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Transport Equations in Incompressible URANS and LES

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1 The Transport Equation for the Reynolds Stresses

The filtered Navier-Stokes equation for \bar{u}_i reads

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_k} (\bar{u}_i \bar{u}_k) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} - \frac{\partial \tau_{ik}^a}{\partial x_k} - g_i \bar{t} \quad (1)$$

$$\tau_{ik} = \overline{u_i u_k} - \bar{u}_i \bar{u}_k, \quad \tau_{ik}^a = \tau_{ik} - \frac{1}{3} \delta_{ik} \tau_{jj}$$

Where τ_{ik} denotes modelled SGS stress or URANS stress. The SGS/URANS turbulent kinetic energy is defined as $k_T = 0.5 \tau_{ii}$. Decompose \bar{u}_i and \bar{p} into a time-averaged (or ensemble-averaged) value and a resolved fluctuation as

$$\begin{aligned} \bar{u}_i &= U_i + u'_i, \quad \bar{p} = P + p', \quad \bar{t} = T + t' \\ U_i &= \langle \bar{u}_i \rangle, \quad P = \langle \bar{p} \rangle, \quad T = \langle \bar{t} \rangle \end{aligned} \quad (2)$$

and insert this in Eq. 1 so that

$$\begin{aligned} \frac{\partial u'_i}{\partial t} + \frac{\partial}{\partial x_k} ((U_i + u'_i)(U_k + u'_k)) &= -\frac{1}{\rho} \frac{\partial (P + p')}{\partial x_i} + \nu \frac{\partial^2 (U_i + u'_i)}{\partial x_k \partial x_k} \\ &\quad - \frac{\partial \tau_{ik}^a}{\partial x_k} - g_i \beta (T + t') \end{aligned} \quad (3)$$

Time (ensemble) averaging of Eq. 3 yields

$$\frac{\partial}{\partial x_k} (U_i U_k) = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k} (\langle u'_i u'_k \rangle + \langle \tau_{ik}^a \rangle) - g_i \beta T \quad (4)$$

Now subtract Eq. 4 from Eq. 3

$$\begin{aligned} \frac{\partial u'_i}{\partial t} + \frac{\partial}{\partial x_k} (U_i u'_k + U_k u'_i + u'_i u'_k) &= \\ -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k} \left(\langle u'_i u'_k \rangle + \underbrace{\langle \tau^a \rangle_{ik} - \tau_{ik}^a}_{-\tau_{ik}^{t'a}} \right) &- g_i \beta t' \end{aligned} \quad (5)$$

Multiply Eq. 5 with u'_j and a corresponding equation for u'_j by u'_i , add them together, and time (ensemble) average

$$\begin{aligned} & \left\langle u'_j \frac{\partial}{\partial x_k} (U_i u'_k + U_k u'_i + u'_i u'_k) \right\rangle + \left\langle u'_i \frac{\partial}{\partial x_k} (U_j u'_k + U_k u'_j + u'_k u'_j) \right\rangle = \\ & - \left\langle \frac{u'_j}{\rho} \frac{\partial p'}{\partial x_i} \right\rangle - \left\langle \frac{u'_i}{\rho} \frac{\partial p'}{\partial x_j} \right\rangle + \nu \left\langle u'_j \frac{\partial^2 u'_i}{\partial x_k \partial x_k} \right\rangle + \nu \left\langle u'_i \frac{\partial^2 u'_j}{\partial x_k \partial x_k} \right\rangle \\ & - \left\langle u'_j \frac{\partial \tau'_{ik}}{\partial x_k} \right\rangle - \left\langle u'_i \frac{\partial \tau'_{jk}}{\partial x_k} \right\rangle - g_i \beta \langle u'_j t' \rangle - g_j \beta \langle u'_i t' \rangle \end{aligned} \quad (6)$$

