Transport Equations in Incompressible URANS and LES

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1 The Transport Equation for the Reynolds Stresses

The filtered Navier-Stokes equation for $\bar{u}_i$ reads

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}_i \bar{u}_k) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} - \frac{\partial \tau_{ik}^a}{\partial x_k} - g_i \beta \bar{t}$$  \hspace{1cm} (1)

where $\tau_{ik}$ denotes modelled SGS stress or URANS stress. The SGS/URANS turbulent kinetic energy is defined as $k_T = 0.5 \bar{\tau}_{ii}$. Decompose $\bar{u}_i$ and $\bar{p}$ into a time-averaged (or ensemble-averaged) value and a resolved fluctuation as

$$\bar{u}_i = U_i + \bar{u}_i', \ \bar{p} = P + \bar{p}', \ \bar{t} = T + \bar{t}'$$

$$U_i = \langle \bar{u}_i \rangle, \ P = \langle \bar{p} \rangle, \ T = \langle \bar{t} \rangle$$  \hspace{1cm} (2)

where $u_i''$ is the SGS fluctuation. Insert this in Eq. 1 so that

$$\frac{\partial \bar{u}_i'}{\partial t} + \frac{\partial}{\partial x_k}((U_i + \bar{u}_i')(U_k + \bar{u}_k')) = -\frac{1}{\rho} \frac{\partial (P + \bar{p})}{\partial x_i} + \nu \frac{\partial^2 (U_i + \bar{u}_i')}{\partial x_k \partial x_k}$$

$$- \frac{\partial \tau_{ik}^a}{\partial x_k} - g_i \beta (T + \bar{t}')$$  \hspace{1cm} (3)

Time (ensemble) averaging of Eq. 3 yields

$$\frac{\partial}{\partial x_k} \langle U_i U_k \rangle = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k} \langle \bar{u}_i' \bar{u}_k' \rangle - \frac{\partial}{\partial x_k} \langle \tau_{ik}^a \rangle - g_i \beta \bar{t}'$$  \hspace{1cm} (4)

Now subtract Eq. 4 from Eq. 3

$$\frac{\partial \bar{u}_i'}{\partial t} + \frac{\partial}{\partial x_k}((U_i \bar{u}_k' + U_k \bar{u}_i') + \bar{u}_i' \bar{u}_k') =$$

$$-\frac{1}{\rho} \frac{\partial \bar{p}'}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k} \left( \langle \bar{u}_i' \bar{u}_k' \rangle + \langle \tau_{ik}^a \rangle - \frac{\partial \tau_{ik}^a}{\partial x_k} \right) - g_i \beta \bar{t}'$$  \hspace{1cm} (5)

Multiply Eq. 5 with $\bar{u}_j'$ and a corresponding equation for $\bar{u}_j'$ by $\bar{u}_i'$, add them together, and time (ensemble) average

$$\langle \bar{u}_j' \frac{\partial}{\partial x_k} (U_i \bar{u}_k' + U_k \bar{u}_i' + \bar{u}_i' \bar{u}_k') \rangle + \langle \bar{u}_i' \frac{\partial}{\partial x_k} (U_j \bar{u}_k' + U_k \bar{u}_j' + \bar{u}_k' \bar{u}_j') \rangle =$$

$$- \langle \bar{u}_j' \frac{\partial \bar{p}'}{\partial x_i} \rangle - \langle \bar{u}_j' \frac{\partial \bar{p}'}{\partial x_j} \rangle + \nu \langle \bar{u}_j' \frac{\partial^2 \bar{u}_i'}{\partial x_k \partial x_k} \rangle + \nu \langle \bar{u}_i' \frac{\partial^2 \bar{u}_j'}{\partial x_k \partial x_k} \rangle$$

