Compressible Inviscid Fluid Flow Equations

Differential Form

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Governing Equations on Integral Form

Eqns. 1 - 3 are the integral form of the continuity, momentum and energy equations, respectively. These equations may be rewritten with the corresponding equations on differential form as a result.

$$\frac{d}{dt}\iiint_{\Omega}\rho d\mathscr{V}+\oint\!\!\!\oint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS=0 \tag{1}$$

$$\frac{d}{dt}\iiint_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oint\!\!\!\oint_{\partial\Omega}\left[(\rho\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint_{\Omega}\rho\mathbf{f}d\mathscr{V} \tag{2}$$

$$\frac{d}{dt}\iiint_{\Omega}\rho e_o d\mathscr{V}+\oint\!\!\!\oint_{\partial\Omega}\rho h_o(\mathbf{v}\cdot\mathbf{n})dS=\iiint_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}+\iiint_{\Omega}\dot{q}\rho d\mathscr{V} \tag{3}$$

Governing Equations on Differential Form

Conservation of Mass

Apply Gauss's divergence theorem on the surface integral in Eqn. 1 gives

$$\oint\!\!\!\oint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS=\iiint_{\Omega}\nabla\cdot(\rho\mathbf{v})d\mathscr{V}$$

Also, if Ω is a fixed control volume

$$\frac{d}{dt}\iiint_{\Omega}\rho d\mathscr{V}=\iiint_{\Omega}\frac{\partial\rho}{\partial t}d\mathscr{V}$$

The continuity equation can now be written as a single volume integral.

$$\iiint_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume and thus

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{4}$$

which is the continuity equation on partial differential form.

Conservation of Momentum

As for the continuity equation, the surface integral terms are rewritten as volume integrals using Gauss's divergence theorem.

$$\oiint_{\partial\Omega} (\rho \mathbf{v} \cdot \mathbf{n}) \mathbf{v} dS = \iiint_{\Omega} \nabla \cdot (\rho \mathbf{v}\mathbf{v}) d\mathscr{V}$$

$$\oiint_{\partial\Omega} p \mathbf{n} dS = \iiint_{\Omega} \nabla p d\mathscr{V}$$

Also, if Ω is a fixed control volume

$$\frac{d}{dt} \iiint_{\Omega} \rho \mathbf{v} d\mathscr{V} = \iiint_{\Omega} \frac{\partial}{\partial t} (\rho \mathbf{v}) d\mathscr{V}$$

The momentum equation can now be written as one single volume integral

$$\iiint_{\Omega} \left[\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p - \rho \mathbf{f} \right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume and thus

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f} \tag{5}$$

which is the momentum equation on partial differential form

Conservation of Energy

Gauss's divergence theorem applied to the surface integral term in the energy equation (Eqn. 3) gives

$$\oiint_{\partial\Omega} \rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint_{\Omega} \nabla\cdot(\rho h_o\mathbf{v})d\mathscr{V}$$

Fixed control volume

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} = \iiint_{\Omega} \frac{\partial}{\partial t}(\rho e_o)d\mathscr{V}$$

The energy equation can now be written as

$$\iiint_{\Omega} \left[\frac{\partial}{\partial t}(\rho e_o) + \nabla\cdot(\rho h_o\mathbf{v}) - \rho\mathbf{f}\cdot\mathbf{v} - \dot{q}\rho\right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume and thus

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla\cdot(\rho h_o\mathbf{v}) = \rho\mathbf{f}\cdot\mathbf{v} + \dot{q}\rho \tag{6}$$

which is the energy equation on partial differential form

Summary

The governing equations for compressible inviscid flow on partial differential form:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p = \rho\mathbf{f}$$

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla\cdot(\rho h_o\mathbf{v}) = \rho\mathbf{f}\cdot\mathbf{v} + \dot{q}\rho$$

The Differential Equations on Non-Conservation Form

The Substantial Derivative

The substantial derivative operator is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{7}$$

where the first term of the right hand side is the local derivative and the second term is the convective derivative.

Conservation of Mass

If we apply the substantial derivative operator to density we get

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$$

From before we have the continuity equation on differential form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

which can be rewritten as

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \rho = 0$$

and thus

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \tag{8}$$

Eqn. 8 says that the mass of a fluid element with a fixed set of fluid particles is constant as the element moves in space.

Conservation of Momentum

We start from the momentum equation on differential form derived above

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}$$

Expanding the first and the second terms gives

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v}(\nabla \cdot \rho \mathbf{v}) + \nabla p = \rho \mathbf{f}$$

Collecting terms, we can identify the substantial derivative operator applied to the velocity vector and the continuity equation.

$$\rho \underbrace{\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right]}_{=\frac{D\mathbf{v}}{Dt}} + \mathbf{v} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}\right]}_{=0} + \nabla p = \rho \mathbf{f}$$

which gives us the non-conservation form of the momentum equation

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f} \tag{9}$$

Conservation of Energy

The last equation on non-conservation differential form is the energy equation. We start by rewriting the energy equation on differential form (Eqn. 6), repeated here for convenience

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \dot{q}\rho$$

Total enthalpy, h_o, is replaced with total energy, e_o

$$h_o = e_o + \frac{p}{\rho}$$

which gives

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho e_o \mathbf{v}) + \nabla \cdot (p\mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \dot{q}\rho$$

Expanding the two first terms as

$$\rho \frac{\partial e_o}{\partial t} + e_o \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla e_o + e_o \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (p\mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \dot{q}\rho$$

Collecting terms, we can identify the substantial derivative operator applied on total energy, De_o/Dt and the continuity equation

$$\rho \underbrace{\left[\frac{\partial e_o}{\partial t} \mathbf{v} \cdot \nabla e_o\right]}_{=\frac{De_o}{Dt}} + e_o \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right]}_{=0} + \nabla \cdot (p\mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \dot{q}\rho$$

and thus we end up with the energy equation on non-conservation differential form

$$\rho \frac{De_o}{Dt} + \nabla \cdot (p\mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \dot{q}\rho \tag{10}$$