The two first lines correspond to the usual $\overline{u_i u_j}$ equation in conventional Reynolds decomposition. The two last terms on line 2 can be re-written as

$$\begin{aligned} & \nu \frac{\partial}{\partial x_k} \left\langle u'_j \frac{\partial u'_i}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle u'_i \frac{\partial u'_j}{\partial x_k} \right\rangle - 2\nu \left\langle \frac{\partial u'_j}{\partial x_k} \frac{\partial u'_i}{\partial x_k} \right\rangle \\ & = \nu \frac{\partial^2}{\partial x_k \partial x_k} \langle u'_i u'_j \rangle - 2\nu \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \right\rangle \end{aligned} \quad (7)$$

The two first terms on the last line in Eq. 6 can be rewritten as

$$- \left\langle \frac{\partial}{\partial x_k} (u'_j \tau'_{ik}) \right\rangle + \left\langle \tau'_{ik} \frac{\partial u'_j}{\partial x_k} \right\rangle - \left\langle \frac{\partial}{\partial x_k} (u'_i \tau'_{jk}) \right\rangle + \left\langle \tau'_{jk} \frac{\partial u'_i}{\partial x_k} \right\rangle \quad (8)$$

Finally, we can now write the transport equation for $\langle u'_i u'_j \rangle$ as

$$\begin{aligned} & \frac{\partial}{\partial x_k} (U_k \langle u'_i u'_j \rangle) = - \langle u'_i u'_k \rangle \frac{\partial U_j}{\partial x_k} - \langle u'_j u'_k \rangle \frac{\partial U_i}{\partial x_k} - \frac{1}{\rho} \left\langle u'_i \frac{\partial p'}{\partial x_j} \right\rangle - \frac{1}{\rho} \left\langle u'_j \frac{\partial p'}{\partial x_i} \right\rangle + \\ & - \frac{\partial}{\partial x_k} \langle u'_i u'_j u'_k \rangle + \nu \frac{\partial^2}{\partial x_k \partial x_k} \langle u'_i u'_j \rangle - 2\nu \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \right\rangle - g_i \beta \langle u'_j t' \rangle - g_j \beta \langle u'_i t' \rangle \\ & - \left\langle \frac{\partial}{\partial x_k} (u'_j \tau'_{ik}) \right\rangle - \left\langle \frac{\partial}{\partial x_k} (u'_i \tau'_{jk}) \right\rangle \\ & + \left\langle \tau'_{ik} \frac{\partial u'_j}{\partial x_k} \right\rangle + \left\langle \tau'_{jk} \frac{\partial u'_i}{\partial x_k} \right\rangle \end{aligned} \quad (9)$$

where the two last lines include all terms related to the SGS/URANS stresses. The third line represents diffusion transport by SGS/URANS stresses and the fourth line represents dissipation by SGS/URANS stresses. For an eddy-viscosity SGS/URANS model

$$\tau_{ij}^a = -2\nu_T \bar{s}_{ij}, \quad \bar{s}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad (10)$$

1.1 Resolved Turbulent Kinetic Energy $\langle k \rangle$

Now we will derive the transport equation for the resolved turbulent kinetic energy $\langle k \rangle = \langle u'_i u'_i \rangle / 2$. Take the trace of Eq. 9 and divide by two

$$\begin{aligned} \frac{\partial}{\partial x_j} (U_j \langle k \rangle) &= -\langle u'_i u'_j \rangle \frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\frac{1}{\rho} \langle u'_j p' \rangle + \frac{1}{2} \langle u'_i u'_i u'_j \rangle \right) + \nu \frac{\partial^2 \langle k \rangle}{\partial x_j \partial x_j} \\ &\quad - \nu \left\langle \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \right\rangle - g_j \beta \langle u'_j t' \rangle - \left\langle \frac{\partial}{\partial x_j} (u'_i \tau_{ij}^a) \right\rangle + \left\langle \tau_{ij}^a \frac{\partial u'_i}{\partial x_j} \right\rangle \end{aligned} \quad (11)$$

The pressure-velocity term was re-written as

$$\left\langle u'_j \frac{\partial p'}{\partial x_j} \right\rangle = \frac{\partial}{\partial x_j} \langle u'_j p' \rangle - \left\langle p' \frac{\partial u'_j}{\partial x_j} \right\rangle \quad (12)$$

where the last term is zero due to continuity.

