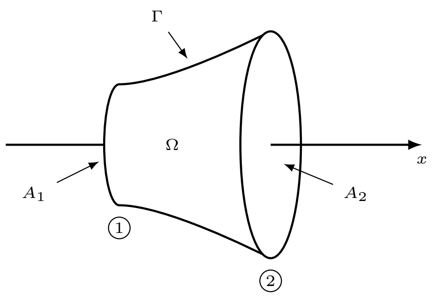
Quasi-One-Dimensional Flow

Governing Equations

Niklas Andersson
Division of Fluid Dynamics
Department of Mechanics and Maritime Sciences
Chalmers University of Technology

Governing Equations for Quasi-one-dimensional Flow



Figur 1: Quasi-one-dimensional flow - control volume

In the following quasi-one-dimensional flow will be assumed. That means that the cross-section is allowed to vary smoothly but flow quantities varies in one direction only. The equations that are derived will thus describe one-dimensional flow in axisymmetric tubes. Let's assume flow in the x-direction, which means that all flow quantities and the cross-section area will vary with the axial coordinate x.

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ ...$$

We will further assume steady-state flow, which means that unsteady terms will be zero.

The equations are derived with the starting point in the governing flow equations on integral form

Continuity Equation

Applying the integral form of the continuity equation on the quasi-one-dimensional flow control volume (Fig. 1) gives

$$\underbrace{\frac{d}{dt}\iiint_{\Omega}\rho d\mathscr{V}}_{=0}+\oiint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS=0 \tag{1}$$

$$\oiint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS=-\rho_1u_1A_1+\rho_2u_2A_2$$

$$\rho_1u_1A_1=\rho_2u_2A_2 \tag{2}$$

Momentum Equation

Applying the integral form of the momentum equation on the quasi-one-dimensional flow control volume (Fig. 1) gives

$$\underbrace{\frac{d}{dt}\iiint_{\Omega}\rho\mathbf{v} d\mathscr{V}}_{=0}+\oiint_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=0 \tag{3}$$

$$\oiint_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=-\rho_1u_1^2A_1+\rho_2u_2^2A_2$$

$$\oiint_{\partial\Omega}p\mathbf{n}dS=-p_1A_1+p_2A_2-\int_{A_1}^{A_2}pdA$$

collecting terms

$$\left(\rho_1u_1^2+p_1\right)A_1+\int_{A_1}^{A_2}pdA=\left(\rho_2u_2^2+p_2\right)A_2 \tag{4}$$

Energy Equation

Applying the integral form of the energy equation on the quasi-one-dimensional flow control volume (Fig. 1) gives

$$\underbrace{\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V}}_{=0} + \oiint_{\partial\Omega} \left[\rho h_o(\mathbf{v}\cdot\mathbf{n})\right] dS = 0 \tag{5}$$

$$\oiint_{\partial\Omega} \left[\rho h_o(\mathbf{v}\cdot\mathbf{n})\right] dS = -\rho_1 u_1 h_{o_1} A_1 + \rho_2 u_2 h_{o_2} A_2$$

$$\rho_1 u_1 h_{o_1} A_1 = \rho_2 u_2 h_{o_2} A_2$$

Now, using the continuity equation $\rho_1 u_1 A_1 = \rho_2 u_2 A_2$ gives

$$h_{o_1} = h_{o_2} \tag{6}$$

Differential Form

The integral term appearing the momentum equation is undesired and therefore the governing equations are converted to differential form.

The continuity equation (Eqn. 2) is rewritten in differential form as

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 = const$$

$$d(\rho u A) = 0 \tag{7}$$

The momentum equation (Eqn. 4) is rewritten in differential form as

$$\left(\rho_1 u_1^2 + p_1\right) A_1 + \int_{A_1}^{A_2} p dA = \left(\rho_2 u_2^2 + p_2\right) A_2 \Rightarrow d\left[(\rho u^2 + p)A\right] = p dA$$

$$d(\rho u^2 A) + d(pA) = pdA$$

$$ud(\rho uA) + \rho uAdu + Adp + \cancel{pdA} = \cancel{pdA}$$

From the continuity equation we have $d(\rho uA)$ and thus

$$\rho u\cancel{A}du + \cancel{A}dp = 0 \Rightarrow$$

$$dp = -\rho udu \tag{8}$$

which is the momentum equation on differential form. Also referred to as Euler's equation. Finally, the energy equation (Eqn. 2) is rewritten in differential form as

$$h_{o_1} = h_{o_2} = const \Rightarrow dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow dh + \frac{1}{2}d(u^2) = 0$$

$$dh + udu = 0 \tag{9}$$

Summary

Continuity:

$$d(\rho u A) = 0$$

Momentum:

$$dp = -\rho u du$$

Energy:

$$dh + u du = 0$$

The equations are valid for:

- quasi-one-dimensional flow
- steady state
- all gas models (no gas model assumptions made)
- inviscid flow

It should be noted that equations are exact but they are applied to a physical model that is approximate, i.e., the approximation that flow quantities varies in one dimension with a varying cross-section area. In reality, a variation of cross-section area would imply flow in three dimensions.