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1 The Transport Equation for the Reynolds Stresses

The filtered Navier-Stokes equation for $\bar{u}_i$ reads

$$\begin{aligned}\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}_i \bar{u}_k) &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} - \frac{\partial \tau^a_{ik}}{\partial x_k} - g_i \beta \bar{t} \\ \tau_{ik} = \overline{u_i u_k} - \bar{u}_i \bar{u}_k,\ &\tau^a_{ik} = \tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{jj}\end{aligned} \tag{1}$$

Where τ_{ik} denotes modelled SGS stress or URANS stress. The SGS/URANS turbulent kinetic energy is defined as $k_T = 0.5\tau_{ii}$. Decompose $\bar{u}_i$ and $\bar{p}$ into a time-averaged (or ensemble-averaged) value and a resolved fluctuation as

$$\begin{aligned}&\bar{u}_i = U_i + \bar{u}'_i,\ \bar{p} = P + \bar{p}',\ \bar{t} = T + \bar{t}' \\ &U_i = \langle \bar{u}_i \rangle,\ P = \langle \bar{p} \rangle,\ T = \langle \bar{t} \rangle\end{aligned} \tag{2}$$

and insert this in Eq. 1 so that

$$\begin{aligned}\frac{\partial \bar{u}'_i}{\partial t} + \frac{\partial}{\partial x_k}((U_i + \bar{u}'_i)(U_k + \bar{u}'_k)) = -\frac{1}{\rho}\frac{\partial (P + \bar{p}')}{\partial x_i} + \nu \frac{\partial^2 (U_i + \bar{u}'_i)}{\partial x_k \partial x_k} \\ -\frac{\partial \tau^a_{ik}}{\partial x_k} - g_i \beta (T + \bar{t}')\end{aligned} \tag{3}$$

Time (ensemble) averaging of Eq. 3 yields

$$\frac{\partial}{\partial x_k}(U_i U_k) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\left(\langle \bar{u}'_i \bar{u}'_k \rangle + \langle \tau^a_{ik} \rangle\right) - g_i \beta T \tag{4}$$

Now subtract Eq. 4 from Eq. 3

$$\begin{aligned}\frac{\partial \bar{u}'_i}{\partial t} + \frac{\partial}{\partial x_k}(U_i \bar{u}'_k + U_k \bar{u}'_i + \bar{u}'_i \bar{u}'_k) = \\ -\frac{1}{\rho}\frac{\partial \bar{p}'}{\partial x_i} + \nu \frac{\partial^2 \bar{u}'_i}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\left(\langle \bar{u}'_i \bar{u}'_k \rangle + \underbrace{\langle \tau^a \rangle_{ik} - \tau^a_{ik}}_{-\tau'^a_{ik}}\right) - g_i \beta \bar{t}'\end{aligned} \tag{5}$$

Multiply Eq. 5 with $\bar{u}'_j$ and a corresponding equation for $\bar{u}'_j$ by $\bar{u}'_i$, add them together, and time (ensemble) average

$$
\begin{aligned}
\left\langle \bar{u}'_j \frac{\partial}{\partial x_k}(U_i \bar{u}'_k + U_k \bar{u}'_i + \bar{u}'_i \bar{u}'_k) \right\rangle + \left\langle \bar{u}'_i \frac{\partial}{\partial x_k}(U_j \bar{u}'_k + U_k \bar{u}'_j + \bar{u}'_k \bar{u}'_j) \right\rangle = \\
- \left\langle \frac{\bar{u}'_j}{\rho} \frac{\partial \bar{p}'}{\partial x_i} \right\rangle - \left\langle \frac{\bar{u}'_i}{\rho} \frac{\partial \bar{p}'}{\partial x_j} \right\rangle + \nu \left\langle \bar{u}'_j \frac{\partial^2 \bar{u}'_i}{\partial x_k \partial x_k} \right\rangle + \nu \left\langle \bar{u}'_i \frac{\partial^2 \bar{u}'_j}{\partial x_k \partial x_k} \right\rangle \\
- \left\langle \bar{u}'_j \frac{\partial \tau'^a_{ik}}{\partial x_k} \right\rangle - \left\langle \bar{u}'_i \frac{\partial \tau'^a_{jk}}{\partial x_k} \right\rangle - g_i \beta \left\langle \bar{u}'_j \bar{t}' \right\rangle - g_j \beta \left\langle \bar{u}'_i \bar{t}' \right\rangle
\end{aligned}
\tag{6}
$$