$$- \langle \bar{u}_j' \frac{\partial \tau_{ik}^a}{\partial x_k} \rangle - \langle \bar{u}_i' \frac{\partial \tau_{ik}^a}{\partial x_k} \rangle - g_i \beta \langle \bar{u}_j' \bar{t}' \rangle - g_j \beta \langle \bar{u}_i' \bar{t}' \rangle$$  \hspace{1cm} (6)
The two first lines correspond to the usual $\bar{u}_i u'_j$ equation in conventional Reynolds decomposition. The two last terms on line 2 can be re-written as

$$\nu \frac{\partial}{\partial x_k} \left\langle \bar{u}_i \frac{\partial u'_j}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle \bar{u}'_i \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle - 2\nu \left\langle \bar{u}'_i \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle = \nu \frac{\partial^2}{\partial x_k \partial x_k} \left\langle \bar{u}'_i \bar{u}'_j \right\rangle - 2\nu \left\langle \bar{u}'_i \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle$$

(7)

The two first terms on the last line in Eq. 6 can be rewritten as

$$- \left\langle \frac{\partial}{\partial x_k} \left( \bar{u}'_j \tau^{\alpha}_{ik} \right) \right\rangle + \left\langle \tau^{\alpha}_{ik} \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle = \left\langle \frac{\partial}{\partial x_k} \left( \bar{u}'_i \tau^{\alpha}_{jk} \right) \right\rangle + \left\langle \tau^{\alpha}_{jk} \frac{\partial \bar{u}'_i}{\partial x_k} \right\rangle$$

(8)

Finally, we can now write the transport equation for $\langle \bar{u}'_i \bar{u}'_j \rangle$ as

$$\frac{\partial}{\partial x_k} \langle U_k \langle \bar{u}'_i \bar{u}'_j \rangle \rangle = -\langle \bar{u}'_i \bar{u}'_j \rangle \frac{\partial U_j}{\partial x_k} - \langle \bar{u}'_i \bar{u}'_j \rangle \frac{\partial U_j}{\partial x_k} - \frac{1}{\rho} \left\langle \bar{u}_i \frac{\partial p'}{\partial x_j} \right\rangle - \frac{1}{\rho} \left\langle \bar{u}'_i \frac{\partial p'}{\partial x_j} \right\rangle +$$

$$- \frac{\partial}{\partial x_k} \left\langle \bar{u}'_i \bar{u}_j \bar{u}'_k \right\rangle + \nu \frac{\partial^2}{\partial x_k \partial x_k} \left( \bar{u}'_i \bar{u}'_j \right) - 2\nu \left\langle \frac{\partial \bar{u}'_i \bar{u}'_j}{\partial x_k} \right\rangle - g_i \beta \left\langle \bar{u}'_i \bar{p}' \right\rangle - g_j \beta \left\langle \bar{u}'_j \bar{p}' \right\rangle -$$

$$- \left\langle \frac{\partial}{\partial x_k} \left( \bar{u}'_j \tau^{\alpha}_{ik} \right) \right\rangle + \left\langle \frac{\partial}{\partial x_k} \left( \bar{u}'_i \tau^{\alpha}_{jk} \right) \right\rangle + \left\langle \tau^{\alpha}_{jk} \frac{\partial \bar{u}'_i}{\partial x_k} \right\rangle + \left\langle \tau^{\alpha}_{ik} \frac{\partial \bar{u}'_j}{\partial x_k} \right\rangle$$

(9)

where the two last lines include all terms related to the SGS/URANS stresses. The third line represents diffusion transport by SGS/URANS stresses and the fourth line represents dissipation by SGS/URANS stresses. For an eddy-viscosity SGS/URANS model

$$\tau^{\alpha}_{ij} = -2\nu T \bar{s}_{ij}, \quad \bar{s}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

(10)

