 \Rightarrow$$

$$\frac{u_2(w^2 + u_1^2) - u_1(w^2 + u_2^2)}{(w^2 + u_2^2)(w^2 + u_1^2)} = 0 \Rightarrow \frac{(u_2 - u_1)(w^2 - u_1u_2)}{(w^2 + u_2^2)(w^2 + u_1^2)} = 0$$

Two solutions:

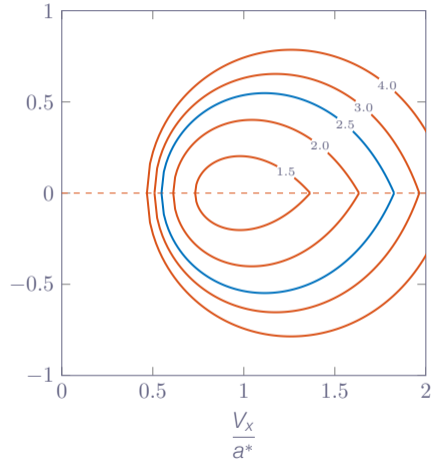
- ▶ $u_2 = u_1$ (no deflection)
- ▶ $w^2 = u_1u_2$ (max deflection)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

No deflection cases:

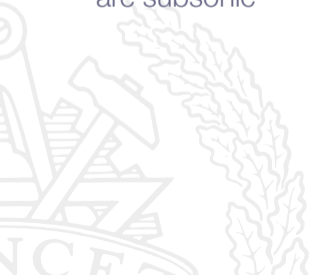
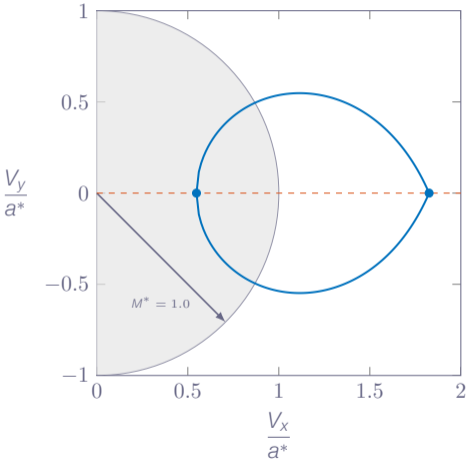
- ▶ normal shock
(reduced shock-normal velocity)
- ▶ Mach wave
(unchanged shock-normal velocity)



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

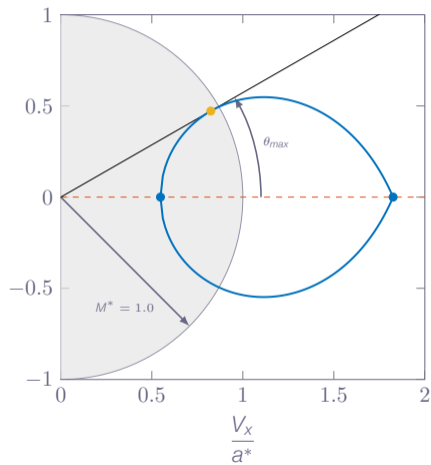
Solutions to the left of the sonic line are subsonic



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

It is not possible to deflect the flow more than θ_{max}



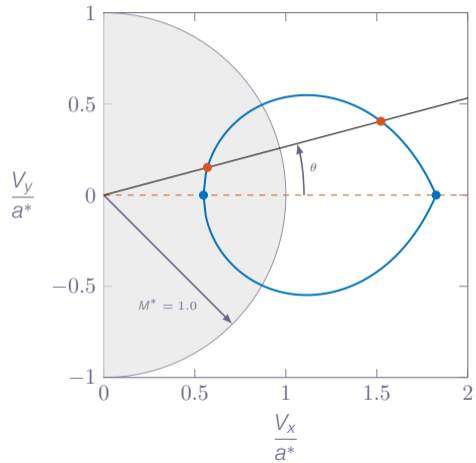
Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

