Compressible Inviscid Fluid Flow Equations

Alternative Forms of the Energy Equation

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The Governing Equations on Differential Non-Conservation Form

Continuity:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \tag{1}$$

Momentum:

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f} \tag{2}$$

Energy:

$$\rho\frac{De_o}{Dt} + \nabla \cdot (p\mathbf{v}) = \rho\mathbf{f} \cdot \mathbf{v} + \dot{q}\rho \tag{3}$$

Internal Energy Formulation

Total internal energy is defined as

$$e_o = e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}$$

Inserted in Eqn. 3, this gives

$$\rho\frac{De}{Dt} + \rho\mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} + \nabla \cdot (p\mathbf{v}) = \rho\mathbf{f} \cdot \mathbf{v} + \dot{q}\rho$$

Now, let's replace the substantial derivative $D\mathbf{v}/Dt$ using the momentum equation on non-conservation form (Eqn. 2).

$$\rho\frac{De}{Dt} - \mathbf{v}\cdot\nabla p + \cancel{\rho\mathbf{f}\cdot\mathbf{v}} + \nabla\cdot(p\mathbf{v}) = \cancel{\rho\mathbf{f}\cdot\mathbf{v}} + \dot{q}\rho$$

Now, expand the term $\nabla\cdot(p\mathbf{v})$ gives

$$\rho\frac{De}{Dt}\cancel{-\mathbf{v}\cdot\nabla p} + \cancel{\mathbf{v}\cdot\nabla p} + p(\nabla\cdot\mathbf{v}) = \dot{q}\rho \Rightarrow \rho\frac{De}{Dt} + p(\nabla\cdot\mathbf{v}) = \dot{q}\rho$$

Divide by ρ

$$\frac{De}{Dt} + \frac{p}{\rho}(\nabla\cdot\mathbf{v}) = \dot{q} \tag{4}$$

Conservation of mass gives

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot\mathbf{v}) = 0 \Rightarrow \nabla\cdot\mathbf{v} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

Insert in Eqn. 5

$$\frac{De}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} = \dot{q} \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$\frac{De}{Dt} + p\frac{D\nu}{Dt} = \dot{q} \tag{5}$$

Compare with the first law of thermodynamics: $de = \delta q - \delta w$

Enthalpy Formulation

$$h = e + \frac{p}{\rho} \Rightarrow \frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

with De/Dt from Eqn. 5

$$\frac{Dh}{Dt} = \dot{q} - \cancel{p\frac{D}{Dt}\left(\frac{1}{\rho}\right)} + \frac{1}{\rho}\frac{Dp}{Dt} + \cancel{p\frac{D}{Dt}\left(\frac{1}{\rho}\right)}$$

$$\frac{Dh}{Dt} = \dot{q} + \frac{1}{\rho}\frac{Dp}{Dt} \tag{6}$$

Total Enthalpy Formulation

$$h_o = h + \frac{1}{2}\mathbf{v}\mathbf{v} \Rightarrow \frac{Dh_o}{Dt} = \frac{Dh}{Dt} + \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt}$$

From the momentum equation (Eqn. 2)

$$\frac{D\mathbf{v}}{Dt} = \mathbf{f} - \frac{1}{\rho}\nabla p$$

which gives

$$\frac{Dh_o}{Dt} = \frac{Dh}{Dt} + \mathbf{v} \cdot \mathbf{f} - \frac{1}{\rho}\mathbf{v} \cdot \nabla p$$

Inserting Dh/Dt from Eqn. 6 gives

$$\frac{Dh_o}{Dt} = \dot{q} + \frac{1}{\rho}\frac{Dp}{Dt} + \mathbf{v} \cdot \mathbf{f} - \frac{1}{\rho}\mathbf{v} \cdot \nabla p = \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v} \cdot \nabla p\right] + \dot{q} + \mathbf{v} \cdot \mathbf{f}$$

The substantial derivative operator applied to pressure

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p$$

and thus

$$\frac{Dp}{Dt} - \mathbf{v} \cdot \nabla p = \frac{\partial p}{\partial t}$$

which gives

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{v} \cdot \mathbf{f}$$

If we assume adiabatic flow without body forces

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

If we further assume the flow to be steady state we get

$$\frac{Dh_o}{Dt} = 0$$

This means that in a steady-state adiabatic flow without body forces, total enthalpy is constant along a streamline.