1.1 Resolved turbulent kinetic energy $\langle k \rangle$

Now we will derive the transport equation for the resolved turbulent kinetic energy $\langle k \rangle = \langle \bar{u}'_i \bar{u}'_i \rangle / 2$. Take the trace of Eq. 9 and divide by two

$$\frac{\partial}{\partial x_j} \langle U_j \langle k \rangle \rangle = -\langle \bar{u}'_i \bar{u}'_j \rangle \frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left( \frac{1}{\rho} \left\langle \bar{u}'_i \bar{p}' \right\rangle + \frac{1}{2} \langle \bar{u}'_i \bar{u}'_j \rangle \right) + \nu \frac{\partial^2 \langle k \rangle}{\partial x_j \partial x_j}$$

$$- \langle \bar{u}'_i \frac{\partial \bar{u}'_j}{\partial x_j} \rangle - g_i \beta \langle \bar{u}'_i \bar{p}' \rangle - \frac{\partial}{\partial x_j} \left( \tau^{\alpha}_{ij} \frac{\partial \bar{u}'_i}{\partial x_j} \right) + \left\langle \tau^{\alpha}_{ij} \frac{\partial \bar{u}'_i}{\partial x_j} \right\rangle$$

(11)

The pressure-velocity term was re-written as

$$\left\langle \bar{u}'_i \frac{\partial \bar{p}'}{\partial x_j} \right\rangle = \frac{\partial}{\partial x_j} \left\langle \bar{u}'_j \bar{p}' \right\rangle - \left\langle \bar{p}' \frac{\partial \bar{u}'_j}{\partial x_j} \right\rangle$$

(12)
where the last term is zero due to continuity.

The last term in Eq. 11 can be both positive and negative. However, if we introduce an eddy-viscosity model it can be shown that it is predominantly negative. If the approximation (using Eq. 10)

$$\tau_{ij}^a = \tau_{ij}^a - \langle \tau_{ij}^a \rangle = -2(\nu_T \bar{s}_{ij} - \langle \nu_T \bar{s}_{ij} \rangle) \simeq -2\nu_T s'_{ij}$$  \hspace{1cm} (13)$$

is made we find that the term is always negative. This is easily seen when inserting Eq. 13 into the last term of Eq. 11

$$\langle \tau_{ij}^a \frac{\partial \bar{u}_i'}{\partial x_j} \rangle \simeq -2\langle \nu_T s'_{ij}(s'_{ij} + \omega'_{ij}) \rangle = -2\langle \nu_T s'_{ij}s'_{ij} \rangle < 0 \hspace{1cm} (14)$$

where $\omega'_{ij} = 0.5(\partial \bar{u}_i'/dx_j - \partial \bar{u}_j'/dx_i)$. In Eq. 14 we have used the fact that the product of a symmetric and anti-symmetric tensor is zero.

The terms in Eq. 11 have the following physical meaning. The term on the left-hand side is the advection. The terms on the right-hand side are production of $\langle k \rangle$, transport of $\langle k \rangle$ by resolved fluctuations, viscous transport of $\langle k \rangle$, viscous dissipation of $\langle k \rangle$, production/destruction of $\langle k \rangle$ by buoyancy, transport of $\langle k \rangle$ by SGS/URANS turbulence and production/destruction of $\langle k \rangle$ by SGS/URANS turbulence.

1.2 Time-averaged kinetic energy $\langle \bar{K} \rangle$

The equation for the time-averaged kinetic energy $\langle \bar{K} \rangle = \frac{1}{2}U_iU_i$ is derived by multiplying the time-averaged (ensemble-averaged) momentum equation, Eq. 4, by $U_i$ so that

$$U_i \frac{\partial}{\partial x_j} (U_iU_j) = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} - U_i \frac{\partial}{\partial x_j} \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i'\bar{u}_j' \rangle \right) - U_i g_i \beta T \hspace{1cm} (15)$$

The left-hand side of Eq. 15 can be rewritten as

$$\frac{\partial}{\partial x_j} (U_iU_j) - U_i \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} = U_i \frac{\partial}{\partial x_j} (U_iU_i) - \frac{1}{2} U_i \frac{\partial}{\partial x_j} (U_iU_i)$$

$$= \frac{1}{2} U_i \frac{\partial}{\partial x_j} (U_iU_i) = \frac{\partial}{\partial x_j} (U_i \langle \bar{K} \rangle) \hspace{1cm} (16)$$