The last term in Eq. 11 can be both positive and negative. However, if we introduce an eddy-viscosity model it can be shown that it is predominantly negative. If the approximation (using Eq. 10)

$$\tau_{ij}^a = \tau_{ij}^a - \langle \tau_{ij}^a \rangle = -2 (\nu_T \bar{s}_{ij} - \langle \nu_T \bar{s}_{ij} \rangle) \simeq -2 \nu_T s'_{ij} \quad (13)$$

is made we find that the term is always negative. This is easily seen when inserting Eq. 13 into the last term of Eq. 11

$$\left\langle \tau_{ij}^a \frac{\partial u'_i}{\partial x_j} \right\rangle \simeq -2 \langle \nu_T s'_{ij} (s'_{ij} + \omega'_{ij}) \rangle = -2 \langle \nu_T s'_{ij} s'_{ij} \rangle < 0 \quad (14)$$

where $\omega'_{ij} = 0.5(\partial u'_i / \partial x_j - \partial u'_j / \partial x_i)$. In Eq. 14 we have used the fact that the product of a symmetric and anti-symmetric tensor is zero.

The terms in Eq. 11 have the following physical meaning. The term on the left-hand side is the advection. The terms on the right-hand side are production of $\langle k \rangle$, transport of $\langle k \rangle$ by resolved fluctuations, viscous transport of $\langle k \rangle$, viscous dissipation of $\langle k \rangle$, production/destruction of $\langle k \rangle$ by buoyancy, transport of $\langle k \rangle$ by SGS/URANS turbulence and production/destruction of $\langle k \rangle$ by SGS/URANS turbulence.

1.2 Modelled Turbulent Kinetic Energy k_T

The equation for the modelled turbulent SGS/RANS kinetic energy reads

$$\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_j k_T) = \frac{\partial}{\partial x_j} \left[(\nu + \nu_T) \frac{\partial k_T}{\partial x_j} \right] + 2\nu_T \bar{s}_{ij} \bar{s}_{ij} - \varepsilon \quad (15)$$

The terms on the right-hand side represent viscous and turbulent diffusion, production and viscous dissipation.

1.3 Mean Kinetic Energy K

The equation for the kinetic energy $K = \frac{1}{2}U_iU_i$ is derived by multiplying the time-averaged (ensemble-averaged) momentum equation, Eq. 4, by U_i so that

$$U_i \frac{\partial}{\partial x_j} (U_i U_j) = -\frac{1}{\rho} U_i \frac{\partial P}{\partial x_i} + \nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} - U_i \frac{\partial}{\partial x_j} (\langle \tau_{ij}^a \rangle + \langle u'_i u'_j \rangle) - U_i g_i \beta T \quad (16)$$

The left-hand side of Eq. 16 can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial x_j} (U_i U_i U_j) - U_i U_j \frac{\partial U_i}{\partial x_j} &= U_j \frac{\partial}{\partial x_j} (U_i U_i) - \frac{1}{2} U_j \frac{\partial}{\partial x_j} (U_i U_i) \\ &= \frac{1}{2} U_j \frac{\partial}{\partial x_j} (U_i U_i) = \frac{\partial}{\partial x_j} (U_j K) \end{aligned} \quad (17)$$

The first term on the right-hand side of Eq. 16 can be written as

$$-U_i \frac{\partial P}{\partial x_i} = -\frac{\partial}{\partial x_i} (U_i P). \quad (18)$$

The viscous term in Eq. 16 is rewritten in the same way as the viscous term in Eq. 7, i.e.

$$\nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} = \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \nu \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \quad (19)$$

