Unsteady Wave Motion

Finite Non-Linear Waves

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Non-Linear One-Dimensional Flow

Starting point: the governing flow equations on partial differential form

Continuity equation:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{1}$$

Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2}$$

Any thermodynamic property can be expressed as a function of two other thermodynamic properties. This means that we can get density as a function of pressure and entropy: $\rho = \rho(p, s)$ and therefore

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds$$

Assuming isentropic flow $ds = 0$ gives

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t} \\ \frac{\partial \rho}{\partial x} &= \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x} \end{aligned} \tag{3}$$

Now, insert 3 in 1 gives

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} = 0 \tag{4}$$

Dividing 4 by ρa gives

$$\frac{1}{\rho a}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) + a\frac{\partial u}{\partial x} = 0 \tag{5}$$

A slightly modified form of the momentum equation is obtained by multiplying and dividing the last term by a

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho a}\left(a\frac{\partial p}{\partial x}\right) = 0 \tag{6}$$

If the continuity equation on the form 5 is added to the momentum equation on the form 6, we get

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0 \tag{7}$$

If, instead, the continuity equation on the form 5 is subtracted from the momentum equation on the form 6, we get

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] = 0 \tag{8}$$

Since $u = u(x,t)$, we have from the definition of a differential

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt \tag{9}$$

Now, let $dx/dt = u + a$

$$du = \frac{\partial u}{\partial t}dt + (u+a)\frac{\partial u}{\partial x}dt = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right]dt \tag{10}$$

which is the change of u in the direction $dx/dt = u + a$

In the same way

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}dx = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}\frac{dx}{dt}dt \tag{11}$$

and thus, in the direction $dx/dt = u + a$

$$dp = \frac{\partial p}{\partial t}dt + (u + a)\frac{\partial p}{\partial x}dt = \left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right] dt \tag{12}$$

If we go back and examine Eqn. 7, we see that Eqns. 10 and 12 appear in the equation and thus it can now be rewritten as follows

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \Rightarrow du + \frac{dp}{\rho a} = 0 \tag{13}$$

Eqn. 13 applies along a C^+ characteristic, i.e., a line in the direction $dx/dt = u + a$ in xt-space and is called the compatibility equation along the C^+ characteristic. If we instead chose a C^- characteristic, i.e., a line in the direction $dx/dt = u - a$ in xt-space, we get

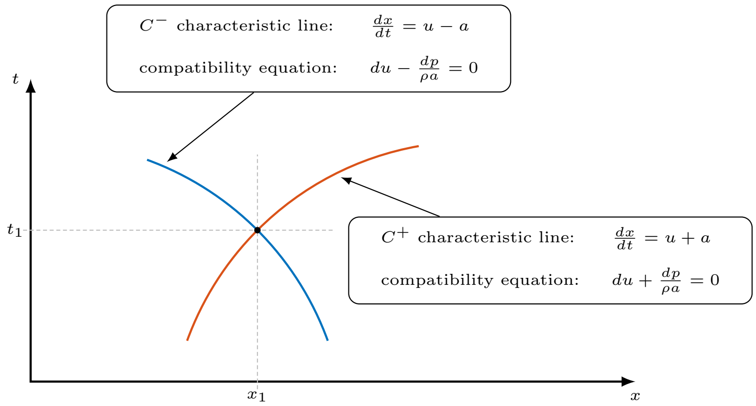
$$du = \left[\frac{\partial u}{\partial t} + (u - a)\frac{\partial u}{\partial x}\right] dt \tag{14}$$

$$dp = \left[\frac{\partial p}{\partial t} + (u - a)\frac{\partial p}{\partial x}\right] dt \tag{15}$$

which can be identified as subsets of Eqn. 8 and thus

$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \Rightarrow du - \frac{dp}{\rho a} = 0 \tag{16}$$

Eqn. 16 applies along a C^- characteristic, i.e., a line in the direction $dx/dt = u - a$ in xt-space and is called the compatibility equation along the C^- characteristic.



Figur 1: Characteristic lines through a point (x_1,t_1)

So, what we have done now is that we have have found paths through a point (x_1,t_1) along which the governing partial differential equations Eqns. 7 and 8 reduces to the ordinary differential equations 13 and 16. The C^+ and C^- characteristic lines are physically the paths of right- and left-running sound waves in the xt-plane.