The viscous diffusion term in Eq. 15 is rewritten in the same way as the viscous term in Eq. 7, i.e.

$$\nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} = \nu \frac{\partial^2 \langle \bar{K} \rangle}{\partial x_j \partial x_j} - \nu \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \hspace{1cm} (17)$$

The turbulent diffusion term is rewritten as

$$U_i \frac{\partial}{\partial x_j} \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i'\bar{u}_j' \rangle \right) = \frac{\partial}{\partial x_j} \left[ U_i \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i'\bar{u}_j' \rangle \right) - \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i'\bar{u}_j' \rangle \right) \right] \frac{\partial U_i}{\partial x_j} \hspace{1cm} (18)$$
Now we can assemble the transport equation for $\langle K \rangle$ by inserting Eqs. 16, 17 and Eq. 18 into Eq. 15

$$\frac{\partial}{\partial x_j}(U_j \langle K \rangle) = \nu \frac{\partial^2}{\partial x_j \partial x_j} \langle K \rangle - \frac{\partial}{\partial x_j}(U_j P) - \frac{\partial}{\partial x_j} \left[ U_i \left( \langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) \right]$$

$$- U_i \frac{\partial P}{\partial x_i} + \left( \langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) \frac{\partial U_i}{\partial x_j} - \nu \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} - g_i \beta U_i T$$

We recognize the usual transport term on the left-hand side due to advection. On the right-hand side we have the main source term (velocity times the pressure gradient) viscous diffusion and transport of $\langle K \rangle$. The term in square brackets represents transport by interaction between the time-averaged (ensemble-averaged) velocity field and turbulence. The term $\langle \bar{u}_i' \bar{u}_j' \rangle \partial U_i / \partial x_j$ is the usual production term of the resolved kinetic energy $0.5\langle \bar{u}_i' \bar{u}_i \rangle$ which usually is negative. This term appears in Eq. 11 but with opposite sign. The term $\langle \tau_{ij}^a \rangle \partial U_i / \partial x_j$ is the production term in the turbulent kinetic energy equation $k_i = 0.5\tau_{ii}$. This term is usually referred to as the SGS/URANS dissipation term, and for an eddy-viscosity model we find (cf. Eqs. 13 and 14)

$$\langle \tau_{ij}^a \rangle \frac{\partial U_i}{\partial x_j} = -2\langle \nu_T \bar{s}_{ij} \rangle (S_{ij} + \Omega_{ij})$$

$$\simeq -2\langle \nu_T \rangle (\bar{s}_{ij}) S_{ij} = -2\langle \nu_T \rangle S_{ij} S_{ij} < 0$$

It is interesting to compare this SGS dissipation term with the viscous dissipation term in Eq. 18. If $\nu_{sgs} \gg \nu$, the SGS dissipation is much larger than the viscous one. If this is not the case, then we’re doing a DNS!

### 1.3 Resolved kinetic energy $K_{res}$

The equation for the time-averaged kinetic energy $K_{res} = \frac{1}{2} \bar{u}_i \bar{u}_i$ is derived by multiplying the filtered momentum equation, Eq. 1, by $\bar{u}_i$, so that

$$\bar{u}_i \left( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_k} (\bar{u}_i \bar{u}_k) \right) = \bar{u}_i \left( -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_i \partial x_i} - \frac{\partial \tau_{ij}^a}{\partial x_j} - g_i \beta \bar{t} \right)$$

