REYNOLDS STRESS TRANSPORT MODELLING OF
TRANSONIC FLOW
AROUND THE RAE2822 AIRFOIL

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This paper presents a second moment closure Reynolds stress turbulence model implementation
applied to a simulation of a well known transonic airfoil flow case (RAE2822). Comparisons are made
between simpler turbulence models and the present implementation regarding its ability to predict the
shock location and the shock-boundary layer interaction. To avoid explicitly added numerical dissipation
in the mean flow equations, we take an approach where the Euler term, containing first derivatives, is
discretized using the third order QUICK scheme. Comparisons are presented between a standard central
scheme and the present approach. The results show that the Reynolds stress turbulence model is superior
to the other turbulence models tested in predicting the shock location and the flowfield downstream of
the shock, and that the implementation of the QUICK scheme significantly increases the accuracy on
coarse meshes, while the need for explicitly added numerical dissipation is greatly reduced. For the
simulations carried out in this work, no numerical dissipation was needed at all.

1 Introduction

Simpler turbulence models, like the Baldwin-Lomax\(^1\) and
the \(k-\varepsilon\) models, which are widely used in Navier-Stokes
solvers, often fail to accurately predict flow cases where
large separation regions occur. Important cases are stall
at high angles of incidence,\(^2\) and separation and reattach-
ment due to shock-boundary layer interaction. Davidson
and Rizzi\(^3\) have succeeded in predicting stall on an air-
foil using an algebraic Reynolds stress model, and this pa-
per investigates whether a more complex turbulence model
is able to accurately predict the shock-boundary layer in-
teraction. The second-moment closure model was imple-
mented by Davidson\(^4\) into a cell-centered finite-volume time
marching Runge-Kutta code, originally written by Rizzi and
Müller.\(^5\) The Reynolds Stress transport equations are de-
coupled from the averaged flow equations and solved sepa-
rately using a semi-implicit solver for the steady equations.\(^6\)
Comparisons are made with results from experiments and
closer implementations of simpler turbulence models into
the same code.\(^6,2\)

It is sometimes argued that the accuracy offered by ad-
vanced turbulence models is overwhelmed by the numerical
dissipation, that has to be added to the averaged discretized
Navier-Stokes equations when central differencing is used.
Indeed, in our earlier work\(^7\) we devised a function to turn

off the artificial dissipation in the boundary layer in order to
preserve the nature of the physical viscosity. This however is
\textit{ad hoc} and requires knowledge of where the wall is located.
In this paper we take another approach, an upwind scheme
with higher spatial resolution. We focus on the boundary
layer where compressibility is negligible and discretize the
inviscid flux terms with the QUICK scheme that is rather
popular for incompressible flow. The intention was that the
third-order truncation error term arising from the QUICK
discretizations\(^8\) would be small enough to not destroy the
physical viscosity, but would be sufficient to stabilize the
solution without the need for numerical dissipation. This
scheme was then used in some numerical simulations, and
our hypothesis was then tested by comparing the results
with those obtained using the earlier implementation.\(^5\)

2 Governing equations

2.1 The compressible Navier-Stokes equations

2.1.1 Averaging procedures

Turbulence modelling is based on the assumption that
turbulent motion in a fluid can be described statistically.
The fluid variables can, if this assumption is valid, be de-
scribed as \(\Phi = \bar{\Phi} + \phi\), where \(\bar{\Phi}\) is the instanta-
neous value of the variable, \(\bar{\Phi}\) is the time average of \(\Phi\) and \(\phi\) is the flu-
cating component, the time average of which, \(\bar{\phi}\), is zero.
To make it clear what is meant by time averaged, the following definition is made: Let be an arbitrary quantity varying in time. Then the time averaged quantity at time \( t_0 \) is defined by

\[
\Phi(t_0) = \frac{1}{2t} \int_{t_0 - t}^{t_0 + t} \Phi(t) dt
\]  

(1)

where \( t \) is a sufficiently long time period to include the largest time scales in the varying quantity.

There is, however, an averaging procedure called mass weighted averaging, which is more convenient for compressible flows. Mass weighted averaging is defined by the following equations:

\[
\dot{\Phi} = \ddot{\Phi} + \dot{\phi}
\]  

(2)

\[
\ddot{\Phi} = \frac{\dot{\Phi}}{\rho} \dot{\rho} \neq 0
\]  

(3)

The two averaging procedures are linked with the relation

\[
\ddot{\Phi} = \frac{\dot{\Phi}}{\rho} + \frac{\dot{\phi}}{\rho}
\]  

(4)

For simplicity, throughout the paper the notation \( \Phi \) is used instead of \( \ddot{\Phi} \) where possible.

