CHALMERS

Publication 2006/05

How to make energy spectra with Matlab

Lars Davidson
http://www.tfd.chalmers.se/~lada

Division of Fluid Dynamics
Department of Applied Mechanics
Chalmers University of Technology
Göteborg, Sweden, October 2006

How to make energy spectra with Matlab

L. Davidson
Division of Fluid Dynamics
Department of Applied Mechanics
Chalmers University of Technology
SE-412 96 Göteborg
http://www.tfd.chalmers.se/~lada

April 23, 2008

1 Introduction

When analysing DNS or LES data, we are interested to look at the energy spectra. From these we can find out in which turbulence scales (i.e. at which wave numbers) the fluctuating kinetic turbulent energy reside. By taking the Fourier transform of the time signal (a fluctuating turbulent velocity) and then taking the square of the Fourier coefficients we obtain the energy spectrum versus frequency.

If we want to have the energy spectrum versus wavenumber, we Fourier transform N instantaneous signals in space and then time average the N Fourier transforms. An alternative way is to Fourier transform of a (time-averaged) two-point correlation, $b(z)$, which is defined as

$$b(z,\zeta) = \overline{w'(\zeta - z)w'(\zeta)} \tag{1}$$

where z is the separation between the two points. Here we assume that z is an homogeneous direction so that b is independent of ζ, i.e. $b = b(z)$. The two-point correlation for an infinite channel flow is shown in Fig. 1. On discrete form the expression for b reads

$$b(k\Delta z) = \frac{1}{M}\sum_{m=1}^{M} w'(\zeta - k\Delta z)w'(\zeta) \tag{2}$$

where m denotes summation in homogenous directions (i.e. time plus spatial homogenoues directions).

In the following section we give a simple example how to use Matlab to Fourier transform a signal where we know the answer. Then we show how to derive the energy spectrum from a spatial two-point correlation. Finally, some comments are given on how to create an energy spectrum versus frequency from an autocorrelation (i.e. from a two-point correlation in time).

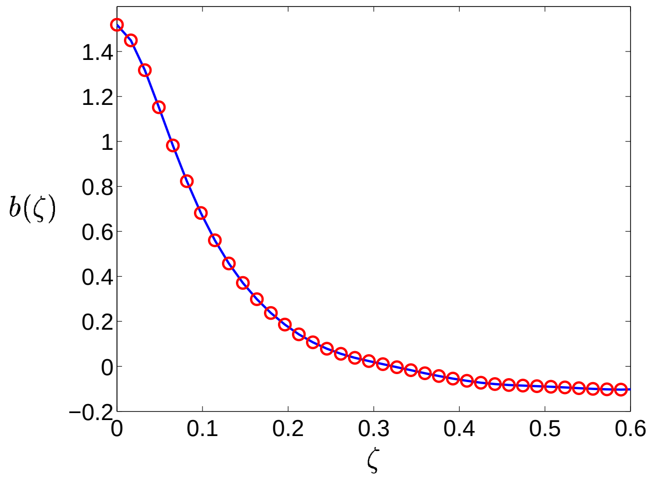

Figure 1: Two-point correlation, $B_{ww}(\zeta) = \langle w'(z)w'(z+\zeta)\rangle$, of DNS data in channel flow taken from [1].

2 An example of using FFT

Here we will present a simple example. Consider the function

$$u = 1 + \cos(2\pi x/L) = 1 + \cos(2\pi(n-1)/N) \tag{3}$$

where L is the length of the domain and $N = 16$ is the number of discrete points, see Fig. 2. Let's use this function as input vector for the discrete Fourier transform (DFT) using Matlab. The function u is symmetric, so we expect the Fourier coefficients to be real. In Matlab the DFT of u is defined as (type `help fft` at the Matlab prompt)

$$U(\kappa) = \sum_{n=1}^{N} u_n \exp\left\{\frac{-\iota 2\pi(\kappa-1)(n-1)}{N}\right\} \qquad 1 \le \kappa \le N \tag{4}$$

where κ is the wavenumber and $\iota = \sqrt{-1}$. In Matlab, we generate the function u in Eq. 3 using the commands

```
N=16;
n=1:1:N;
u=1+cos(2*pi*(n-1)/N);
```

The u function is shown in Fig. 2. 16 nodes are used and node 1 is located at $x = L/16$ and node 16 is located at $15L/16$.

Now we take the discrete Fourier transform of u. Type

```
U=fft(u);
```

Instead of using the built-in `fft` command in Matlab we can program it

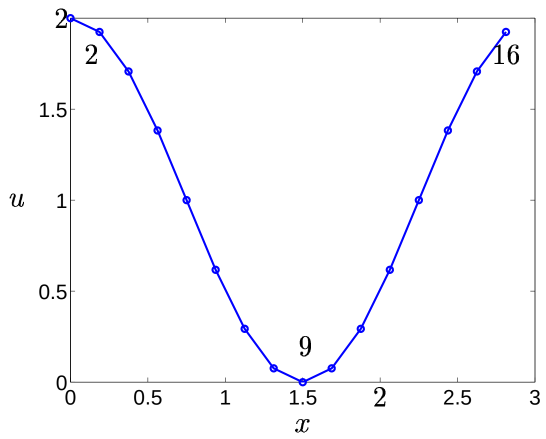

Figure 2: The u function.

directly in Matlab as

```
U=zeros(1,N);
for k=1:N
for n=1:N
  a=u(n);
  arg1=2*pi*(k-1)*(n-1)/N;
  U(k)=U(k)+a*cos(arg1);
end
end
```

Note that since u is symmetric, we have only used $\cos(x)$ (the symmetric part of $\exp(-\iota x)$).