For each deflection angle $\theta < \theta_{max}$, there are two solutions

- ▶ strong shock solution
- ▶ weak shock solution

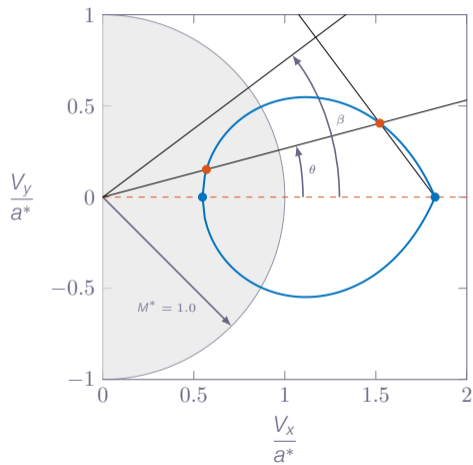
Weak shocks give lower losses and therefore the preferred solution



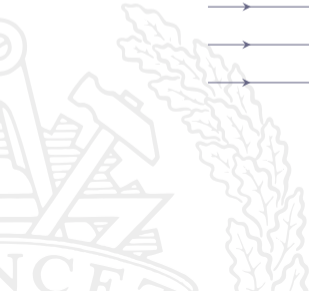
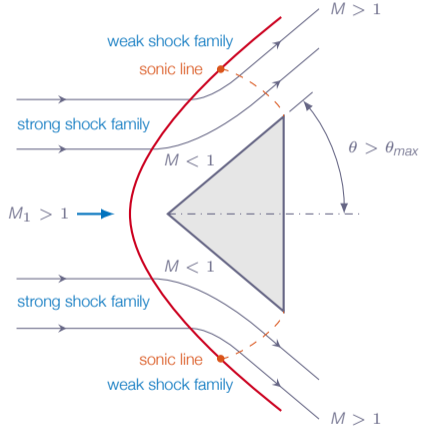
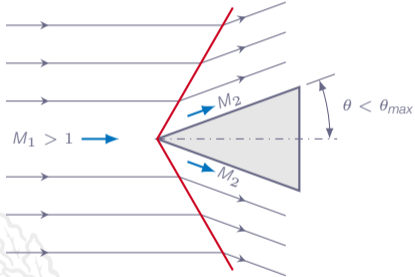
Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

The shock polar can be used to calculate the shock angle β for a given deflection angle θ



Flow Deflection

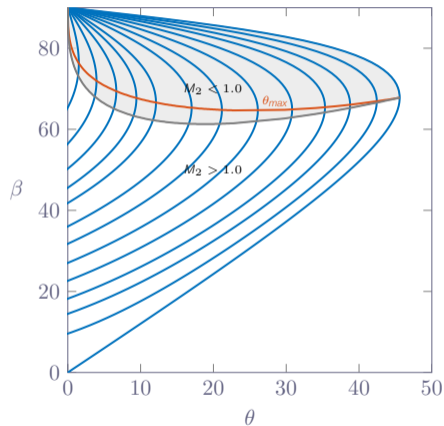


The θ - β -Mach Relation

A relation between:

- ▶ flow deflection angle θ
- ▶ shock angle β
- ▶ upstream flow Mach number M_1

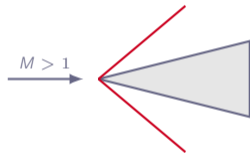
$$\tan(\theta) = \frac{2 \cot(\beta)(M_1^2 \sin^2(\beta) - 1)}{M_1^2(\gamma + \cot(2\beta)) + 2}$$