The two first lines correspond to the usual $\overline{u_i u_j}$ equation in conventional Reynolds decomposition. The two last terms on line 2 can be re-written as

$$
\begin{aligned}
\nu \frac{\partial}{\partial x_k} \left\langle \bar{u}'_j \frac{\partial \bar{u}'_i}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle \bar{u}'_i \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle - 2\nu \left\langle \frac{\partial \bar{u}'_j}{\partial x_k} \frac{\partial \bar{u}'_i}{\partial x_k} \right\rangle \\
= \nu \frac{\partial^2}{\partial x_k \partial x_k} \left\langle \bar{u}'_i \bar{u}'_j \right\rangle - 2\nu \left\langle \frac{\partial \bar{u}'_i}{\partial x_k} \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle
\end{aligned}
\tag{7}
$$

The two first terms on the last line in Eq. 6 can be rewritten as

$$
- \left\langle \frac{\partial}{\partial x_k} (\bar{u}'_j \tau'^a_{ik}) \right\rangle + \left\langle \tau'^a_{ik} \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle - \left\langle \frac{\partial}{\partial x_k} (\bar{u}'_i \tau'^a_{jk}) \right\rangle + \left\langle \tau'^a_{jk} \frac{\partial \bar{u}'_i}{\partial x_k} \right\rangle \tag{8}
$$

Finally, we can now write the transport equation for $\langle \bar{u}'_i \bar{u}'_j \rangle$ as

$$
\begin{aligned}
\frac{\partial}{\partial x_k}(U_k \langle \bar{u}'_i \bar{u}'_j \rangle) = -\langle \bar{u}'_i \bar{u}'_k \rangle \frac{\partial U_j}{\partial x_k} - \langle \bar{u}'_j \bar{u}'_k \rangle \frac{\partial U_i}{\partial x_k} - \frac{1}{\rho} \left\langle \bar{u}'_i \frac{\partial \bar{p}'}{\partial x_j} \right\rangle - \frac{1}{\rho} \left\langle \bar{u}'_j \frac{\partial \bar{p}'}{\partial x_i} \right\rangle + \\
- \frac{\partial}{\partial x_k} \langle \bar{u}'_i \bar{u}'_j \bar{u}'_k \rangle + \nu \frac{\partial^2}{\partial x_k \partial x_k} \langle \bar{u}'_i \bar{u}'_j \rangle - 2\nu \left\langle \frac{\partial \bar{u}'_i}{\partial x_k} \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle - g_i \beta \left\langle \bar{u}'_j \bar{t}' \right\rangle - g_j \beta \left\langle \bar{u}'_i \bar{t}' \right\rangle \\
- \left\langle \frac{\partial}{\partial x_k} (\bar{u}'_j \tau'^a_{ik}) \right\rangle - \left\langle \frac{\partial}{\partial x_k} (\bar{u}'_i \tau'^a_{jk}) \right\rangle \\
+ \left\langle \tau'^a_{ik} \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle + \left\langle \tau'^a_{jk} \frac{\partial \bar{u}'_i}{\partial x_k} \right\rangle
\end{aligned}
\tag{9}
$$

where the two last lines include all terms related to the SGS/URANS stresses. The third line represents diffusion transport by SGS/URANS stresses and the fourth line represents dissipation by SGS/URANS stresses. For an eddy-viscosity SGS/URANS model

$$
\tau^a_{ij} = -2\nu_T \bar{s}_{ij}, \; \bar{s}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \tag{10}
$$

1.1 Resolved Turbulent Kinetic Energy $\langle k \rangle$

Now we will derive the transport equation for the resolved turbulent kinetic energy $\langle k \rangle = \langle \bar{u}'_i \bar{u}'_i \rangle / 2$. Take the trace of Eq. 9 and divide by two

$$\frac{\partial}{\partial x_j}(U_j \langle k \rangle) = -\langle \bar{u}'_i \bar{u}'_j \rangle \frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\left\langle \bar{u}'_j \bar{p}' \right\rangle + \frac{1}{2}\langle \bar{u}'_i \bar{u}'_i \bar{u}'_j \rangle\right) + \nu \frac{\partial^2 \langle k \rangle}{\partial x_j \partial x_j}$$
$$-\nu \left\langle \frac{\partial \bar{u}'_i}{\partial x_j} \frac{\partial \bar{u}'_i}{\partial x_j} \right\rangle - g_j \beta \left\langle \bar{u}'_j \bar{t}' \right\rangle - \left\langle \frac{\partial}{\partial x_j}(\bar{u}'_i \tau'^a_{ij}) \right\rangle + \left\langle \tau'^a_{ij} \frac{\partial \bar{u}'_i}{\partial x_j} \right\rangle \tag{11}$$