The resulting Fourier coefficients get the value (cf. Eq. 4) $U(1)/N = 1$, $U(2)/N = 0.5$, $U(16)/N = 0.5$, and the remaining coefficients are zero. In Fig. 4 the U is plotted versus wavenumber. With $\kappa = 2\pi \cdot (n-1)/L$ the Fourier coefficients are plotted versus wavenumber in Fig. 4. The first Fourier coefficients corresponds to the mean of u, i.e. $U(1) = \langle u \rangle$. The second and $16th$ Fourier coefficients correspond to $\cos\{2\pi(n-1/N\} = \cos(2\pi x/L)$.

The kinetic energy, k, of the signal in Fig. 2 can be computed as

$$k = \frac{1}{L}\int_0^L \frac{1}{2}u^2(x)dx = \frac{1}{2}\sum_{n=1}^{N} u_n^2/N = 1.5 \qquad (5)$$

We can find k also from Fig. 3. In wavenumber space the kinetic energy is equal to the integral of the square of the Fourier coefficients, i.e.

$$k = \frac{1}{L}\int_0^\infty \frac{1}{2}U^2(\kappa)d\kappa = \frac{1}{N}\sum_{n=1}^{N} \frac{1}{2}U_n^2/N = 1.5 \qquad (6)$$

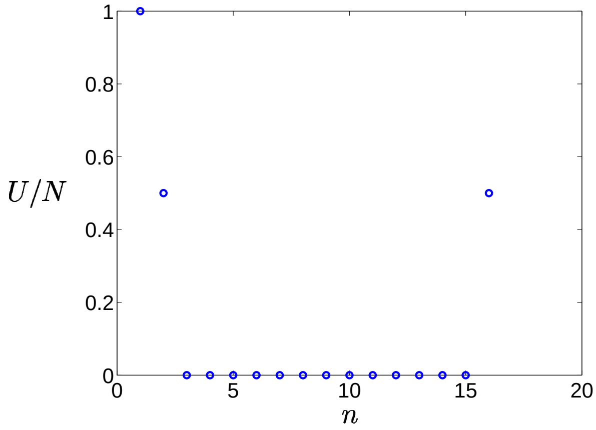

Figure 3: The U coefficients.

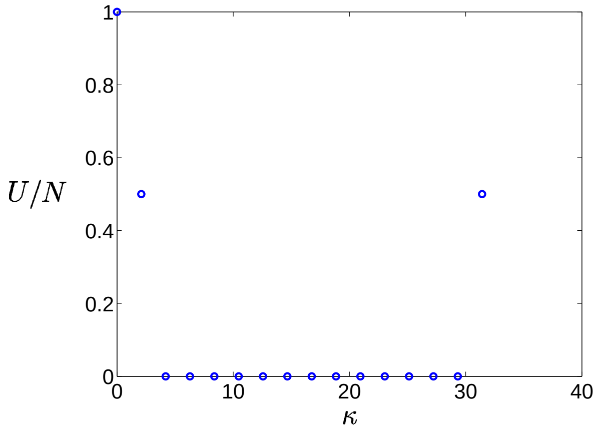

Figure 4: The U coefficients versus wavenumber, κ.

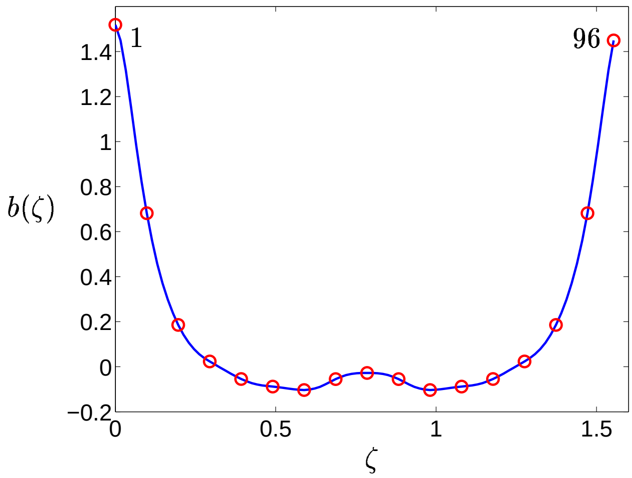

Figure 5: Periodic two-point correlation, $b_{ww}^{per}(\zeta) = \langle w'(z)w'(z+\zeta)\rangle$, of DNS data in channel flow taken from [1].