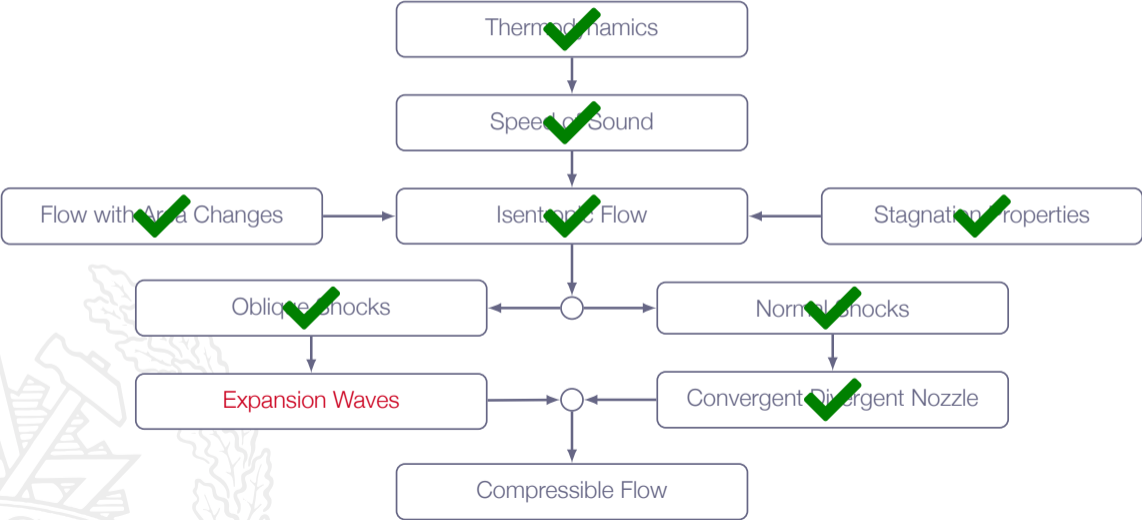
The θ - β -Mach Relation - Wedge Flow

Wedge flow oblique shock analysis:

1. θ - β - M relation $\Rightarrow \beta$ for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of $M_1 \Rightarrow M_{n_2}$ (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of $M_1 \Rightarrow \rho_2 / \rho_1, p_2 / p_1$, etc
6. upstream conditions + $\rho_2 / \rho_1, p_2 / p_1$, etc \Rightarrow downstream conditions



Roadmap - Compressible Flow



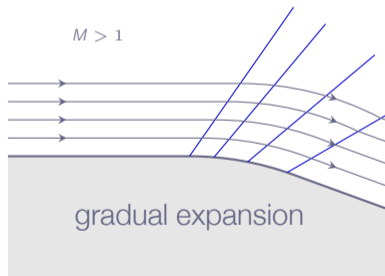
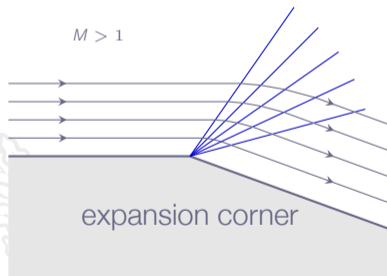
Expansion Waves

- ▶ Gradual change of flow angle
- ▶ Increasing flow area
- ▶ Increasing Mach number
- ▶ Accumulation of infinitesimal flow deflections - isentropic



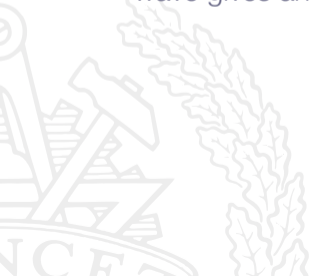
Expansion Waves

What is an expansion wave or expansion region?



The Prandtl-Meyer Function

- ▶ The change of flow properties over an expansion region can be calculated using the Prandtl-Meyer function
- ▶ The Prandtl-Meyer function derivation is based on the fact that each expansion wave gives an infinitesimal change in flow angle and flow properties



Prandtl-Meyer Function Derivation (*for the interested*)

For small deflection angles, linearization of the θ - β -Mach relation gives

$$\frac{dp}{p} \approx \frac{\gamma M^2}{(M^2 - 1)^{1/2}} d\theta$$

The momentum equation for inviscid flows gives

$$dp = -d(\rho V^2) = -\rho V dV - V d(\rho V) = -\rho V dV = -\rho V^2 \frac{dV}{V} = -\rho a^2 M^2 \frac{dV}{V} \Rightarrow$$

$$\frac{dp}{p} = -\gamma M^2 \frac{dV}{V}$$

Prandtl-Meyer Function Derivation (*for the interested*)