The turbulent term is rewritten as

$$U_i \frac{\partial}{\partial x_j} (\langle \tau_{ij}^a \rangle + \langle u'_i u'_j \rangle) = \frac{\partial}{\partial x_j} [U_i (\langle \tau_{ij}^a \rangle + \langle u'_i u'_j \rangle)] - (\langle \tau_{ij}^a \rangle + \langle u'_i u'_j \rangle) \frac{\partial U_i}{\partial x_j} \quad (20)$$

Now we can assemble the transport equation for K by inserting Eqs. 17, 18, 19 and Eq. 20 into Eq. 16

$$\begin{aligned} \frac{\partial}{\partial x_j} (U_j K) &= \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (U_j P) - \frac{\partial}{\partial x_j} [U_i (\langle \tau_{ij}^a \rangle + \langle u'_i u'_j \rangle)] \\ &\quad + (\langle \tau_{ij}^a \rangle + \langle u'_i u'_j \rangle) \frac{\partial U_i}{\partial x_j} - \nu \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} - g_i \beta U_i T \end{aligned} \quad (21)$$

We recognize the usual transport term on the left-hand side due to advection. On the right-hand side we have viscous diffusion, transport of K by time-averaged (ensemble-averaged) pressure-velocity interaction. The term in square brackets represents transport by interaction between the time-averaged (ensemble-averaged) velocity field and turbulence. The term $\langle u'_i u'_j \rangle \partial U_i / \partial x_j$ is the usual production term of the resolved kinetic energy $0.5 \langle u'_i u'_i \rangle$ which usually is negative. This term appears in Eq. 11 but with opposite sign. The term $\langle \tau_{ij}^a \rangle \partial U_i / \partial x_j$ is the production term in the turbulent kinetic energy equation $k_T = 0.5 \tau_{ii}$. This term is usually referred to as

the SGS/URANS dissipation term, and for an eddy-viscosity model we find (cf. Eqs. 13 and 14)

$$\begin{aligned} \langle \tau_{ij}^a \rangle \frac{\partial U_i}{\partial x_j} &= -2 \langle \nu_T \bar{s}_{ij} \rangle (S_{ij} + \Omega_{ij}) \\ &\simeq -2 \langle \nu_T \rangle \langle \bar{s}_{ij} \rangle S_{ij} = -2 \langle \nu_T \rangle S_{ij} S_{ij} < 0 \end{aligned} \quad (22)$$

It is interesting to compare this SGS dissipation term with the viscous dissipation term in Eq. 20. If $\nu_{sgs} \gg \nu$, the SGS dissipation is much larger than the viscous one. If this is not the case, then we're doing a DNS!