3 Energy spectrum from the two-point correlation

Now that we have learnt how to use the FFT command in Matlab, let's use it on our two-point correlation in Eq. 1 and Fig. 1. Equation 4 reads

$$B_\kappa = \sum_{n=1}^{N} b_n \exp\left\{\frac{-\iota 2\pi(\kappa-1)(n-1)}{N}\right\} \tag{7}$$

First we make it symmetric, see Fig. 5, i.e. we assume the signal to be periodic and we pick one period. Hence, Eq. 7 will be cosin-transform, i.e.

$$B_\kappa = \sum_{n=1}^{N} b_n \cos\left\{\frac{2\pi(\kappa-1)(n-1)}{N}\right\} \tag{8}$$

In this special case (DNS of fully developed channel flow), the flow is periodic in z direction, which means that we could have obtained the symmetric two-point correlation directly. However, in general this is not the case. In general we have to start with a two-point as that in Fig. 1. Then we mirror it to get a symmetric two-point correlation as in Fig. 5. Since FFT works best when the number of nodes can be expressed as 2^r, where r is an integer, the two-point correlation is created using 48 nodes so that the symmetric two-point correlation has 96. Node 1 and 96 are indicated in Fig. 5. Note that $b_{ww,n=97}^{per} = b_{ww,n=1}^{per}$ is omitted, because node number 97 is the first node in the next period.

In Fig. 6 the Fourier coefficients B_{κ_z} are presented versus wavenumber $\kappa_z = 2\pi\cdot(n-1)/\zeta_{max}$, where $\zeta_{max} \simeq 1.55$, see Fig. 5.

As usual, the Fourier coefficient for the first wavenumber, i.e. B_1, is equal to the mean of b, $\langle b\rangle$, i.e.

$$\langle b\rangle = \frac{1}{N}\sum_{n=1}^{N} b_n \equiv \frac{1}{N}B_1 \tag{9}$$

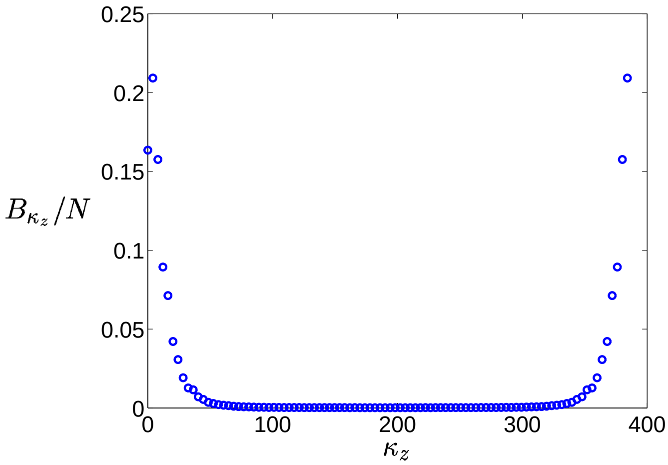

Figure 6: The energy spectrum of w_{rms}^2 versus wavenumber, κ_z.

compare with Eq. 8. Note that this is almost the same expression as that for the integral length scale [2] which is defined as

$$\mathcal{L} = \frac{1}{w_{rms}^2} \int_0^\infty b(z)dz = \frac{1}{N w_{rms}^2} \sum_{n=1}^{N} b_n \tag{10}$$

Hence the integral length scale is related to the first Fourier mode B_1 as

$$\mathcal{L} = \frac{1}{N w_{rms}^2} B_1 \tag{11}$$

The two-point correlation for no separation is equal to square of w_{rms}, i.e. $b(0) = w_{rms}^2 = 1.51$. Another way to obtain w_{rms}^2 is to integrate the energy spectrum in Fig. 6, i.e.

$$w_{rms}^2 = \frac{1}{\zeta_{max}} \int_0^\infty B(\kappa)d\kappa = \frac{1}{N} \sum_{n=1}^{N} B_n = 1.51 \tag{12}$$

4 Energy spectra from the autocorrelation

When computing the energy spectra of the w' velocity, say, versus frequency, the time series of $w'(t)$ is commonly Fourier transformed and the energy spectrum is obtained by plotting the square of the Fourier coefficients versus frequency, f. In the previous section we computed the energy spectrum versus wavenumber by Fourier transforming the two-point correlation. We can use the same approach in time. First we create the autocorrelation $b(\tau) = \langle w'(t)w'(t+\tau)\rangle$ (this can be seen as a two-point correlation in time). Then $w'(t)$ is Fourier transformed to get $W(f)$, and $W(f)$ is now twice the energy spectrum of w'.

References

[1] L. Davidson and M. Billson. Hybrid LES/RANS using synthesized turbulence for forcing at the interface. *International Journal of Heat and Fluid Flow*, 27(6):1028–1042, 2006.

[2] G. Comte-Bellot and S. Corrsin. Simple Eularian time correlation of full- and narrow-band velocity signals in grid-generated "isotropic" turbulence. *Journal of Fluid Mechanics*, 48(2):273–337, 1971.