Now, setting the two expressions for dp/p equal

$$-\gamma M^2 \frac{dV}{V} = \frac{\gamma M^2}{(M^2 - 1)^{1/2}} d\theta \Rightarrow d\theta = -(M^2 - 1)^{1/2} \frac{dV}{V}$$

$$V = Ma \Rightarrow dV = a dM + M da \Rightarrow \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

$$d\theta = -(M^2 - 1)^{1/2} \left(\frac{dM}{M} + \frac{da}{a} \right)$$

Prandtl-Meyer Function Derivation (*for the interested*)

$$d\theta = -(M^2 - 1)^{1/2} \left(\frac{dM}{M} + \frac{da}{a} \right)$$

$$\frac{a_o}{a} = \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{1/2}$$

$$da = \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-1/2} da_o + a_o d \left[\left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-1/2} \right]$$

isentropic $\Rightarrow da_o = 0$

$$\frac{da}{a} = \frac{d \left[\left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-1/2} \right]}{\left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-1/2}} = \frac{-\frac{1}{2} \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-3/2} (\gamma - 1) M dM}{\left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-1/2}}$$

Prandtl-Meyer Function Derivation (*for the interested*)

$$d\theta = -(M^2 - 1)^{1/2} \left(\frac{dM}{M} + \frac{da}{a} \right)$$

$$\frac{da}{a} = \frac{-\frac{1}{2}(\gamma - 1)M dM}{1 + \frac{\gamma - 1}{2}M^2} \Rightarrow d\theta = -\frac{2(M^2 - 1)^{1/2}}{2 + (\gamma - 1)M^2} \frac{dM}{M}$$

Introducing ω defined such that: $d\omega = -d\theta$, $\omega = 0$ when $M = 1$

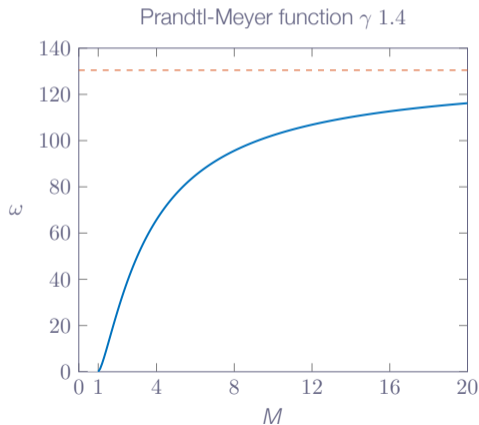
$$\int_0^\omega d\omega = \int_1^M \frac{2(M^2 - 1)^{1/2}}{2 + (\gamma - 1)M^2} \frac{dM}{M}$$

$$\omega(M) = \left(\frac{\gamma + 1}{\gamma - 1} \right)^{1/2} \tan^{-1} \left(\frac{M^2 - 1}{(\gamma + 1)/(\gamma - 1)} \right)^{1/2} - \tan^{-1}(M^2 - 1)^{1/2}$$

The Prandtl-Meyer Function

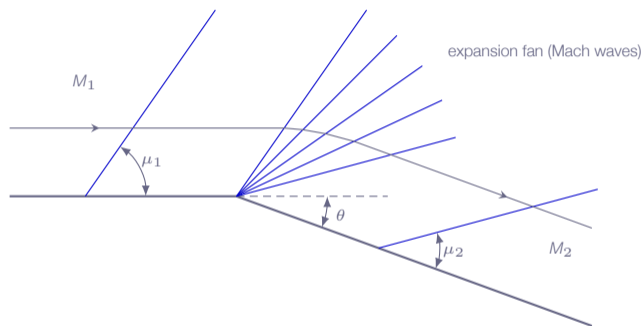
$$\omega(M) = \left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2} \tan^{-1} \left(\frac{M^2 - 1}{(\gamma + 1)/(\gamma - 1)} \right)^{1/2} - \tan^{-1}(M^2 - 1)^{1/2}$$

$$\omega(M)|_{M \rightarrow \infty} = 130.45^\circ$$



Prandtl-Meyer Expansion Waves

Example:



1. $\theta_1 = 0$, $M_1 > 1$ is given
2. θ_2 is given
3. find M_2 such that $\theta_2 = \omega(M_2) - \omega(M_1)$

Prandtl-Meyer Expansion Waves

Since flow is isentropic, the usual isentropic relations apply:

(p_o and T_o are constant)

Calorically perfect gas:

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2 \right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2 \right]$$