The pressure-velocity term was re-written as

$$\left\langle \bar{u}'_j \frac{\partial \bar{p}'}{\partial x_j} \right\rangle = \frac{\partial}{\partial x_j}\left\langle \bar{u}'_j \bar{p}' \right\rangle - \left\langle \bar{p}' \frac{\partial \bar{u}'_j}{\partial x_j} \right\rangle \tag{12}$$

where the last term is zero due to continuity.

The last term in Eq. 11 can be both positive and negative. However, if we introduce an eddy-viscosity model it can be shown that it is predominantly negative. If the approximation (using Eq. 10)

$$\tau'^a_{ij} = \tau^a_{ij} - \langle \tau^a_{ij} \rangle = -2\left(\nu_T \bar{s}_{ij} - \langle \nu_T \bar{s}_{ij} \rangle\right) \simeq -2\nu_T s'_{ij} \tag{13}$$

is made we find that the term is always negative. This is easily seen when inserting Eq. 13 into the last term of Eq. 11

$$\left\langle \tau'^a_{ij} \frac{\partial \bar{u}'_i}{\partial x_j} \right\rangle \simeq -2\left\langle \nu_T s'_{ij}(s'_{ij} + \omega'_{ij}) \right\rangle = -2\left\langle \nu_T s'_{ij} s'_{ij} \right\rangle < 0 \tag{14}$$

where $\omega'_{ij} = 0.5(\partial \bar{u}'_i / dx_j - \partial \bar{u}'_j / dx_i)$. In Eq. 14 we have used the fact that the product of a symmetric and anti-symmetric tensor is zero.

The terms in Eq. 11 have the following physical meaning. The term on the left-hand side is the advection. The terms on the right-hand side are production of $\langle k \rangle$, transport of $\langle k \rangle$ by resolved fluctuations, viscous transport of $\langle k \rangle$, viscous dissipation of $\langle k \rangle$, production/destruction of $\langle k \rangle$ by buoyancy, transport of $\langle k \rangle$ by SGS/URANS turbulence and production/destruction of $\langle k \rangle$ by SGS/URANS turbulence.

1.2 Modelled Turbulent Kinetic Energy k_T

The equation for the modelled turbulent SGS/RANS kinetic energy reads

$$\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j k_T) = \frac{\partial}{\partial x_j}\left[(\nu + \nu_T)\frac{\partial k_T}{\partial x_j}\right] + 2\nu_T \bar{s}_{ij}\bar{s}_{ij} - \varepsilon \tag{15}$$

The terms on the right-hand side represent viscous and turbulent diffusion, production and viscous dissipation.

1.3 Mean Kinetic Energy K

The equation for the kinetic energy $K = \frac{1}{2}U_i U_i$ is derived by multiplying the time-averaged (ensemble-averaged) momentum equation, Eq. 4, by U_i so that

$$U_i \frac{\partial}{\partial x_j}(U_i U_j) = -\frac{1}{\rho} U_i \frac{\partial P}{\partial x_i} + \nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} - U_i \frac{\partial}{d x_j}\left(\langle \tau^a_{ij}\rangle + \langle \bar{u}'_i \bar{u}'_j\rangle\right) - U_i g_i \beta T \tag{16}$$

The left-hand side of Eq. 16 can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial x_j}(U_i U_i U_j) - U_i U_j \frac{\partial U_i}{\partial x_j} &= U_j \frac{\partial}{\partial x_j}(U_i U_i) - \frac{1}{2} U_j \frac{\partial}{\partial x_j}(U_i U_i) \\ &= \frac{1}{2} U_j \frac{\partial}{\partial x_j}(U_i U_i) = \frac{\partial}{\partial x_j}(U_j K) \end{aligned} \tag{17}$$

The first term on the right-hand side of Eq. 16 can be written as

$$-U_i \frac{\partial P}{\partial x_i} = -\frac{\partial}{\partial x_i}(U_i P). \tag{18}$$

The viscous term in Eq. 16 is rewritten in the same way as the viscous term in Eq. 7, i.e.

$$\nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} = \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \nu \frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j} \tag{19}$$

The turbulent term is rewritten as

$$U_i \frac{\partial}{d x_j}\left(\langle \tau^a_{ij}\rangle + \langle \bar{u}'_i \bar{u}'_j\rangle\right) = \frac{\partial}{d x_j}\left[U_i\left(\langle \tau^a_{ij}\rangle + \langle \bar{u}'_i \bar{u}'_j\rangle\right)\right] - \left(\langle \tau^a_{ij}\rangle + \langle \bar{u}'_i \bar{u}'_j\rangle\right)\frac{\partial U_i}{d x_j} \tag{20}$$

Now we can assemble the transport equation for K by inserting Eqs. 17, 18, 19 and Eq. 20 into Eq. 16

$$\begin{aligned} \frac{\partial}{\partial x_j}(U_j K) = \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(U_j P) - \frac{\partial}{d x_j}\left[U_i\left(\langle \tau^a_{ij}\rangle + \langle \bar{u}'_i \bar{u}'_j\rangle\right)\right] \\ + \left(\langle \tau^a_{ij}\rangle + \langle \bar{u}'_i \bar{u}'_j\rangle\right)\frac{\partial U_i}{d x_j} - \nu \frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j} - g_i \beta U_i T \end{aligned} \tag{21}$$

We recognize the usual transport term on the left-hand side due to advection. On the right-hand side we have viscous diffusion, transport of K by time-averaged (ensemble-averaged) pressure-velocity interaction. The term in square brackets represents transport by interaction between the time-averaged (ensemble-averaged) velocity field and turbulence. The term $\langle \bar{u}'_i \bar{u}'_j\rangle \partial U_i / d x_j$ is the usual production term of the resolved kinetic energy $0.5\langle \bar{u}'_i \bar{u}'_i\rangle$ which usually is negative. This term appears in Eq. 11 but with opposite sign. The term $\langle \tau^a_{ij}\rangle \partial U_i / d x_j$ is the production term in the turbulent kinetic energy equation $k_T = 0.5\tau_{ii}$. This term is usually referred to as the SGS/URANS dissipation term, and for an eddy-viscosity model we find (cf. Eqs. 13 and 14)

$$\begin{aligned}\langle \tau^a_{ij}\rangle \frac{\partial U_i}{dx_j} &= -2\langle \nu_T \bar{s}_{ij}\rangle \left(S_{ij} + \Omega_{ij}\right) \\ &\simeq -2\langle \nu_T\rangle\langle \bar{s}_{ij}\rangle S_{ij} = -2\langle \nu_T\rangle S_{ij} S_{ij} < 0\end{aligned} \tag{22}$$

It is interesting to compare this SGS dissipation term with the viscous dissipation term in Eq. 20. If $\nu_{sgs} \gg \nu$, the SGS dissipation is much larger than the viscous one. If this is not the case, then we're doing a DNS!

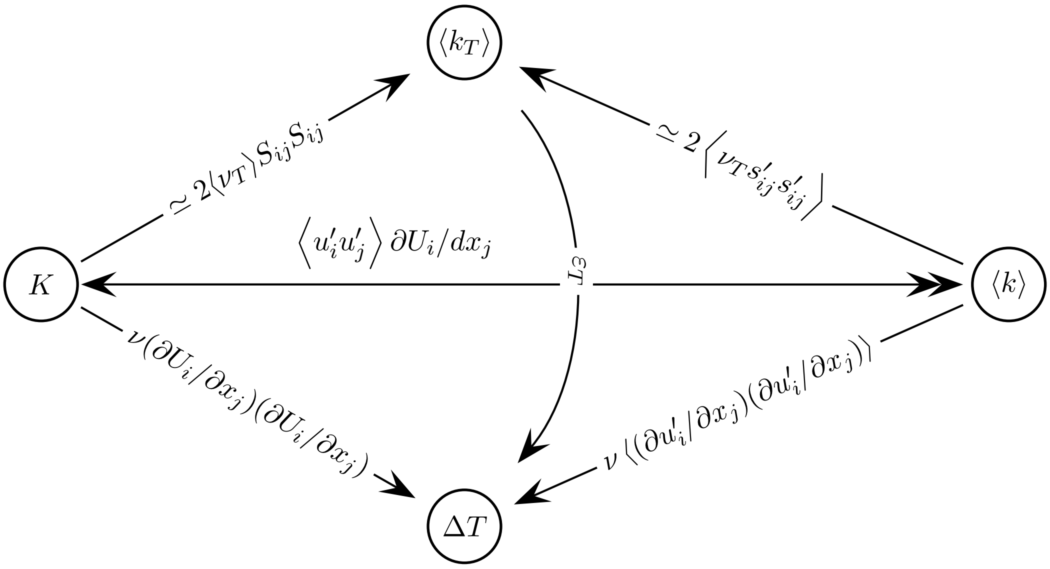

Figure 1: Transfer of kinetic turbulent energy. Often the energy is transferred in both directions. A double arrow indicated in which direction the net energy is transferred.

2 The Transport Equation for the Heat Fluxes

The filtered temperature equation for $\bar{t}$ reads

$$\begin{aligned}\frac{\partial \bar{t}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}_k \bar{t}) &= \frac{\nu}{Pr}\frac{\partial^2 \bar{t}}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k} \\ h_k &= \overline{u_k t} - \bar{u}_k t\end{aligned} \tag{23}$$

Use Eq. 2 in Eq. 23 so that

$$\frac{\partial}{\partial t}(T + \bar{t}'_i) + \frac{\partial}{\partial x_k}((U_k + \bar{u}'_k)(T + \bar{t}')) = \frac{\nu}{Pr}\frac{\partial^2 (T + \bar{t}')}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k} \tag{24}$$

Time (ensemble) averaging of Eq. 24 yields

$$\frac{\partial}{\partial x_k}(U_k T) = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\left(\langle \bar{u}'_k \bar{t}'\rangle + \langle h_k\rangle\right) \tag{25}$$

Now subtract Eq. 25 from Eq. 24

$$\frac{\partial \bar{t}'}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}'_k T + U_k \bar{t}' + \bar{u}'_k \bar{t}') =$$
$$\frac{\nu}{Pr}\frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\left(\langle \bar{u}'_k \bar{t}'\rangle + \underbrace{\langle h\rangle_k - h_k}_{-h'_k}\right) \tag{26}$$

Multiply Eq. 26 with $\bar{u}'_i$ and multiply Eq. 5 with $\bar{t}'$, add them together and time (ensemble) average

$$\left\langle \bar{u}'_i \frac{\partial}{\partial x_k}(\bar{u}'_k T + U_k \bar{t}' + \bar{u}'_k \bar{t}') + \bar{t}' \frac{\partial}{\partial x_k}(U_i \bar{u}'_k + U_k \bar{u}'_i + \bar{u}'_i \bar{u}'_k) \right\rangle$$
$$= -\left\langle \frac{\bar{t}'}{\rho}\frac{\partial \bar{p}'}{\partial x_i}\right\rangle + \frac{\nu}{Pr}\left\langle \bar{u}'_i \frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k}\right\rangle + \nu \left\langle \bar{t}' \frac{\partial^2 \bar{u}'_i}{\partial x_k \partial x_k}\right\rangle - g_i \beta \langle \bar{t}' \bar{t}'\rangle \tag{27}$$
$$- \left\langle \bar{u}'_i \frac{\partial h'_k}{\partial x_k}\right\rangle - \left\langle \bar{t}' \frac{\partial \tau'^a_{ik}}{\partial x_k}\right\rangle$$

The two first lines correspond to the conventional heat flux equation. The two terms in the middle on line 2 can be re-written as

$$\frac{\nu}{Pr}\frac{\partial}{\partial x_k}\left\langle \bar{u}'_i \frac{\partial \bar{t}'}{\partial x_k}\right\rangle - \frac{\nu}{Pr}\left\langle \frac{\partial \bar{u}'_i}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle + \nu \frac{\partial}{\partial x_k}\left\langle \bar{t}' \frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle - \nu \left\langle \frac{\partial \bar{u}'_i}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle =$$
$$\frac{\nu}{Pr}\frac{\partial}{\partial x_k}\left\langle \bar{u}'_i \frac{\partial \bar{t}'}{\partial x_k}\right\rangle + \nu \frac{\partial}{\partial x_k}\left\langle \bar{t}' \frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle - \left(\nu + \frac{\nu}{Pr}\right)\left\langle \frac{\partial \bar{u}'_i}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle \tag{28}$$

Using Eq. 28 in Eq. 27 and at the same time re-writing the SGS/URANS terms we get

$$\frac{\partial}{\partial x_k}\langle U_k \bar{u}'_i \bar{t}'\rangle$$
$$= -\langle \bar{u}'_i \bar{u}'_k\rangle \frac{\partial T}{\partial x_k} - \langle \bar{u}'_k \bar{t}'\rangle \frac{\partial U_i}{\partial x_k} - \left\langle \frac{\bar{t}'}{\rho}\frac{\partial \bar{p}'}{\partial x_i}\right\rangle - \frac{\partial}{\partial x_k}\langle \bar{u}'_k \bar{u}'_i \bar{t}'\rangle$$
$$+ \frac{\nu}{Pr}\frac{\partial}{\partial x_k}\left\langle \bar{u}'_i \frac{\partial \bar{t}'}{\partial x_k}\right\rangle + \nu \frac{\partial}{\partial x_k}\left\langle \bar{t}' \frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle - \left(\nu + \frac{\nu}{Pr}\right)\left\langle \frac{\partial \bar{u}'_i}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle - g_i \beta \langle t'^2\rangle$$
$$- \frac{\partial}{\partial x_k}\langle \bar{u}'_i h'_k\rangle + \left\langle h'_k \frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle - \frac{\partial}{\partial x_k}\langle \bar{t}' \tau'^a_{ik}\rangle + \left\langle \tau'^a_{ik}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle \tag{29}$$

The SGS/URANS heat fluxes are commonly obtain from an eddy-viscosity model

$$h_i = -\frac{\nu_T}{Pr_T}\frac{\partial \bar{t}}{\partial x_i} \tag{30}$$

3 The Transport Equation for the Temperature Variance

Multiply Eq. 26 with $\bar{t}'$ and time (ensemble) average

$$\left\langle \bar{t}' \frac{\partial}{\partial x_k}(\bar{u}'_k T + U_k \bar{t}' + \bar{u}'_k \bar{t}') \right\rangle = \frac{\nu}{Pr} \left\langle \bar{t}' \frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} \right\rangle - \left\langle \bar{t}' \frac{\partial h'_k}{\partial x_k} \right\rangle \tag{31}$$

The first term on the right-hand side can be re-written as

$$\frac{\nu}{Pr} \left\langle \frac{\partial}{\partial x_k} \left(\bar{t}' \frac{\partial \bar{t}'}{\partial x_k} \right) \right\rangle - \frac{\nu}{Pr} \left\langle \frac{\partial \bar{t}'}{\partial x_k} \frac{\partial \bar{t}'}{\partial x_k} \right\rangle = \frac{1}{2} \frac{\nu}{Pr} \frac{\partial^2}{\partial x_k \partial x_k} \left\langle t'^2 \right\rangle - \frac{\nu}{Pr} \left\langle \frac{\partial \bar{t}'}{\partial x_k} \frac{\partial \bar{t}'}{\partial x_k} \right\rangle \tag{32}$$

Using Eq. 32 and re-writing the SGS/URANS term, Eq. 31 can now be written as

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial x_k} \left\langle U_k t'^2 \right\rangle &= - \left\langle \bar{u}'_k \bar{t}' \right\rangle \frac{\partial T}{\partial x_k} - \frac{1}{2} \frac{\partial}{\partial x_k} \left\langle \bar{u}'_k t'^2 \right\rangle \\ &+ \frac{1}{2} \frac{\nu}{Pr} \frac{\partial^2}{\partial x_k \partial x_k} \left\langle t'^2 \right\rangle - \frac{\nu}{Pr} \left\langle \frac{\partial \bar{t}'}{\partial x_k} \frac{\partial \bar{t}'}{\partial x_k} \right\rangle - \frac{\partial}{\partial x_k} \left\langle \bar{t}' h'_k \right\rangle + \left\langle h'_k \frac{\partial \bar{t}'}{\partial x_k} \right\rangle \end{aligned} \tag{33}$$

Multiply Eq. 31 by 2 and we get

$$\begin{aligned} \frac{\partial}{\partial x_k} \left\langle U_k t'^2 \right\rangle &= -2 \left\langle \bar{u}'_k \bar{t}' \right\rangle \frac{\partial T}{\partial x_k} - \frac{\partial}{\partial x_k} \left\langle \bar{u}'_k t'^2 \right\rangle \\ &+ \frac{\nu}{Pr} \frac{\partial^2}{\partial x_k \partial x_k} \left\langle t'^2 \right\rangle - 2 \frac{\nu}{Pr} \left\langle \frac{\partial \bar{t}'}{\partial x_k} \frac{\partial \bar{t}'}{\partial x_k} \right\rangle - 2 \frac{\partial}{\partial x_k} \left\langle \bar{t}' h'_k \right\rangle + 2 \left\langle h'_k \frac{\partial \bar{t}'}{\partial x_k} \right\rangle \end{aligned} \tag{34}$$