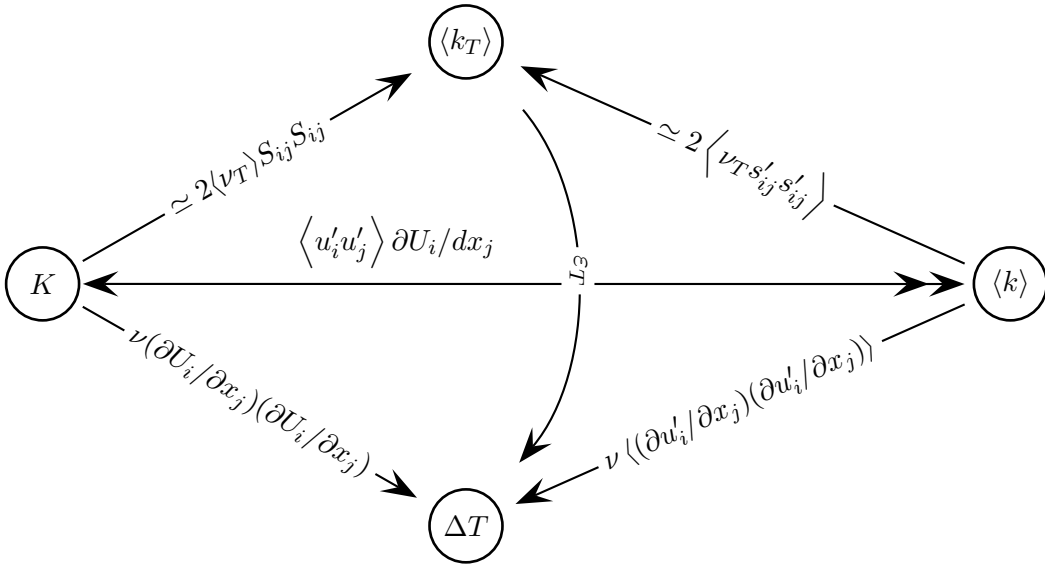


Figure 1: Transfer of kinetic turbulent energy. Often the energy is transferred in both directions. A double arrow indicated in which direction the net energy is transferred.

2 The Transport Equation for the Heat Fluxes

The filtered temperature equation for \bar{t} reads

$$\begin{aligned} \frac{\partial \bar{t}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{u}_k \bar{t}) &= \frac{\nu}{Pr} \frac{\partial^2 \bar{t}}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k} \\ h_k &= \overline{u_k t} - \bar{u}_k \bar{t} \end{aligned} \quad (23)$$

Use Eq. 2 in Eq. 23 so that

$$\frac{\partial}{\partial t} (T + t'_i) + \frac{\partial}{\partial x_k} ((U_k + u'_k)(T + t')) = \frac{\nu}{Pr} \frac{\partial^2 (T + t')}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k} \quad (24)$$

Time (ensemble) averaging of Eq. 24 yields

$$\frac{\partial}{\partial x_k} (U_k T) = \frac{\nu}{Pr} \frac{\partial^2 T}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k} (\langle u'_k t' \rangle + \langle h_k \rangle) \quad (25)$$

Now subtract Eq. 25 from Eq. 24

$$\begin{aligned} \frac{\partial t'}{\partial t} + \frac{\partial}{\partial x_k} (u'_k T + U_k t' + u'_k t') = \\ \frac{\nu}{Pr} \frac{\partial^2 t'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k} \left(\langle u'_k t' \rangle + \underbrace{\langle h \rangle_k - h_k}_{-h'_k} \right) \end{aligned} \quad (26)$$

Multiply Eq. 26 with u'_i and multiply Eq. 5 with t' , add them together and time (ensemble) average

$$\begin{aligned} & \left\langle u'_i \frac{\partial}{\partial x_k} (u'_k T + U_k t' + u'_k t') + t' \frac{\partial}{\partial x_k} (U_i u'_k + U_k u'_i + u'_i u'_k) \right\rangle \\ = & - \left\langle \frac{t'}{\rho} \frac{\partial p'}{\partial x_i} \right\rangle + \frac{\nu}{Pr} \left\langle u'_i \frac{\partial^2 t'}{\partial x_k \partial x_k} \right\rangle + \nu \left\langle t' \frac{\partial^2 u'_i}{\partial x_k \partial x_k} \right\rangle - g_i \beta \langle t' t' \rangle \\ & - \left\langle u'_i \frac{\partial h'_k}{\partial x_k} \right\rangle - \left\langle t' \frac{\partial \tau'_{ik}}{\partial x_k} \right\rangle \end{aligned} \quad (27)$$

The two first lines correspond to the conventional heat flux equation. The two terms in the middle on line 2 can be re-written as

$$\begin{aligned} \frac{\nu}{Pr} \frac{\partial}{\partial x_k} \left\langle u'_i \frac{\partial t'}{\partial x_k} \right\rangle - \frac{\nu}{Pr} \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle t' \frac{\partial u'_i}{\partial x_k} \right\rangle - \nu \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle = \\ \frac{\nu}{Pr} \frac{\partial}{\partial x_k} \left\langle u'_i \frac{\partial t'}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle t' \frac{\partial u'_i}{\partial x_k} \right\rangle - \left(\nu + \frac{\nu}{Pr} \right) \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle \end{aligned} \quad (28)$$