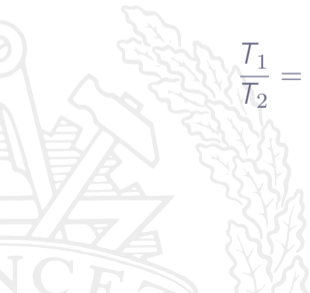


Prandtl-Meyer Expansion Waves

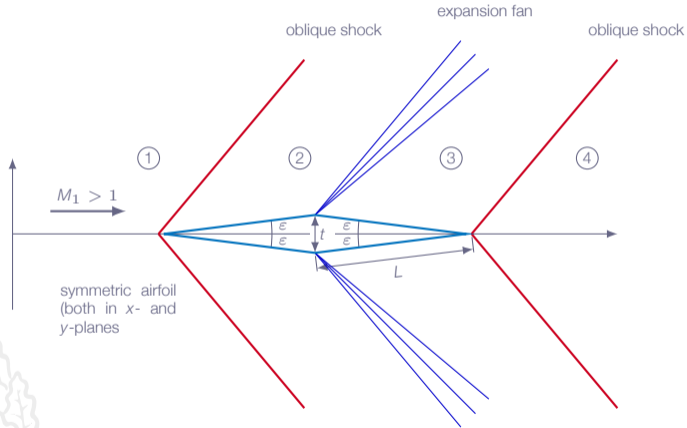
since $p_{o1} = p_{o2}$ and $T_{o1} = T_{o2}$

$$\frac{p_1}{p_2} = \frac{p_{o2}}{p_{o1}} \frac{p_1}{p_2} = \left(\frac{p_{o2}}{p_2} \right) / \left(\frac{p_{o1}}{p_1} \right) = \left[\frac{1 + \frac{1}{2}(\gamma - 1)M_2^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2} \right]^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{T_1}{T_2} = \frac{T_{o2}}{T_{o1}} \frac{T_1}{T_2} = \left(\frac{T_{o2}}{T_2} \right) / \left(\frac{T_{o1}}{T_1} \right) = \left[\frac{1 + \frac{1}{2}(\gamma - 1)M_2^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2} \right]$$



Diamond-Wedge Airfoil



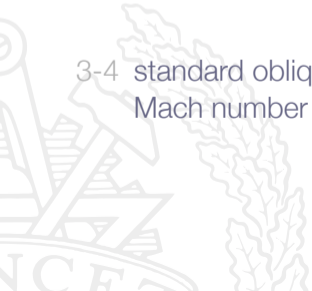
Note! symmetric airfoil at zero incidence \Rightarrow zero lift but what about drag?

Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3



Diamond-Wedge Airfoil - Wave Drag

Since conditions 2 and 3 are constant in their respective regions, we obtain:

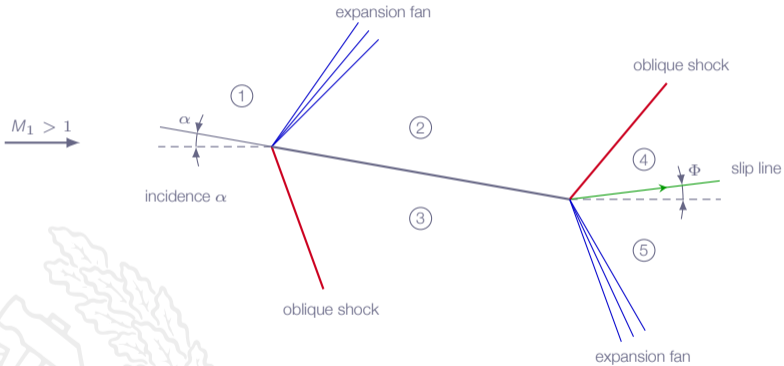
$$D = 2 [\rho_2 L \sin(\varepsilon) - \rho_3 L \sin(\varepsilon)] = 2(\rho_2 - \rho_3) \frac{t}{2} = (\rho_2 - \rho_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $\rho_2 > \rho_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