Looking at the derivation in Section 1.1 and the final equation (Eq. 19) we get

$$\frac{\partial K_{res}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_j K_{res}) = \nu \frac{\partial^2 K_{res}}{\partial x_j \partial x_j} - \bar{u}_j \frac{\partial \bar{p}}{\partial x_j} - \frac{\partial \bar{u}_i \tau_{ij}^a}{\partial x_j}$$

$$+ \tau_{ij}^a \frac{\partial \bar{u}_i}{\partial x_j} - \nu \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} - g_i \beta \bar{u}_i \bar{t}$$

### 1.4 Equation for $K = u_i u_i / 2$

The equation for $K$ is derived by multiplying Navier-Stokes (i.e. Eq. 1 without SGS stresses and non-filtered variables) by $u_i$, i.e.

$$u_i \left( \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} (u_i u_k) \right) = u_i \left( -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_i \partial x_i} - g_i \beta \bar{t} \right)$$
Looking at the derivation in Section 1.1 and the final equation (Eq. 19) we get

\[
\frac{\partial K}{\partial t} + \frac{\partial}{\partial x_j} (u_j K) = \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (u_j p) - \nu \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - g_i \beta u_i t \tag{24}
\]

This is the same equation as in Section 2.3 in [1] but there it is expressed in the stress tensor, \( \sigma_{ij} \).

1.5 SGS turbulent kinetic energy, \( k_T = 0.5(u_i u_i - \bar{u}_i \bar{u}_i) \)

The SGS turbulent kinetic energy is defined as

\[
k_T = 0.5(u_i u_i - \bar{u}_i \bar{u}_i) = \bar{K} - \bar{K}_{res} \tag{25}
\]

It is obtained by subtracting Eq. 22 from the filtered Eq. 24

\[
\frac{\partial(\bar{K} - \bar{K}_{res})}{\partial t} + \frac{\partial}{\partial x_j} (u_j \bar{K} - \bar{u}_j \bar{K}_{res}) = \nu \frac{\partial^2 (\bar{K} - \bar{K}_{res})}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (u_j \bar{p} - \bar{u}_j \bar{p} - \bar{u}_i \tau_{ij}^a) - \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \nu \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} \right) - g_i \beta (\bar{u}_i t - \bar{u}_i \bar{t})
\]

Adding the term \( \partial/\partial x_j (\bar{u}_j \bar{K} - u_j \bar{K}) \) on both sides and using Eq. 25 gives

\[
\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_j k_T) = \nu \frac{\partial^2 k_T}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (u_j \bar{p} - \bar{u}_j \bar{p} - \bar{u}_i \tau_{ij}^a + u_j \bar{K} - \bar{u}_j \bar{K}) - \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \nu \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} \right) - g_i \beta (\bar{u}_i t - \bar{u}_i \bar{t})
\]

Line 1: convection and viscous diffusion.

Line 2: turbulent diffusion.

Line 3: production; it appears with opposite sign in Eq. 22.

Line 4: viscous dissipation.

Line 5: buoyancy.
1.6 Equation for modeled $k_T$

The equation for the modelled turbulent SGS/RANS kinetic energy reads

$$\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_j k_T) = \frac{\partial}{\partial x_j} \left[ (\nu + \nu_T) \frac{\partial k_T}{\partial x_j} \right] + 2\nu_T \bar{s}_{ij} \bar{s}_{ij} - \varepsilon$$

(26)

The terms on the right-hand side represent viscous and turbulent diffusion, production and viscous dissipation.

1.7 Equation for resolved heat flux, $\langle \bar{u}_i \bar{T} \rangle$

The filtered temperature equation for $\bar{t}$ reads

$$\frac{\partial \bar{t}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{u}_k \bar{t}) = \nu \frac{\partial^2 \bar{t}}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k}$$

(27)

Use Eq. 2 in Eq. 27 so that

$$\frac{\partial}{\partial t} (T + \bar{t}) + \frac{\partial}{\partial x_k} ((U_k + \bar{u}_k')(T + \bar{t}')) = \nu \frac{\partial^2 (T + \bar{t})}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k}$$

(28)