For the flow cases studied in this paper, it is assumed that the second term in equation (4) is small compared with the first term, and thus \( \Phi \approx \ddot{\Phi} \).

2.1.2 Scaling of the variables

To make the flow variables functions of the freestream Mach-, Reynolds-, and Prandt numbers only, they are made dimensionless by scaling as described in, for example [5].

2.1.3 The averaged Navier-Stokes equations

The Navier-Stokes equations for the averaged variables read

\[
\frac{\partial}{\partial t} \int_V \Phi dV + \int_{\partial V} (F_i(\Phi) + F_i^* (\Phi)) n_i dS = 0
\]  

(5)

where \( V \) is an arbitrary control volume with boundary \( \partial V \) and outward-pointing boundary unit normal \( n \), and

\[
\Phi = \begin{bmatrix}
1 \\
U_1 \\
U_2 \\
E
\end{bmatrix} : F_i - U_i \Phi + P
\]  

(6)

\[
F_i^* = - \begin{bmatrix}
0 \\
\Sigma_{i}^{ij} f_j \\
\Sigma_{i}^{ij} C_j \\
\frac{\Sigma_{i}^{i} U_j - \rho u_i c}{\Theta_i}
\end{bmatrix}
\]  

(7)

and

\[
E = \frac{1}{2} |U|^2 + c_\theta \Theta + k
\]  

(8)

\[
P = (\gamma - 1) \rho c_\theta \Theta
\]  

(9)

\[
\Sigma_{ij}^{ij} f_j = \mu \left( (U_{j,i} + U_{i,j}) - \frac{2}{3} \delta_{ij} U_{m,m} \right) - \rho u_i u_j
\]  

(10)

\[
k = \frac{1}{2} \rho u_i u_i
\]  

(11)

The correlation \( \Sigma_{ij} U_j \), the turbulent diffusion of energy, \( \rho u_i c \), and the last term in equation (10), \( \rho u_i u_j \), or the Reynolds stress tensor, are left for the turbulence models to provide.

2.1.4 Boundary conditions

The farfield boundary conditions are based on the theory of characteristics for the locally one-dimensional problem normal to the boundary. Details can be found in, for example, [6]. On the wall, the no-slip boundary condition is imposed for the velocity field, \( U_i = 0 \) and a homogeneous Neumann boundary condition is imposed for the temperature, \( T \), and the pressure, \( P \).

2.1.5 Modelling of the energy equation

The modelling of the unknown correlations in the energy equation is in this work the same for both turbulence models used, and it is expressed as

\[
\Sigma_{ij} U_j - \rho u_i c + \kappa \Theta_i = \Sigma_{ij}^{ij} U_j + \kappa^{ij} U_j
\]  

(12)

where a new quantity, the effective heat conductivity or \( \kappa^{ij} \), has been introduced. The definition of this quantity is deferred to section 2.2.

The main approximation made in this model of the averaged energy equation is, that the turbulent diffusion of the turbulent kinetic energy, \( k \), is considered small compared to the other terms in the equation and it is therefore neglected.

2.2 The eddy-viscosity assumption

Most of the simpler turbulence models do not explicitly calculate the Reynolds stress tensor, but use instead the mathematical concept of eddy viscosity or turbulent viscosity, \( \mu_t \). In the eddy-viscosity assumption, the definition of the stress tensor \( \Sigma_{ij}^{ij} \) in equation (10) becomes

\[
\Sigma_{ij}^{ij} = (\mu + \mu_t) \left( (U_{j,i} + U_{i,j}) - \frac{2}{3} \delta_{ij} U_{m,m} \right)
\]  

(13)
The effective heat capacity, $\kappa^{eff}$, left undefined above, is, when used together with the eddy-viscosity assumption defined as:

$$\kappa^{eff} = \frac{\gamma}{\gamma - 1} \left( \frac{\mu}{Pr_t} + \frac{\mu_s}{Pr_t^s} \right)$$  \hspace{1cm} (14)$$

where $Pr_t$, the turbulent Prandtl number, is usually set to 0.9.

The only variable now left for the turbulence model to predict is the turbulent viscosity, $\mu_t$.

### 2.3 The $k-\varepsilon$ model

In the $k-\varepsilon$ model, two transport equations are solved in addition to the averaged flow equations. One for the turbulent energy, $k$, and one for its dissipation rate, $\varepsilon$. The Reynolds stress tensor is then calculated using the Boussinesq assumption:

$$-\overline{\rho u_i u_j} = -\nu \frac{2}{3} \delta_{ij} U_{mm} - \frac{2}{3} \delta_{ij} \overline{\rho k}$$  \hspace{1cm} (15)$$

where the turbulent viscosity is given by

$$\mu_t = C_{\mu} \frac{k^2}{\varepsilon}$$  \hspace{1cm} (16)$$

The constant $C_{\mu}$ is usually taken to 0.09.