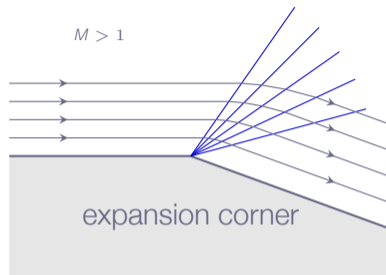
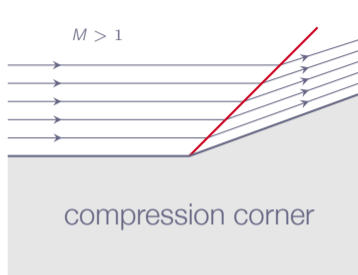
For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the mismatch of flow angles out



Flat-Plate Airfoil

- ▶ Flow states 4 and 5 must satisfy:
 - ▶ $\rho_4 = \rho_5$
 - ▶ flow direction 4 equals flow direction 5 (Φ)
- ▶ Shock between 2 and 4 as well as expansion fan between 3 and 5 will adjust themselves to comply with the requirements
- ▶ For calculation of lift and drag only states 2 and 3 are needed
- ▶ States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

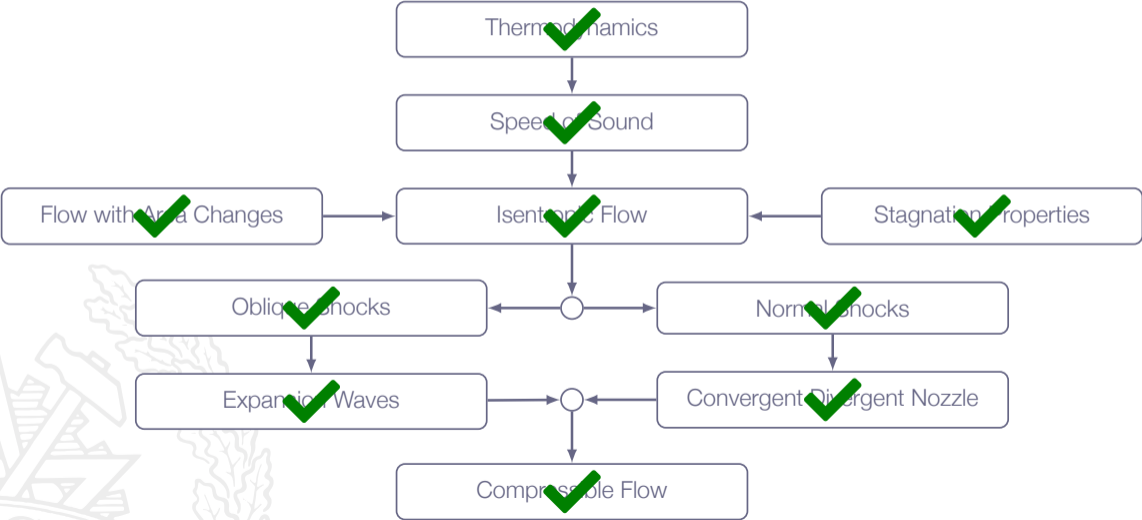
Oblique Shocks and Expansion Waves



M decrease
 V decrease
 ρ increase
 ρ increase
 T increase

M increase
 V increase
 ρ decrease
 ρ decrease
 T decrease

Roadmap - Compressible Flow



Supersonic Stereo

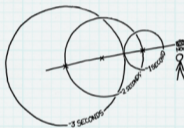
What if you somehow managed to make a stereo travel at twice the speed of sound, would it sound backwards to someone who was just casually sitting somewhere as it flies by?

—Tim Currie

Yes.

Technically, anyway. It would be pretty hard to hear.

The basic idea is pretty straightforward. The stereo is going faster than its own sound, so it will reach you first, followed by the sound it emitted one second ago, followed by the sound it emitted two seconds ago, and so forth.



The problem is that the stereo is moving at Mach 2, which means that two seconds ago, it was over a kilometer away. It's hard to hear music from that distance, particularly when your ears were just hit by (a) a sonic boom, and (b) pieces of a rapidly disintegrating stereo.

Wind speeds of Mach 2 would messily disassemble most consumer electronics. The force of the wind on the body of the stereo is roughly comparable to that of a dozen people standing on it:

