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Approximation of subgrid-scale stresses based on the Leonard expansion

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Abstract — Based on the reconstruction series for subgrid-scale (SGS) stress tensor, SGS modelling is revisited. It is shown that, along with the first Leonard term in the series, the second term is also exploitable in relation to the viscous dissipation rate tensor, ε_{ij}, being further subjected to a Leonard expansion. The approximation of ε_{ij} is discussed in analogy to RANS modelling. With the assumption of anisotropy dissipation, it is shown that the second term can be approximated in terms of an eddy-viscosity formulation, which, together with the first Leonard term, forms a two-term mixed model. The resulting mixed model has been analyzed in LES of turbulent channel flow. The emphasis in the present work has been placed on the effect of model coefficients. The Leonard term may induce negative diffusion associated to energy backscatter, while the second Smagorinsky term reinforces energy dissipation. Moreover, the modelled Leonard stresses have also been highlighted in the computation.

1. Introduction

Subgrid scale (SGS) modeling remains one of the major issues in large eddy simulation of turbulent flows. By applying a spatial filtering to the governing equations, the large-scale turbulent structure is *distinguished* from unresolved subgrid scales and, consequently, leading to the closure problem to represent the interaction between subgrid-scale turbulence and resolved large-scale turbulent motion. The separation of resolved large and unresolved subgrid scales is a *conceptual definition*, but can plausibly be manifested on the basis of turbulence energy transfer, namely, the exchange between the resolved and sub-grid turbulent kinetic energy.

We consider incompressible flows, the filtered Navier-Stokes equations read

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \tau_{ij}\right) \tag{1}$$

where $\tau_{ij} = (\overline{u_i u_j} - \bar{u}_i \bar{u}_j)$ is the SGS stress tensor. Conventionally, the modelling for τ_{ij} has been to introduce a representation of forward energy drain from large-scale to subgrid-scale eddies. In this pallet, a typical example is the well-known Smagorinsky model [1] based on the SGS eddy viscosity concept. It has long been recognized that the Smagorinsky model displays little correlation with the real SGS stresses, as demonstrated in *a priori* testing by, e.g., Clark [2] using DNS data and by Liu et al. [3] with experimental data. Moreover, it has also been shown that, while the product $-\tau_{ij}\bar{S}_{ij}$ renders global energy dissipation, it may become negative locally for reverse energy transfer from small to large eddies, namely, energy backscatter, see, e.g., [4]. A well-known model that is able to reflect energy backscatter is the scale-similarity model by Bardina et al. [5]. In spite of this capability and its high correlation with the true SGS stresses [3], it is known that the scale-similarity model alone may fail to provide sufficient energy dissipation in actual LES. By a combination of the scale-similarity model and the Smagorinsky eddy-viscosity model, this has led to the mixed model [6] and a number of variants in different forms [7, 8, 9].

Since local reverse energy transfer, along with energy dissipation, is an inherent physical phenomenon in turbulent flows, a realistic SGS model should be able to describe both the forward and backward energy scatter. Instead of directly invoking the Bardina-type scale-similarity model, Leonard [10, 11] expanded the SGS stress into a reconstruction series, resulting in the tensor-diffusivity model by taking the first (leading) term from this series. Using experimental data, Liu et al. has shown that the correlation between the tensor-diffusivity model and the real SGS stress is comparable to (or even larger than) the Bardina similarity model [3]. The Leonard tensor-diffusivity model is equivalent to using a tensorial time scaling or a tensorial SGS viscosity in terms of the resolved large-scale velocity gradients and the filter width. To make up for the the truncated higher-order terms in the reconstruction series, the Leonard one-term model (i.e. the tensor-diffusivity model) has been supplemented by the Smagorinsky model as done by Clark [2] and, for further improvement, by the dynamic Smagorinsky model as proposed in Vreman et al. [12] and in Winckelmans et al. [13, 14].

In the present work, the SGS modelling stemmed from the Leonard expansion [10] of filtered residuals is revisited. We will show that the viscous dissipation of τ_{ij} is actually incurred in the second (forth-order) term in the reconstruction series. Upon the modelling of the viscous dissipation rate tensor, an eddy viscosity model may readily be brought in the formulation, but not added in an *arbitrary* manner. This leads consequently to a Clark-type two-term mixed model. The resulting model is then examined and analyzed in large eddy simulation of turbulent channel flow to highlight its properties.

2. Modelling Approximation

In previous studies [10, 11, 13, 14], it has been shown that, with an isotropic Gaussian filter, the filtered product of variables f and g can be written in an infinite reconstruction series, which reads

$$\overline{fg} = \bar{f}\bar{g} + \bar{\Delta}^2 \frac{\partial \bar{f}}{\partial x_k}\frac{\partial \bar{g}}{\partial x_k} + \frac{\bar{\Delta}^4}{2!}\frac{\partial^2 \bar{f}}{\partial x_k \partial x_l}\frac{\partial^2 \bar{g}}{\partial x_k \partial x_l} + \cdots \tag{2}$$

A reconstruction series as such is hereafter referred to as the *Leonard expansion*, since this series was explored first by Leonard [10]. For convenience in the following modelling approximation, moreover, it is noted here that a uniform filter width, $\bar{\Delta}$, has been assumed in the reconstruction series. Applying the Leonard expansion to the SGS stress tensor, τ_{ij}, one has

$$\begin{aligned} \tau_{ij} &= (\overline{u_i u_j} - \bar{u}_i \bar{u}_j) \\ &= \bar{\Delta}^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \frac{\bar{\Delta}^4}{2!}\frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_l}\frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_l} + \frac{\bar{\Delta}^6}{3!}\frac{\partial^3 \bar{u}_i}{\partial x_k \partial x_l \partial x_m}\frac{\partial^3 \bar{u}_j}{\partial x_k \partial x_l \partial x_m} + \cdots \end{aligned} \tag{3}$$

The first term on the right-hand side in Eq. (3) is the Leonard tensor-diffusivity model. This term has a similar form (but not the coefficient) to that in the model by Clark et al. [2] and by Vreman et al. [12] derived for the top-hat filter. Carati et al. [15] has further shown that, for all symmetric filters that are C^{∞} in Fourier space and have non-zero second moment, the Leonard term is preserved as the first term in the reconstruction series. Obviously, this holds true for most of the filters that are currently used in LES, for example, the top-hat, the Gaussian and the spectral cut-off filter.