Using the assumption that, for a turbulent quantity, $\Phi \approx \Phi\kappa$, the transport equations for $k$ and $\varepsilon$ read:

$$\frac{\partial}{\partial t} \left( \overline{\rho k} \right) = -\left( \overline{\rho U_i U_j} \right)_j + \left( \rho + \frac{\mu}{\sigma_k} \right) k_j + P_k - \overline{\rho \varepsilon}$$  \hspace{1cm} (17)$$

$$\frac{\partial}{\partial t} \left( \overline{\rho \varepsilon} \right) = -\left( \overline{\rho U_i \varepsilon} \right)_j + \left( \rho + \frac{\mu}{\sigma_\varepsilon} \right) \varepsilon_j + \frac{\varepsilon}{k} (c_{1\varepsilon} P_k - c_{2\varepsilon} \overline{\rho \varepsilon})$$  \hspace{1cm} (18)$$

where

$$P_k = -\overline{\rho u_i u_j} U_{ij}$$  \hspace{1cm} (19)$$

and the turbulent Prandtl numbers are set to $\sigma_k = 1.0$ and $\sigma_\varepsilon = 1.3$. The constants $c_{1\varepsilon}$ and $c_{2\varepsilon}$ are set to $c_{1\varepsilon} = 1.44; c_{2\varepsilon} = 1.92$.

#### 2.3.1 Near-wall treatment

The transport equations for $k$ and $\varepsilon$ are designed for high Reynolds-number flow. Therefore a special treatment is needed near the wall. Several near-wall models have been designed, and the one used here is a model by Wolfstein,\textsuperscript{8} modified by Chen and Patel.\textsuperscript{9} It is a one-equation model, in which only the $k$-equation is solved. The dissipation rate, $\varepsilon$, and the turbulent viscosity, $\mu_t$, are then calculated using

$$\varepsilon = \frac{k^2}{l_e}; \mu_t = C_{\mu} \sqrt{k \nu}$$  \hspace{1cm} (20)$$

with the turbulent length scales, $l_t$ and $l_e$, prescribed as

$$l_t = c_{\mu} \left( 1 - e^{-\frac{n^+}{\delta}} \right); l_e = C_{\mu} n \left( 1 - e^{-\frac{n^+}{\delta}} \right)$$  \hspace{1cm} (21)$$

with

$$R_n = \frac{\overline{\rho \sqrt{k \nu}}}{\mu}$$  \hspace{1cm} (22)$$

and the constants defined as

$$C_{\mu} = 0.09; c_t = \frac{k^2}{\varepsilon}; A_n = 70; A_e = 2 c_t$$

The near-wall model is used out to a distance of order $n^+ - 50$ from the wall, and the matching line is, in this work, chosen along a preselected grid line. It should be noted, that the value of $\varepsilon$ calculated using the near-wall model serves as a boundary condition for the transport equation for $\varepsilon$ used in the outer field. Thus, a discontinuity in the value of $\varepsilon$ cannot occur.

#### 2.3.2 Boundary conditions

The far-field boundary conditions for $k$ and $\varepsilon$ are a homogenous Dirichlet boundary condition at the inflow boundary and a homogenous Neumann condition at the outflow boundary.

The wall boundary condition for $k$ is $k = 0$, and the near-wall boundary condition for $\varepsilon$ transport equation has been discussed above.

### 2.4 The Reynolds stress model

The compressibility effects are neglected and the modelled Reynolds stress equations become:

$$\frac{\partial}{\partial t} \left( \overline{\rho u_i u_j} \right) = -\left( \overline{\rho u_i u_j} \right)_j + \left( \rho + \frac{\mu}{\sigma} \right) \frac{\partial \overline{u_i u_j}}{\partial x_j}$$  \hspace{1cm} (23)$$

#### 2.4.1 Second moment closure

The dissipation term, $(c_{ij})$, the diffusion term, $(D_{ij})$, and the pressure-strain term $(P_{ij})$, which promotes isotropy turbulence, all include unknown correlations and are modelled in a standard way.\textsuperscript{10}

Two different ways of modelling the diffusion term, $(D_{ij})$, have been tested. These are:

$$(D_{ij}) = -\left( \mu + \frac{\mu_s}{\sigma} \right) \frac{\partial \overline{u_i u_j}}{\partial x_j}$$  \hspace{1cm} (24)$$

$$(D_{ij}) = -\left( \mu + \frac{\mu_s}{\sigma} \right) \frac{\partial \overline{u_i u_j}}{\partial x_j}$$  \hspace{1cm} (25)$$

the latter expression being somewhat simpler than the first. Both models have been used in this work; the closure model using equation (24) is hereafter called RSM-GGNDH, where
GGDH stands for “Generalized Gradient Diffusion Hypothesis”, and the closure model using equation (25) is simply called RSM.