Using Eq. 28 in Eq. 27 and at the same time re-writing the SGS/URANS terms we get

$$\begin{aligned} & \frac{\partial}{\partial x_k} \langle U_k u'_i t' \rangle \\ = & - \langle u'_i u'_k \rangle \frac{\partial T}{\partial x_k} - \langle u'_k t' \rangle \frac{\partial U_i}{\partial x_k} - \left\langle \frac{t'}{\rho} \frac{\partial p'}{\partial x_i} \right\rangle - \frac{\partial}{\partial x_k} \langle u'_k u'_i t' \rangle \\ + & \frac{\nu}{Pr} \frac{\partial}{\partial x_k} \left\langle u'_i \frac{\partial t'}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle t' \frac{\partial u'_i}{\partial x_k} \right\rangle - \left(\nu + \frac{\nu}{Pr} \right) \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle - g_i \beta \langle t'^2 \rangle \\ & - \frac{\partial}{\partial x_k} \langle u'_i h'_k \rangle + \left\langle h'_k \frac{\partial u'_i}{\partial x_k} \right\rangle - \frac{\partial}{\partial x_k} \langle t' \tau'_{ik} \rangle + \left\langle \tau'_{ik} \frac{\partial t'}{\partial x_k} \right\rangle \end{aligned} \quad (29)$$

The SGS/URANS heat fluxes are commonly obtain from an eddy-viscosity model

$$h_i = - \frac{\nu_T}{Pr_T} \frac{\partial \bar{t}}{\partial x_i} \quad (30)$$

3 The Transport Equation for the Temperature Variance

Multiply Eq. 26 with t' and time (ensemble) average

$$\left\langle t' \frac{\partial}{\partial x_k} (u'_k T + U_k t' + u'_k t') \right\rangle = \frac{\nu}{Pr} \left\langle t' \frac{\partial^2 t'}{\partial x_k \partial x_k} \right\rangle - \left\langle t' \frac{\partial h'_k}{\partial x_k} \right\rangle \quad (31)$$

The first term on the right-hand side can be re-written as

$$\frac{\nu}{Pr} \left\langle \frac{\partial}{\partial x_k} \left(t' \frac{\partial t'}{\partial x_k} \right) \right\rangle - \frac{\nu}{Pr} \left\langle \frac{\partial t'}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle = \frac{1}{2} \frac{\nu}{Pr} \frac{\partial^2}{\partial x_k \partial x_k} \langle t'^2 \rangle - \frac{\nu}{Pr} \left\langle \frac{\partial t'}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle \quad (32)$$

Using Eq. 32 and re-writing the SGS/URANS term, Eq. 31 can now be written as

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial x_k} \langle U_k t'^2 \rangle &= - \langle u'_k t' \rangle \frac{\partial T}{\partial x_k} - \frac{1}{2} \frac{\partial}{\partial x_k} \langle u'_k t'^2 \rangle \\ + \frac{1}{2} \frac{\nu}{Pr} \frac{\partial^2}{\partial x_k \partial x_k} \langle t'^2 \rangle &- \frac{\nu}{Pr} \left\langle \frac{\partial t'}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle - \frac{\partial}{\partial x_k} \langle t' h'_k \rangle + \left\langle h'_k \frac{\partial t'}{\partial x_k} \right\rangle \end{aligned} \quad (33)$$

Multiply Eq. 31 by 2 and we get

$$\begin{aligned} \frac{\partial}{\partial x_k} \langle U_k t'^2 \rangle &= -2 \langle u'_k t' \rangle \frac{\partial T}{\partial x_k} - \frac{\partial}{\partial x_k} \langle u'_k t'^2 \rangle \\ + \frac{\nu}{Pr} \frac{\partial^2}{\partial x_k \partial x_k} \langle t'^2 \rangle &- 2 \frac{\nu}{Pr} \left\langle \frac{\partial t'}{\partial x_k} \frac{\partial t'}{\partial x_k} \right\rangle - 2 \frac{\partial}{\partial x_k} \langle t' h'_k \rangle + 2 \left\langle h'_k \frac{\partial t'}{\partial x_k} \right\rangle \end{aligned} \quad (34)$$