We have previously discussed the modelling formulation based on the Leonard expansion stemmed from the Gaussian filter [16]. The top-hat filter is another commonly used filter in LES using finite difference or finite volume method. Following the generalized expansion by Carati et al. [15], when the kernel of the top-hat filter is used, the reconstruction series for τ_{ij} can be written as

$$\begin{aligned} \tau_{ij} &= (\overline{u_i u_j} - \bar{u}_i \bar{u}_j) \\ &= \bar{\Delta}^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \frac{\bar{\Delta}^4}{5}\frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_l}\frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_l} - \frac{\bar{\Delta}^4}{5}\frac{\partial^3 \bar{u}_i}{\partial x_k \partial x_k \partial x_l}\frac{\partial \bar{u}_j}{\partial x_l} + \cdots \end{aligned} \tag{4}$$

It is observed that, for both the Gaussian and the top-hat filter, the first two terms in Eqs (3) and (4) take, respectively, similar forms except the coefficient in the second term. Peng and Davidson [16] have shown that the second term in the above reconstruction series is actually related to the viscous dissipation rate tensor in the transport equation of τ_{ij}, which can be further exploited in the reconstruction-based modelling. By truncating the rest of higher-order terms, τ_{ij} is approximated in a two-term formulation, namely,

$$\tau_{ij} \approx \bar{\Delta}^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + C_\varepsilon \bar{\Delta}^4 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_l}\frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_l} \tag{5}$$

where a coefficient, C_ε, has been used in the second term in order to cast this term in a unified form for different filters. Obviously, the terms truncated, respectively, from Eq. (3) and Eq. (4) are different due to different filters, of which the effect should be appropriately accounted for by the modelling of the first two leading terms in Eq. (5). As it is, Eq. (5) represents a *raw* approximation of the SGS stress tensor in terms of the first- and second-order velocity derivatives. Nonetheless, the truncation of Eq. (3) or Eq. (4) has made Eq. (5) deviate from the accurate representation of the real SGS stress. Therefore, it would be awkward to directly use Eq. (5) in actual LES, unless additional modelling approximation is introduced.

In the present work, instead of computing the higher-order velocity derivatives, the second term in Eq. (5) is further approximated. Note that the viscous dissipation rate tensor is defined by $\varepsilon_{ij} = 2\nu\Upsilon_{ij}$, and

$$\Upsilon_{ij} = \left(\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} - \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} \right) \tag{6}$$

Applying the Leonard expansion to Υ_{ij} and taking the first term from the resulting reconstruction series for Υ_{ij}, we have

$$\Upsilon_{ij} = \left(\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} - \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} \right) \approx \bar{\Delta}^2 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_l}\frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_l} \tag{7}$$

Using Eq. (7) in Eq. (5) gives rise of a two-term model for τ_{ij}, which reads

$$\tau_{ij} \approx \tau_{L,ij} + C_\varepsilon \bar{\Delta}^2 \Upsilon_{ij} = \bar{\Delta}^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \frac{C_\varepsilon \bar{\Delta}^2}{2\nu}\varepsilon_{ij} \tag{8}$$

The Leonard tensor-diffusivity model, $\tau_{L,ij}$, is kept in Eq. (8), which has shown a favorable function in energy backscatter similar to a scale-similarity model of the Bardina type. In actual LES, it is found that this term may introduce excessively directional negative diffusion causing numerical instability problem. A model constant, C_l, is thus multiplied to this term with $C_l \in (0, 1.0]$. This yields, for the Leonard term,

$$\tau_{L,ij} = \left(C_l\bar{\Delta}\right)^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} \tag{9}$$

The presence of ε_{ij} in the second term of Eq. (8) suggests that this term may accommodate a major part of energy dissipation. Upon the modelling for the viscous dissipation rate tensor, ε_{ij}, Peng and Davidson [16] have shown that the resulting approximation in Eq.(8) may lead to a formulation of a mixed SGS model, when an anisotropic model is taken for ε_{ij} and the incurred energy dissipation is modelled with an eddy viscosity model (e.g. the Smagorinsky model). Obviously, more complicated nonlinear model could also be derived from Eq. (8), provided that ε_{ij} would be modelled in a sophisticated nonlinear form.

We take first the simplest form of the classical *local-isotropy* model for ε_{ij}, a concept that has been well exploited in RANS modelling. This gives $\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$, where ε is the SGS dissipation rate of SGS turbulence energy. Consequently, the two-term model takes the form of

$$\tau_{ij} = \left(C_l\bar{\Delta}\right)^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \frac{C_\varepsilon \bar{\Delta}^2}{3\nu}\varepsilon\delta_{ij} \tag{10}$$

The contribution by $\tau_{L,ij}$ to the local energy flux, $\varepsilon_L = -\tau_{L,ij}\bar{S}_{ij}$, may become negative (and thus energy backscatter). With the local-isotropy assumption, however, the second term does not make any contribution to energy dissipation or backscatter due to continuity, but altering the diagonal SGS stresses. In view of the energy dissipation/backscatter, the model as expressed in Eq. (10) is thus similar to the one-term Leonard model.

Alternatively, Rotta's anisotropic dissipation model, proposed originally for RANS modelling [17], can be used analogously for ε_{ij}. It is assumed that

$$\varepsilon_{ij} = \frac{\varepsilon}{k}\tau_{ij}^M \tag{11}$$

where τ_{ij}^M is used for convenience of further discussion. Using Eq. (11) in Eq. (8), one gets

$$\tau_{ij} = \left(C_l\bar{\Delta}\right)^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \frac{C_\varepsilon \bar{\Delta}^2}{2\nu}\frac{\varepsilon}{k}\tau_{ij}^M \tag{12}$$

In Eq. (12), if we are willing to (recursively) model τ_{ij}^M in the same form as for τ_{ij}, namely, $\tau_{ij}^M = \tau_{ij}$, the model will return to a form proportional to $\tau_{L,ij}$, but with a different coefficient. This is, however, not the purpose with the present work. Instead, we have estimated τ_{ij}^M here using an eddy viscosity model, namely, $\tau_{ij}^M = -2\nu_{sgs}\bar{S}_{ij}$. Incorporating the relations of $k \sim \nu_{sgs}^2/\bar{\Delta}^2$ and $\varepsilon \sim \nu_{sgs}^3/\bar{\Delta}^4$, Eq. (12) is further approximated in the following form.

$$\tau_{ij} = \left(C_l\bar{\Delta}\right)^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} - C_d R_{sgs}\nu_{sgs}\bar{S}_{ij} \tag{13}$$

where C_d is a model coefficient and $R_{sgs} = \nu_{sgs}/\nu$ is the SGS turbulence Reynolds number, indicating the intensity of modelled SGS turbulence. Obviously, the second term in the model, Eq. (13), contributes a positive part to the total local energy flux, $\varepsilon = -\tau_{ij}\bar{S}_{ij}$. Consequently, this term plays a role in energy forwardscatter from large to small eddies, which makes the model, as a whole, similar to a mixed model. When SGS turbulence kinetic energy, k, is concerned, moreover, one should be able to deduce k from the trace of τ_{ij} by taking $k = \tau_{kk}/2$. In this case, Eqs (10) and (13) should thus be written with an inclusion of a trace, τ_{ij}^{tr}, on the right-hand side in the form of

$$\tau_{ij}^{tr} = -\frac{1}{3}\tau_{kk}\delta_{ij} + \frac{2}{3}k\delta_{ij} \tag{14}$$

Note that the previous mixed models by, e.g., Clark [2], Vreman et al. [12] and Winckelmans et al.[14], have been constructed by keeping only the first term in the reconstruction series (or a Taylor expansion) of the SGS stress tensor, whereas the Smagorinsky model is added as a supplement, or as a compensation of the truncated higher-order terms. By a Leonard expansion for the viscous dissipation rate tensor, ε_{ij}, it is shown here that the second term in the reconstruction series of τ_{ij} is actually exploitable in terms of ε_{ij}. The present approximation provides thus a more plausible modelling argumentation, by which the SGS eddy-viscosity formulation may declare its root in the viscous dissipation rate tensor incurred in the second term of the reconstruction series.

The presence of an eddy viscosity term in the resulting two-term model depends on the modelling of ε_{ij}. Different from previous Clark-type mixed models, it should be noted that the the SGS turbulence Reynolds number, R_{sgs}, has been brought in the present formulation, being multiplied to the eddy-viscosity term, as shown in Eq.(13).

The SGS eddy viscosity, ν_{sgs}, in the model, Eq.(13), can be estimated from the Smagorinsky model or from any other existing eddy-viscosity models. The remaining issue for the model to be used in actual LES is the model coefficients, C_l and C_d, as expressed in Eq.(13). In the Clark model $C_l = 1/\sqrt{12}$, and $C_l = 1.0$ in the model by Winckelmans et al. using explicit filtering. The dynamic procedure can be well exploited to determine the coefficient of the second term in a similar manner as by Vreman et al. [12] and Winckelmans et al. [14]. Another issue is the filter width, $\bar{\Delta}$, which has been assumed being isotropic in the present reconstruction series. It is noted here that anisotropic filter width is also applicable, as shown by Carati et al. [15]. For simplicity, in the present computation we have taken the isotropic filter width of $\bar{\Delta} = (\bar{\Delta}_x\bar{\Delta}_y\bar{\Delta}_z)^{1/3}$ with the top-hat filter kernel implicitly incorporated in a LES solver using finite volume method.

3. Analysis in LES for Channel Flow

In *a priori* test and/or actual LES with explicit filtering for decaying isotropic turbulence and turbulent channel flow, Winckelmans et al. [13, 14] have made a thoroughout investigation on the Leonard tensor-diffusivity term supplemented by a dynamic Smagorinsky model. Some of the major findings in their studies include [14]:

- The Leonard term is able to provide significant local energy backscatter, while remaining globally dissipative.

- In spite of high correlation between the Leonard term and the true SGS stress in *a priori* tests with DNS data, this term alone is not able to produce good LES prediction, due to insufficient global dissipation.

- The mixed model, using the Leonard term supplemented by a dynamic Smagorinsky model, provides (slightly) better or similar LES predictions, as compared to the dynamic Smagorinsky model used alone.

- The Leonard term may well suffice for practical reconstruction of the SGS stress. However, the truncating modelling needs to be further addressed, apart from being supplemented with the Smagorinsky model.

In the present work, the second term is exploited in terms of the viscous dissipation rate tensor, ε_{ij}. It is thus expected that its modelling should be more *guidable* with a model for ε_{ij}. Instead of using a sophisticated formulation, we have adopted here a simple anisotropy assumption for ε_{ij} in analogy to the Rotta assumption for RANS modelling. The second term has thus been approximated in an eddy-viscosity formulation, as given in Eq. (13).

It should be further noted that, on the basis of the expressions derived in Section 2, one may reach a hierarchy of different reconstruction-based models for τ_{ij} upon the approximation used, respectively, for ε_{ij} in Eq. (8), for τ_{ij}^M in Eq. (12) and for ν_{sgs} in Eq. (13). In this work, only is the modelling approximation in Eq. (13) explored, using the Smagorinsky model for ν_{sgs}, namely, $\nu_{sgs} = (C_s\bar{\Delta})^2|\bar{S}|$.

The dynamic procedure is not used in this work to determine the model coefficient for the second Smagorinsky term of Eq. (13). Instead, we have taken constant model coefficients to calibrate their effects in LES for turbulent channel flow. This serves well the primary purpose with the present work: to analyze the two-term modelling approximation and to highlight its potential development in engineering LES.

We consider the channel flow at $Re_\tau = 550$ with available DNS data [18]. The computational domain is $(L_x, L_y, L_z) = (6.4, 2, 3.2)$ meshed with $64 \times 80 \times 64$ cells. This resolution is comparable to the simulation by Winckelmans et al.[14] for the channel flow of $Re_\tau = 395$, but the streamwise resolution is relatively coarse in the present computation ($\Delta x^+ = 55$ compared to $\Delta x^+ = 39.5$). All the results presented below have been normalized using the friction velocity, u_τ, and the half-channel height, $h = L_y/2$, denoted with a superscript "+".

The computation starts with an exploration of the effect of the model coefficients, C_l and C_d, by keeping the Smagorinsky constant $C_s = 0.1$ in ν_{sgs} in the second term of Eq.(13). It was found that the solution blow up by setting $C_l = 1.0$, whether the Leonard term is used alone or combined with the second term. This term has induced large and negative local diffusion, triggering numerical instability, which is consistent with the observation by Winckelmans et al. [14]. Such a numerical instability problem may be remedied partly by using anisotropic filter width in each direction for the first term, as demonstrated by Vreman [19]. In the present work, however, an isotropic filter width has been invoked for both the first and the second terms. Moreover, an empirical damping function has been invoked for the two terms, using $f_d = 1 - \exp(-y^+/10)$. This helps also to reduce large directional diffusion in the vicinity of the wall, particularly, by the Leonard term. Consequently, the reconstruction-based mixed model adopted in the present study takes the following form.

$$\tau_{ij} = \tau_{L,ij} + \tau_{S,ij} = \left(C_l\bar{\Delta}\right)^2 f_d \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} - C_d R_{sgs} f_d^2 \nu_{sgs} \bar{S}_{ij} \quad \text{with} \quad \nu_{sgs} = (C_s\bar{\Delta})^2|\bar{S}| \qquad (15)$$

The Leonard term plays, locally and instantaneously, a role in energy backscatter, but globally and statistically, this term renders energy dissipation. In the LES calibration with the Leonard term alone, it was found that a large value of C_l provided insufficient energy dissipation, particularly, in the log-layer. As $C_l \geq 0.3$, in the computation with only the Leonard tensor-diffusivity model, the model introduces negative diffusion, which has to be limited so that it does not exceed the magnitude of the viscous diffusion to avoid numerical instability problem. Such a limit is however inappropriate, which tends to cancel the viscous diffusion, leading to a "locally inviscid" flow simulation for large values of C_l. Figure 1 shows the LES predictions using only the Leonard term with different values of C_l. With $C_l \leq 0.2$, the Leonard model alone has induced only marginal difference in the profile for the predicted mean velocity, as compared to the simulation with no model, but having brought the prediction closer to the DNS data for the resolved stresses, particularly, in the log-layer region.

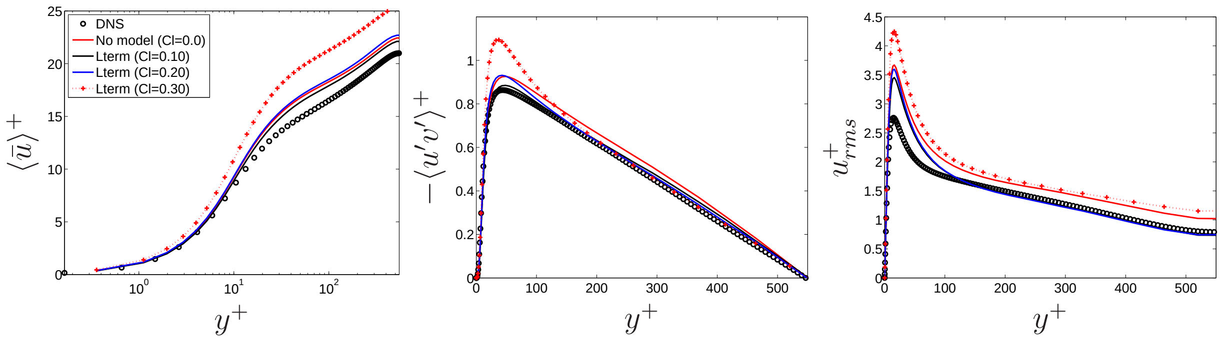

Figure 1: LES using only the Leonard term with different values of C_l. For $C_l = 0.3$, the Leonard term has been limited by viscous diffusion for negative values. Left: Mean streamwise velocity; Mid: Resolved shear stress; Right: Streamwise velocity fluctuation.

With the two-term mixed model, Eq. (15), the presence of R_{sgs} in the second term enhances the modelled SGS turbulent diffusion for $R_{sgs} > 1$. In the computation with $C_l > 0.75$ and $C_d \leq 1.0$, it was found that the solution became numerically unstable, due to large negative diffusion caused by the Leonard term. The effect of R_{sgs} in the Smagorinsky term can be removed by setting the model coefficient $C_d = 2/R_{sgs}$, which makes consequently the second term a *standard* Smagorinsky model. In this case, it was found that C_l should have a value of $C_l \leq 0.5$ to maintain a stable numerical procedure.

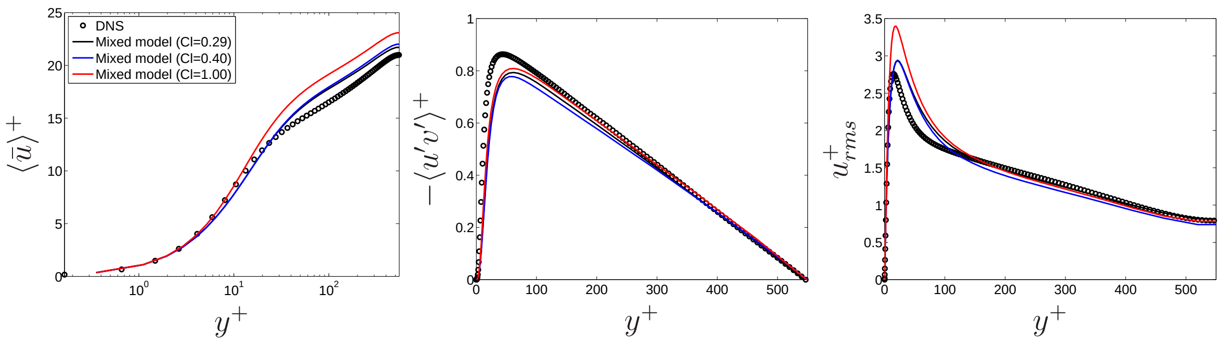

Figure 2: LES using the two-term mixed model with different values of C_l. $C_d = 2/R_{sgs}$ and $C_s = 0.1$ have been used in Eq. (15). For $C_l = 1.0$, the Leonard term is limited by turbulent diffusion for negative values. Left: Mean streamwise velocity; Mid: Resolved shear stress; Right: Streamwise velocity fluctuations.

In Figure 2, the effect of C_l is illustrated with the two-term mixed model, where $C_s = 0.1$ and $C_d = 2/R_{sgs}$ have been used. For $C_l = 1.0$, local negative values of the Leonard term have been "cut-off" when their magnitudes are larger than SGS turbulent diffusion. It is shown that a relative large contribution of the Leonard term (with a large C_l value) may enhance the velocity fluctuations (only the streamwise fluctuation is shown here, but the same is observed for the velocity fluctuations in the other two directions). This implies that, by means of energy backscatter, the Leonard term has indeed contributed to the suppression of global energy dissipation in the mixed model.

Obviously, there is an inherent connection between the first Leonard term and the second Smagorinsky term. Together, they should compensate with each other in terms of energy back and forward scatter. To render good predictions, the model coefficient should be set as a function of local flow properties. Investigation on this has been carrying out in a separate work. In the present work, we focus on an analysis of the behavior of each term in the mixed model with constant model coefficients.

After a number of testing, we have set $C_l = 1/\sqrt{12}$, which is the constant for the Leonard term employed in the mixed model by Clark et al. [2]. The effect of C_d has then been explored. In Figure 3, the LES predictions obtained, respectively, with $C_d = 1.0$, $C_d = 4.0$ and $C_d = 2/R_{sgs}$ in Eq. (15), are compared. Note that, for $C_d = 2/R_{sgs}$, the mixed model has actually been composed of the Leonard term and the *standard* Smagorinsky term. With $C_d = 1.0$, the two-term model gives appreciably improved predictions for the mean streamwise velocity and for its fluctuations, as compared with the Smagorinsky model ($C_s = 0.1$, and the damping function is incorporated in ν_{sgs}). Increasing the values of C_d does not provide any significant improvement in the prediction. When a too large values of C_d is used, e.g., $C_d = 4.0$, the Smagorinsky term in the mixed model becomes overall dominant and gives excessive SGS turbulent diffusion. This has consequently dampened to some extent the near-wall velocity fluctuations in all directions. Moreover, the resolved streamwise velocity presents sensible discrepancies from the DNS data in the viscous sublayer and in the buffer layer, in spite of slight improvement in the log-layer. With $C_d = 2/R_{sgs}$, the predicted profiles for the streamwise velocity and for its fluctuations are similar to those with $C_d = 1.0$, but the wall-normal and spanwise velocity fluctuations have been more dampened in the near-wall layer (for $y^+ < 100$). This is partly due to the presence of R_{sgs} in the second term for $C_d = 1.0$, which has quickly dropped down to smaller values ($R_{sgs} < 1$) away from the buffer layer. Consequently, the energy dissipation caused by the Smagorinsky term is reduced, whereas by setting $C_d = 2/R_{sgs}$ the effect of R_{sgs} is removed from the Smagorinsky term. It should be admitted that, with the current model coefficients (either $C_d = 1.0$ or $C_d = 2/R_{sgs}$), the function of energy transfer inherent in the Leonard and the Smagorinsky term needs to be further addressed.

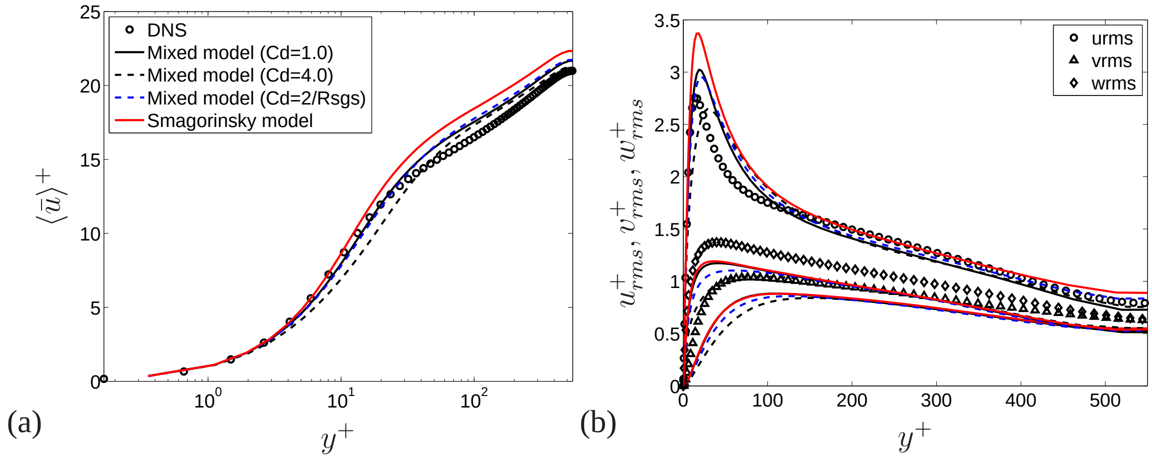

Figure 3: LES using the two-term mixed model (Eq. (15)) with different C_d values by setting $C_l = 1/\sqrt{12}$ and $C_s = 0.1$. (a) Mean streamwise velocity. (b) Velocity fluctuations (only every other DNS data have been plotted).

In what follows, the mixed model using $C_l = 1/\sqrt{12}$ and $C_d = 1.0$ in Eq. (15) is further explored to highlight the property of the Leonard term. In Figure 4 (a), the modelled diagonal Leonard stresses, $\tau_{L,kk}$, are plotted. As shown, the Leonard term gives rise of a large diagonal stress in the streamwise direction, $\tau_{L,11}$, due to large wall-normal gradient of the streamwise velocity in the near-wall layer. The wall-damping effect has also been reflected in $\tau_{L,22}$. By contrast, the Smagorinsky diagonal SGS stresses, $\tau_{S,kk}$, are much smaller, as illustrated in Figure 4 (b). Note that, due to continuity, the sum of the Smagorinsky diagonal SGS stresses should be zero, as shown in Figure 4 (b).

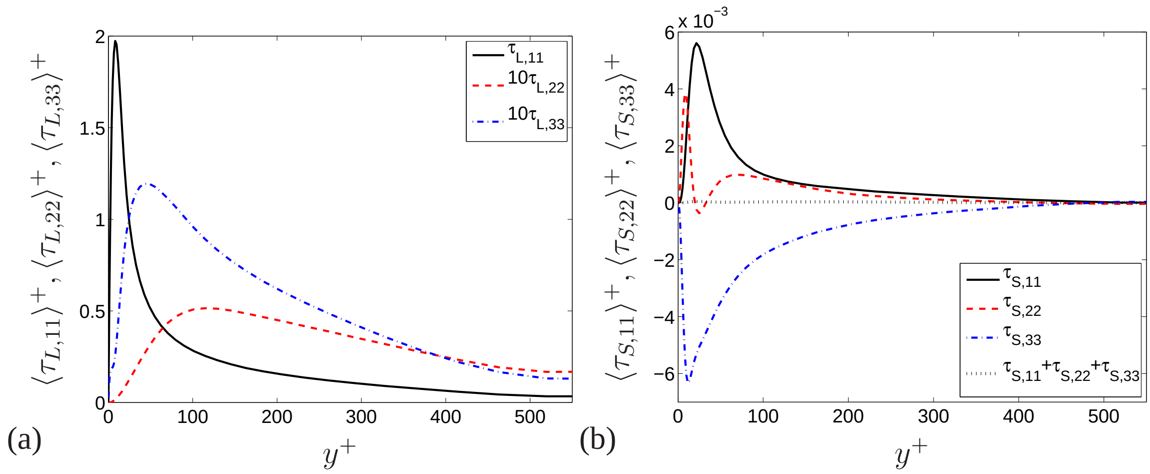

Figure 4: Two-term mixed model, with $C_d = 1.0$, $C_l = 1/\sqrt{12}$ and $C_s = 0.1$ in Eq. (15). (a) Modelled diagonal stresses by the Leonard term. (b) Modelled diagonal stresses by the Smagorinsky term.

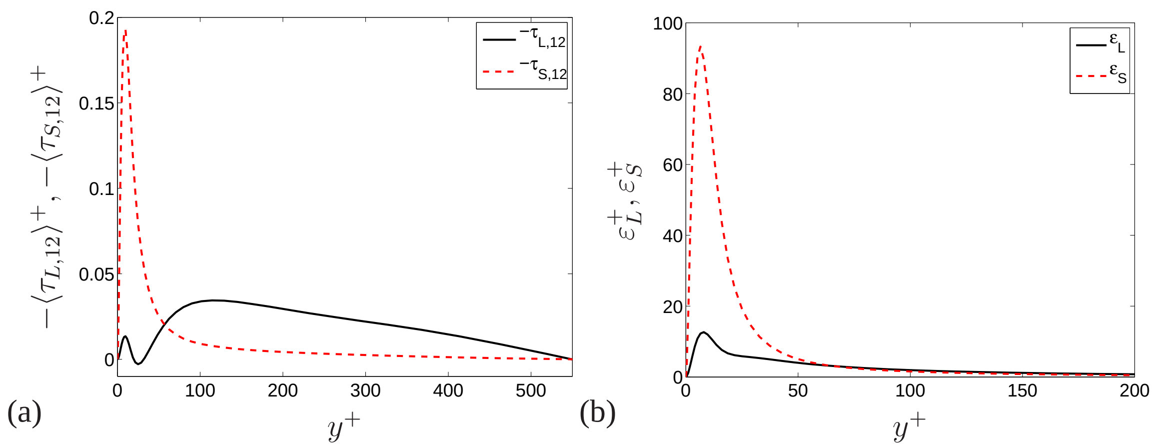

Figure 5: Two-term mixed model with $C_d = 1.0$, $C_l = 1/\sqrt{12}$ and $C_s = 0.1$ in Eq. (15). (a) Modelled shear stress by the Leonard ($\tau_{L,12}$) and the Smagorinsky ($\tau_{S,12}$) term, respectively. (b) Modelled SGS dissipation by the two terms.

In Figure 5 (a), a comparison is made for the modelled SGS shear stress given by the Leonard term ($\tau_{L,12}$) and by the Smagorinsky term ($\tau_{S,12}$), respectively. As seen, the Smagorinsky term is dominant over the Leonard term in the near-wall layer, while it is becoming smaller in the log layer. It should be noted that the Leonard shear stress may become negative close to the wall, a behavior similar to the counter-gradient diffusion. In Figure 5 (b), the time-averaged modelled SGS dissipation is compared, due to the Leonard term and the Smagorinsky term, respectively.

As expected, the energy dissipation induced by the Leonard term, $\varepsilon_L = -\tau_{L,ij}\bar{S}_{ij}$, is much smaller close the wall than by the Smagorinsky term, $\varepsilon_S = -\tau_{S,ij}\bar{S}_{ij}$, generally less than 15% of the total energy dissipation for $y^+ \leq 15$. In the log-layer after $y^+ = 50 - 60$ away from the wall, however, ε_L becomes comparable to (and even slightly larger than) ε_S. This is partly due to the fact that an appreciable amount of energy backscatter induced by the Leonard term occurs usually near the wall. In the future work, this will be further explored by distinguishing energy backscatter of ε_L from the global energy dissipation.

