One-Dimensional Steady Flow

Normal Shock Relations

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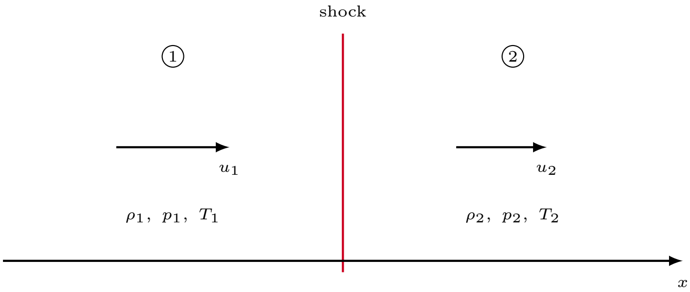
Normal Shock



Figur 1: Stationary normal shock

The starting point is to set up the governing equations for one-dimensional steady compressible flow over a control volume enclosing the normal shock (Fig. 1).

continuity:

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

momentum:

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{2}$$

energy:

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 \tag{3}$$

Divide the momentum equation by $\rho_1 u_1$

$$\frac{1}{\rho_1 u_1}\left(\rho_1 u_1^2 + p_1\right) = \frac{1}{\rho_1 u_1}\left(\rho_2 u_2^2 + p_2\right) = \{\rho_1 u_1 = \rho_2 u_2\} = \frac{1}{\rho_2 u_2}\left(\rho_2 u_2^2 + p_2\right) \Rightarrow$$

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1 \tag{4}$$

For a calorically perfect gas $a = \sqrt{\gamma p/\rho}$, which if implemented in Eqn. 4 gives

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \tag{5}$$

The energy equation (Eqn. 3) with $h = C_p T$

$$C_p T_1 + \frac{1}{2} u_1^2 = C_p T_2 + \frac{1}{2} u_2^2 \tag{6}$$

Replacing C_p with $\gamma R/(\gamma - 1)$ gives

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{1}{2} u_1^2 = \frac{\gamma R T_2}{\gamma - 1} + \frac{1}{2} u_2^2 \tag{7}$$

With $a = \sqrt{\gamma R T}$ this becomes

$$\frac{a_1^2}{\gamma - 1} + \frac{1}{2} u_1^2 = \frac{a_2^2}{\gamma - 1} + \frac{1}{2} u_2^2 \tag{8}$$

Eqn. 8 can be set up between any two points in the flow. Specifically, we can use the relation to relate the flow velocity, u, and speed of sound, a, in any point to the corresponding flow properties at sonic conditions ($u = a = a^*$).

$$\frac{a^2}{\gamma - 1} + \frac{1}{2} u^2 = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} \tag{9}$$

If Eqn. 9 is evaluated in locations 1 and 2, we get

$$\begin{aligned} a_1^2 &= \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma - 1}{2} u_1^2 \\ a_2^2 &= \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma - 1}{2} u_2^2 \end{aligned} \tag{10}$$

Since the change in flow conditions over the shock is adiabatic (no heat is added inside the shock), critical properties will be constant over the shock. Especially a^* will be constant.

Eqn. 10 inserted in 5 gives

$$\frac{1}{\gamma u_1}\left(\frac{\gamma+1}{2}a^{*2}-\frac{\gamma-1}{2}u_1^2\right)-\frac{1}{\gamma u_2}\left(\frac{\gamma+1}{2}a^{*2}-\frac{\gamma-1}{2}u_2^2\right)=u_2-u_1 \Rightarrow$$

$$\left(\frac{\gamma+1}{2\gamma}\right)a^{*2}\left(\frac{1}{u_1}-\frac{1}{u_2}\right)=\left(\frac{\gamma+1}{2\gamma}\right)(u_2-u_1) \Rightarrow$$

$$a^{*2}\left(\frac{1}{u_1}-\frac{1}{u_2}\right)=(u_2-u_1) \Rightarrow$$

$$a^{*2}\left(\frac{u_2}{u_1u_2}-\frac{u_1}{u_1u_2}\right)=(u_2-u_1) \Rightarrow$$

$$\frac{1}{u_1u_2}a^{*2}(u_2-u_1)=(u_2-u_1) \Rightarrow$$

$$a^{*2}=u_1u_2 \tag{11}$$

Eqn. 11 is sometimes referred to as the Prandtl relation. Divide the Prandtl relation by a^{*2} on both sides gives

$$1=\frac{u_1}{a^*}\frac{u_2}{a^*}=M_1^*M_2^*$$

or

$$M_2^*=\frac{1}{M_1^*} \tag{12}$$

The relation between M^* and M is given by

$$M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \tag{13}$$

from which is can be seen that M^* will follow the Mach number M in the sense that

- $M = 1 \Rightarrow M^* = 1$
- $M < 1 \Rightarrow M^* < 1$
- $M > 1 \Rightarrow M^* > 1$

The Mach number ahead of the shock must be greater than one and thus Eqn. 12 shows that the Mach number downstream of the shock must be less than one.

Eqn. 13 inserted in Eqn. 12 gives

$$\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} = \frac{2+(\gamma-1)M_2^2}{(\gamma+1)M_2^2}$$

$$M_2^2 = \frac{1+[(\gamma-1)/2]\,M_1^2}{\gamma M_1^2-(\gamma-1)/2} \tag{14}$$

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Rewriting the continuity equation (Eqn. 1)

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \{a^* = u_1 u_2\} = \frac{u_1^2}{a^{*2}} = M_1^{*2} \tag{15}$$

Eqn. 13 in Eqn. 15 gives

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \tag{16}$$

To get a corresponding relation for the pressure ratio over the shock, we go back to the momentum equation (Eqn. 2)

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_1\} = \rho_1 u_1(u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{p_2 - p_1}{p_1} = \frac{\rho_1 u_1^2}{p_1}\left(1 - \frac{u_2}{u_1}\right) = \left\{a_1 = \sqrt{\frac{\gamma p_1}{\rho_1}}\right\} = \gamma \frac{u_1^2}{a_1^2}\left(1 - \frac{u_2}{u_1}\right) = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{p_2}{p_1} - 1 = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right) = \left\{\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2}\right\} = \gamma M_1^2 \left(1 - \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right)$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \tag{17}$$

The temperature ratio over the shock can be obtained using the already derived relations for pressure ratio and density ratio together with the equation of state $p = \rho R T$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right) \tag{18}$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\left[\frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}\right] \tag{19}$$