Riemann Invariants

If the compatibility equations are integrated along respective characteristic line, i.e., integrate 13 along the C^+ characteristic and 16 along the C^- characteristic, we get the Riemann invariants J^+ and J^-.

$$J^+ = u + \int \frac{dp}{\rho a} = const \tag{17}$$

$$J^- = u - \int \frac{dp}{\rho a} = const \tag{18}$$

The Riemann invariants are constants along the associated characteristic line.

We have assumed isentropic flow and thus we may use the isentropic relations

$$p = C_1 T^{\gamma/(\gamma-1)} = C_2 a^{2\gamma/(\gamma-1)} \tag{19}$$

where C_1 and C_2 are constants. Differentiating Eqn. 19 gives

$$dp = C_2 \left(\frac{2\gamma}{\gamma - 1} \right) a^{[2\gamma/(\gamma-1)-1]} da \tag{20}$$

Now, if we further assume the gas to be calorically perfect

$$a^2 = \gamma RT = \frac{\gamma p}{\rho} \Rightarrow \rho = \frac{\gamma p}{a^2} \tag{21}$$

Eqn. 19 in 21 gives

$$\rho = C_2 \gamma a^{[2\gamma/(\gamma-1)-2]} \tag{22}$$

and thus

$$J^+ = u + \int \frac{C_2 \left(\frac{2\gamma}{\gamma-1} \right) a^{[2\gamma/(\gamma-1)-1]}}{C_2 \gamma a^{[2\gamma/(\gamma-1)-2]} a} da = u + \left(\frac{2}{\gamma - 1} \right) \int da$$

$$J^+ = u + \frac{2a}{\gamma - 1} \tag{23}$$

$$J^- = u - \frac{2a}{\gamma - 1} \tag{24}$$

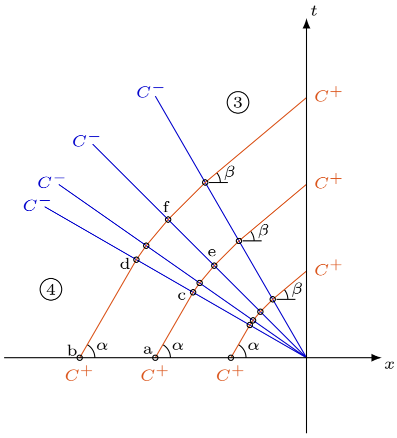
Eqns. 23 and 24 are the Riemann invariants for a calorically perfect gas. The Riemann invariants are constants along C^+ and C^- characteristics and if the situation shown in Fig. 2 appears, that fact can be used to calculate the flow velocity and speed of sound in the location (x_1,t_1).

$$J^+ + J^- = u + \frac{2a}{\gamma - 1} + u - \frac{2a}{\gamma - 1} = 2u \Rightarrow u = \frac{1}{2}(J^+ + J^-) \tag{25}$$

$$J^+ = u + \frac{2a}{\gamma - 1} = \frac{1}{2}(J^+ + J^-) + \frac{2a}{\gamma - 1} \Rightarrow a = \frac{\gamma - 1}{4}(J^+ - J^-) \tag{26}$$

Expansion Wave

The expansion wave propagation into the driver section in a shock tube can be described using characteristic lines.



Figur 2: Expansion fan centered at $(x, t) = (0.0, 0.0)$

The expansion is propagating into stagnant fluid in region four (the driver section), which means that the flow properties ahead of the expansion wave are constant.

$$J_a^+ = J_b^+$$

J^+ invariants constant along C^+ characteristics

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

Since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$. In fact, since the flow properties ahead of the expansion are constant, all C^+ lines will have the same J^+ value.

J^- invariants constant along C^- characteristics

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

$$\left.\begin{aligned} u_e &= \frac{1}{2}(J_e^+ + J_e^-) \\ u_f &= \frac{1}{2}(J_f^+ + J_f^-) \\ J_e^- &= J_f^- \\ J_e^+ &= J_f^+ \end{aligned}\right\} \Rightarrow u_e = u_f \Rightarrow a_e = a_f$$

Due to the fact the J^+ is constant in the entire expansion region, u and a will be constant along each C^- line.