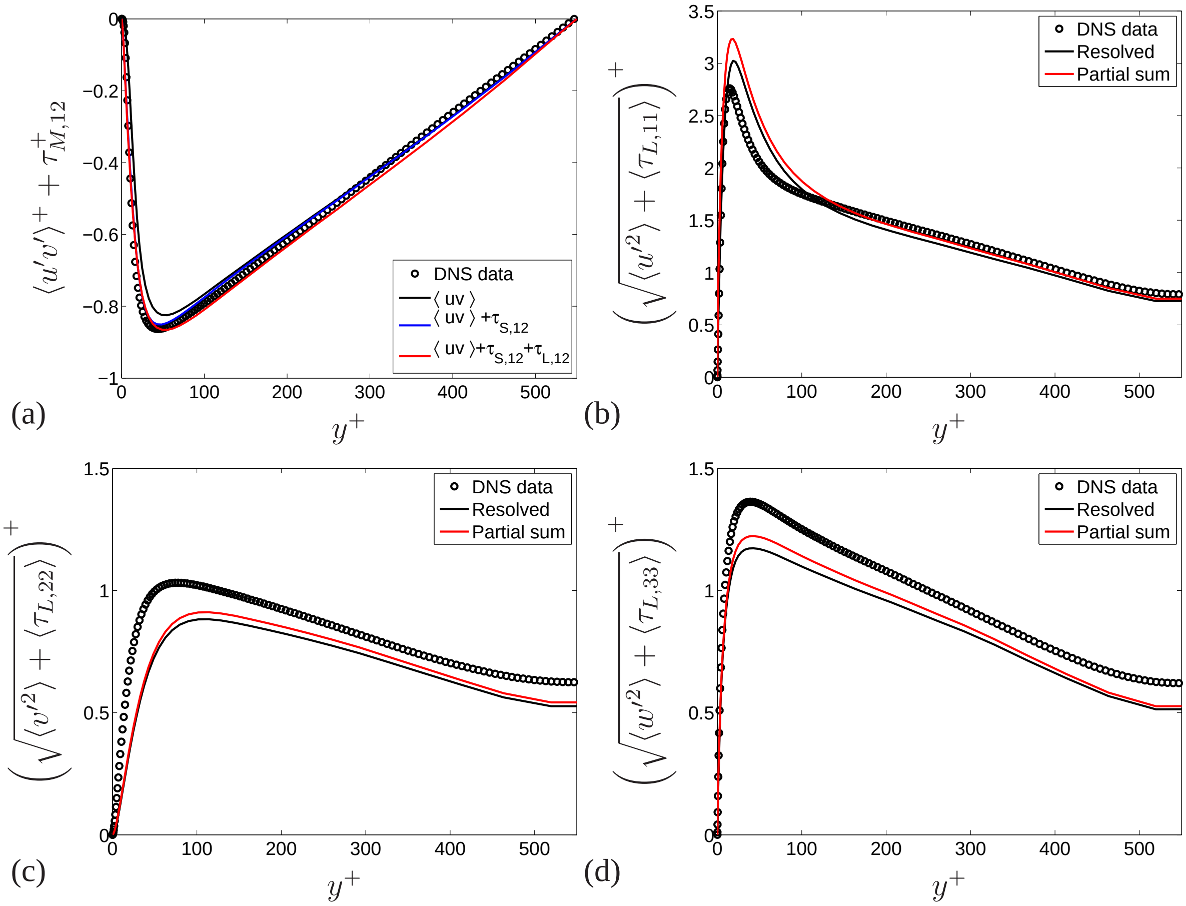

Figure 6: Two-term mixed model with $C_d = 1.0$, $C_l = 1/\sqrt{12}$ and $C_s = 0.1$ in Eq. (15). (a) Sum of resolved and modelled turbulent shear stress. (b) Streamwise velocity fluctuations, (c) Wall-normal velocity fluctuations. (d) Spanwise velocity fluctuations.

The summation of resolved and modelled turbulent shear stress is illustrated in Figure 6 (a). As shown, the resolved turbulent shear stress, $\langle u'v' \rangle$, is compensated by the modelled SGS shear stress contributed, respectively, by the Leonard and by the Smagorinsky term, resulting in an improved comparison with DNS data. In the vicinity of the wall the major contribution to the total turbulent shear stress is due to the Smagorinsky term, and in the outer part the contribution from the Leonard term becomes more sensible. As illustrated in Figure 4, the modelled Leonard diagonal stresses possess fairly large values. It is thus interesting to observe their contribution to the velocity fluctuations. Assuming that the resolved fluctuation is ϕ_r and its unresolved SGS counterpart is ϕ_s, the R.M.S. of the total fluctuation should then be $\phi_{rms} = \sqrt{\langle (\phi_r + \phi_s)^2 \rangle} = \sqrt{\langle \phi_r^2 \rangle + \langle \phi_s^2 \rangle + 2\langle \phi_r \phi_s \rangle}$, in which the first term is obtainable from the resolved part and the second term can be approximated from the SGS modelled part, whereas the third term is not extracted herewith. We consider here only the effect of the Leonard term and the unresolved fluctuations is estimated simply by a contraction of $\tau_{L,ij}$. In Figure 6, therefore, only an estimated sum of the first two terms has been plotted for the velocity fluctuations, namely, $u_{i,rms} = \sqrt{\langle u_i'^2 \rangle + \langle \tau_{L,ii} \rangle}$ (no summation for i here). As shown, the modelled diagonal stresses may make sensible contributions to the velocity fluctuations.

4. Conclusions

An approximation of the SGS stress tensor has been discussed on the basis of its reconstruction series. Unlike previous modelling approximation, stemmed from the Leonard expansion and truncating all the higher-order terms but the first Leonard term, it is shown that the second term in the reconstruction series can actually be further exploited in relation to the viscous dissipation rate tensor present in the transport equation of τ_{ij}, being further subjected to a Leonard expansion. This has consequently led to a two-term model for the SGS stresses.

The approximation of the second term is accomplished by formulating the viscous dissipation rate tensor, ε_{ij}, in analogy to RANS modelling. The local-isotropy assumption of ε_{ij} renders the second term lack of the desirable function for energy dissipation, but altering only the diagonal SGS stresses, and the resulting two-term model returns to a scale-similarity type model. With an anisotropy assumption for ε_{ij}, nonetheless, the second term can be cast in an eddy-viscosity formulation. Consequently, the resulting two-term model attains to a mixed model. The present approximation indicates that the use of an eddy-viscosity (dissipative) term in the reconstruction-based mixed model is rooted in an anisotropy assumption of the viscous dissipation rate tensor in the transport equation for τ_{ij}.

The two-term mixed model has been investigated in LES for turbulent channel flow. The emphasis is placed on the exploration of the effect of model coefficients. It was found that, when the Leonard term is used alone in LES, the model coefficient to this term has to be kept below $C_l \approx 0.3$ to maintain a stable numerical procedure for locally negative diffusion. This is due partly to the truncation of higher-order terms from the reconstruction series, and partly to the use of an isotopic filter width in present LES. Nonetheless, this has also reflected the important function of the Leonard term to account for energy backscatter. The inclusion of the second Smagorinsky term helps to reinforce the energy dissipation, and enabling a relatively large value for C_l. The two terms in the mixed model interact with each other in terms of energy back and forward scatter. Usually, when the second term is increased (with an increased value of C_d), the first term can use a large value of C_l. The value of C_d should be restricted, however, in order to appropriately represent SGS turbulent diffusion and energy dissipation. On the other hand, increasing the value of C_l (namely, increasing the Leonard term) may usually enhance the streamwise velocity fluctuations but dampen the velocity fluctuations in the other two directions. With $C_l = 1/\sqrt{12}$ and $C_d = 1.0$ in present LES for turbulent channel flow, the two-term mixed model is able to provide reasonable predictions.

Furthermore, the Leonard shear stress displays a tendency of becoming negative, indicating a potential behavior similar to the counter-gradient diffusion in the near-wall layer. The global energy dissipation introduced by the Smagorinsky term is much larger than by the Leonard term. In the vicinity of the wall ($y^+ \leq (10-15)$), in general, the Leonard term accounts for only about $10 - 15\%$ of the total energy dissipation. In the log-layer, this term may contribute an amount of energy dissipation that is comparable to the Smagorinsky term. Further investigation of the reconstruction-based mixed model has been being undertaken on the function of the Leonard term in terms of energy backscatter.

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