Time (ensemble) averaging of Eq. 28 yields

$$\frac{\partial}{\partial x_k} (U_k T) = \nu \frac{\partial^2 T}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k} (\langle \bar{u}_k' \bar{T} \rangle + \langle h_k \rangle)$$

(29)

Now subtract Eq. 29 from Eq. 28

$$\frac{\partial \bar{t}'}{\partial t} + \frac{\partial}{\partial x_k} (\bar{u}_k' + \bar{u}_k T + \bar{u}_k' \bar{T}) =$$

$$\nu \frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k} \left( \langle \bar{u}_k' \bar{T} \rangle + \langle h_k \rangle - h_k \right)$$

(30)

Multiply Eq. 30 with $\bar{u}_i'$ and multiply Eq. 5 with $t'$, add them together and time (ensemble) average

$$\langle \bar{u}_i' \frac{\partial}{\partial x_k} (\bar{u}_k' T + U_k \bar{T} + \bar{u}_k' \bar{T}) + \bar{t}' \frac{\partial}{\partial x_k} (U_i \bar{u}_k' + U_k \bar{u}_i' + \bar{u}_i' \bar{u}_k') \rangle$$

$$= - \frac{\bar{t}'}{\rho} \frac{\partial \bar{p}'}{\partial x_i} + \frac{\nu}{\Pr} \left( \bar{u}_i' \frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} + \nu \left\langle \bar{t}' \frac{\partial^2 \bar{u}_i'}{\partial x_k \partial x_k} \right\rangle - g_i \beta \left\langle \bar{t}' \bar{t}' \right\rangle \right)$$

$$- \left\langle \bar{u}_i' \frac{\partial h_k'}{\partial x_k} \right\rangle - \left\langle \bar{t}' \frac{\partial g_i}{\partial x_k} \right\rangle$$

(31)

The two first lines correspond to the conventional heat flux equation. The two terms in the middle on line 2 can be re-written as

$$\frac{\nu}{\Pr} \frac{\partial}{\partial x_k} \left\langle \bar{u}_i' \frac{\partial \bar{t}'}{\partial x_k} \right\rangle - \frac{\nu}{\Pr} \left\langle \bar{u}_i' \frac{\partial \bar{t}'}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle \bar{t}' \frac{\partial \bar{u}_i'}{\partial x_k} \right\rangle - \nu \left\langle \bar{u}_i' \frac{\partial \bar{t}'}{\partial x_k} \right\rangle$$

$$= \frac{\nu}{\Pr} \frac{\partial}{\partial x_k} \left\langle \bar{u}_i' \frac{\partial \bar{t}'}{\partial x_k} \right\rangle + \nu \frac{\partial}{\partial x_k} \left\langle \bar{t}' \frac{\partial \bar{u}_i'}{\partial x_k} \right\rangle - \left( \nu + \frac{\nu}{\Pr} \right) \left\langle \bar{u}_i' \frac{\partial \bar{t}'}{\partial x_k} \right\rangle$$

(32)
Using Eq. 32 in Eq. 31 and at the same time re-writing the SGS/URANS terms we get

\[
\frac{\partial}{\partial x_k} U_k \langle \bar{u}_i' T \rangle = - \langle \bar{u}_i' \rangle \frac{\partial T}{\partial x_k} - \langle \bar{u}_i' T \rangle \frac{\partial U_i}{\partial x_k} - \frac{\bar{p} \partial \bar{p}'}{\partial x_i} - \frac{\partial}{\partial x_k} \langle \bar{u}_i' \bar{u}_i' \rangle \\
+ \frac{\nu}{P_r} \frac{\partial}{\partial x_k} \left( \langle \bar{u}_i' \bar{p} \rangle \right) + \nu \frac{\partial}{\partial x_k} \left( \frac{\partial T}{\partial x_k} \right) - \left( \nu + \frac{\nu}{P_r} \right) \left( \frac{\partial \bar{u}_i'}{\partial x_k} \frac{\partial \bar{p}'}{\partial x_k} \right) - g_i \beta \left( \bar{r}_i'^2 \right) \\
- \frac{\partial}{\partial x_k} \langle \bar{u}_i' h_i' \rangle + \left( h_k \frac{\partial \bar{u}_i'}{\partial x_k} \right) - \frac{\partial}{\partial x_k} \langle \bar{t}_i'^a \rangle + \frac{\partial \bar{t}_i'^a}{\partial x_k} \right) \\
\] (33)