The constant J^+ value can be used to obtain relations for the variation of flow properties through the expansion region. Evaluation of the J^+ invariant at any position within the expansion region should give the same value as in region 4.

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = 0 + \frac{2a_4}{\gamma - 1}$$

and thus

$$\frac{a}{a_4} = 1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right) \tag{27}$$

Eqn. 27 and $a = \sqrt{\gamma R T}$ gives

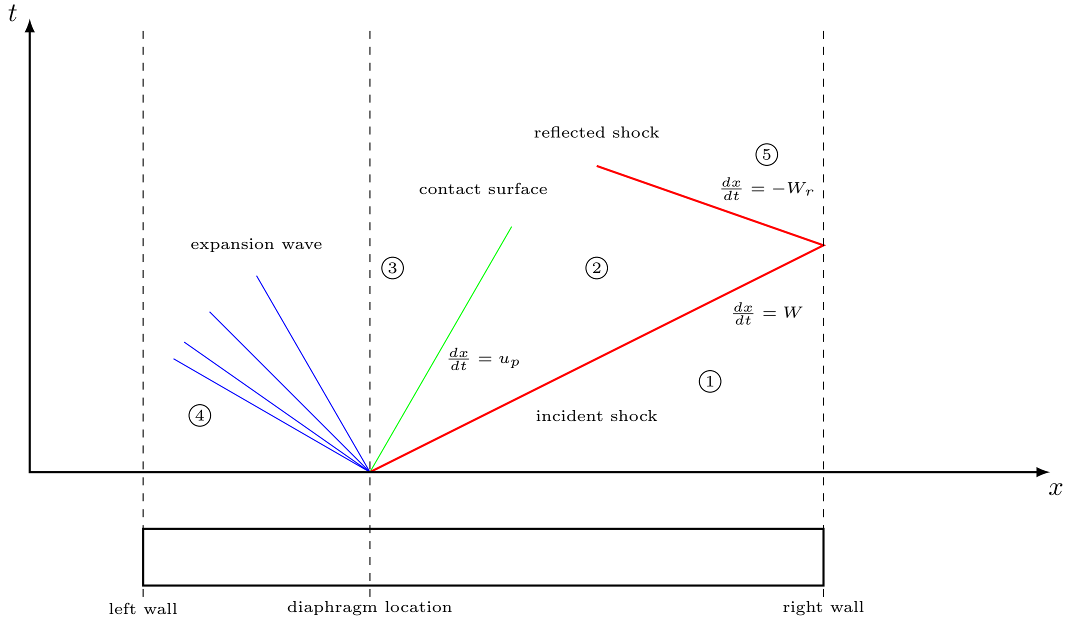
$$\frac{T}{T_4} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right)\right]^2 \tag{28}$$

Using isentropic relations, we can get pressure ratio and density ratio

$$\frac{p}{p_4} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right)\right]^{2\gamma/(\gamma-1)} \tag{29}$$

$$\frac{\rho}{\rho_4} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right)\right]^{2/(\gamma-1)} \tag{30}$$

Shock Tube



Figur 3: traveling waves in a shock tube

From the analysis of the incident shock, we have a relation for the induced flow behind the shock

$$u_2 = u_p = \frac{a_1}{\gamma_1}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\left(\frac{2\gamma_1}{\gamma_1+1}\right)}{\left(\frac{\gamma_1-1}{\gamma_1+1}\right) + \left(\frac{p_2}{p_1}\right)}\right)^{1/2} \tag{31}$$

The velocity in region 3 can be obtained from the expansion relations

$$\frac{p_3}{p_4} = \left[1 - \frac{\gamma_4 - 1}{2}\left(\frac{u_3}{a_4}\right)\right]^{2\gamma_4/(\gamma_4-1)} \tag{32}$$

Solving for u_3 gives

$$u_3 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right] \tag{33}$$

There is no change in pressure or velocity over the contact surface, which means $u_2 = u_3$ and $p_2 = p_3$.

$$u_2 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right] \tag{34}$$

Now, we have two ways of calculating u_2. Setting Eqn. 31 equal to Eqn. 34 leads to the shock tube relation

$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left\{1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1\left[2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1)\right]}}\right\}^{-2\gamma_4/(\gamma_4-1)} \tag{35}$$