The SGS/URANS heat fluxes are commonly obtain from an eddy-viscosity model

\[ h_i = - \frac{\nu_T}{P_r} \frac{\partial \bar{T}}{\partial x_i} \] (34)

### 1.8 Equation for resolved temperature variance, \( \langle \bar{t}^2 \rangle \)

Multiply Eq. 30 with \( \bar{t}' \) and time (ensemble) average

\[
\left( \langle \bar{t}' \frac{\partial}{\partial x_k} (\bar{u}_k' T + U_k \bar{t}' + \bar{u}_k' \bar{t}') \rangle \right) = \frac{\nu}{P_r} \left( \langle \bar{t}' \frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} \rangle \right) - \langle \bar{t}' \frac{\partial h_i'}{\partial x_k} \rangle \\
\] (35)

The first term on the right-hand side can be re-written as

\[
\frac{\nu}{P_r} \left( \frac{\partial}{\partial x_k} \left( \langle \bar{t} \frac{\partial \bar{t}}{\partial x_k} \rangle \right) \right) - \frac{\nu}{P_r} \left( \frac{\partial \bar{t}}{\partial x_k} \frac{\partial \bar{t}}{\partial x_k} \right) = \frac{1}{2} \frac{\nu}{P_r} \frac{\partial^2}{\partial x_k \partial x_k} \langle \bar{t}^2 \rangle - \frac{\nu}{P_r} \left( \frac{\partial \bar{t}}{\partial x_k} \frac{\partial \bar{t}}{\partial x_k} \right) \\
\] (36)

Using Eq. 36 and re-writing the SGS/URANS term, Eq. 35 can now be written as

\[
\frac{1}{2} \frac{\partial}{\partial x_k} \langle U_k \langle \bar{t}^2 \rangle \rangle = - \langle \bar{u}_k' \rangle \frac{\partial T}{\partial x_k} - \frac{1}{2} \frac{\partial}{\partial x_k} \langle \bar{u}_k' \bar{t}^2 \rangle \\
+ \frac{1}{2} \frac{\nu}{P_r} \frac{\partial^2}{\partial x_k \partial x_k} \langle \bar{t}^2 \rangle - \frac{\nu}{P_r} \left( \frac{\partial \bar{t}'}{\partial x_k} \frac{\partial \bar{t}'}{\partial x_k} \right) - \frac{\partial}{\partial x_k} \langle \bar{h}_i' \bar{t}' \rangle + \frac{\partial \bar{h}_i'}{\partial x_k} \right) \\
\] (37)

Multiply Eq. 35 by 2 and we get

\[
\frac{\partial}{\partial x_k} \langle U_k \langle \bar{t}^2 \rangle \rangle = -2 \langle \bar{u}_k' \rangle \frac{\partial T}{\partial x_k} - \frac{\partial}{\partial x_k} \langle \bar{u}_k' \bar{t}^2 \rangle \\
+ \frac{\nu}{P_r} \frac{\partial^2}{\partial x_k \partial x_k} \langle \bar{t}^2 \rangle - 2 \frac{\nu}{P_r} \left( \frac{\partial \bar{t}'}{\partial x_k} \frac{\partial \bar{t}'}{\partial x_k} \right) - 2 \frac{\partial}{\partial x_k} \langle \bar{h}_i' \bar{t}' \rangle + 2 \frac{\partial \bar{h}_i'}{\partial x_k} \right) \\
\] (38)

### References