The constants in the model are

\[ c_1 = 1.8; \quad c_2 = 0.6; \quad \dot{c}_1 = 0.5; \quad \dot{c}_2 = 0.18; \quad C_p = 0.09; \quad \sigma_p = 1; \quad c_s = 0.22 \]

2.4.2 Modification of the \( k - \varepsilon \) equations

In addition to the Reynolds Stress transport equations, the \( k \) and \( \varepsilon \) transport equations have also to be solved. They have the same form as discussed in Section 2.3, with the exception that the production term, \( P_k \), now uses the Reynolds stress tensor calculated from the RSM equations instead of the Boussinesq assumption in the outer region. In the inner region, where the one-equation model is used, the Boussinesq assumption is still used.

In the RSM-GGDH closure model, the diffusion in the \( k \) and \( \varepsilon \) equations is also modelled using the generalized gradient diffusion hypothesis. The \( k \) and \( \varepsilon \) equations then take the form

\[
\frac{\partial}{\partial \mathbf{t}} (\overline{\rho k}) = - \left( \overline{\rho u_j k} \right)_j + \\
+ \left( \left( \mu \varepsilon_m + c_k \frac{k}{\varepsilon_m} \right) \varepsilon_m \right)_{\mathbf{j}} + P_k - \overline{\rho \varepsilon} \tag{26}
\]

\[
\frac{\partial}{\partial \mathbf{t}} (\overline{\rho \varepsilon}) = - \left( \overline{\rho u_j \varepsilon} \right)_j + \\
+ \left( \left( \mu \varepsilon_m + c_k \frac{k}{\varepsilon_m} \right) \varepsilon_m \right)_{\mathbf{j}} + \\
+ \frac{\varepsilon}{k} (c_s P_k - c_s \overline{\rho \varepsilon}) \tag{27}
\]

\[ c_k = 0.22; \quad c_s = 0.17 \]

2.4.3 Near-wall treatment

The modelling of the unknown correlations in the previous section is designated for high Reynolds number flow. Therefore, in the vicinity of a wall, the one-equation model discussed in Section 2.3.1, is used, and the Reynolds stress tensor is then calculated using the Boussinesq assumption.

The matching line, inside of which the one-equation model is used instead of the RSM equations, is in this work chosen to be the same as in the \( k - \varepsilon \) model. In this way, the Reynolds stress tensor, calculated by the Boussinesq assumption and the one-equation model, serves as a boundary condition for the RSM transport equations at the matching line.

2.4.4 Farfield boundary conditions

For the components of the Reynolds stress tensor, a homogenous Dirichlet condition is applied on the inflow boundary and a homogenous Neumann condition on the outflow boundary.

3 The solver

3.1 Solution procedure

When the \( k - \varepsilon \) solver was first implemented in this code, attempts were carried out to couple the \( k - \varepsilon \) transport equations to the averaged flow equations and solve the whole system with a finite-volume time-marching Runge-Kutta procedure. However, no stable convergent solution could be obtained, although a recent paper shows that this is possible. Instead a semi-implicit solver, well known from the SIMPLE algorithm, was implemented to solve these transport equations. This method is also used to solve the transport equations for the Reynolds stress tensor. The solution strategy becomes:

\[ * \text{Obtain Start-approximation for all variables.} \]

REPEAT

\[ * \text{Perform a Runge-Kutta timestep for the averaged flow variables using old values of the Reynolds stress tensor and effective heat conductivity.} \]

\[ * \text{Perform an iteration of the steady \( \varepsilon \) transport equation solver using old values of \( k \) and the Reynolds stress tensor and new values of the averaged flow variables.} \]

\[ * \text{Perform an iteration of the steady \( k \) transport equation solver using old values of the Reynolds stress tensor and new values of the averaged flow variables and \( \varepsilon \).} \]

\[ * \text{Perform an iteration of the steady \( \overline{\mu u^2} \) transport equation solver using old values of the correlations \( \overline{\mu u^2} \) and \( \overline{\mu \varepsilon} \) and new values of \( k \), \( \varepsilon \), \( \overline{\mu u^2} \) and the averaged flow variables.} \]

\[ * \text{Perform an iteration of the steady \( \overline{\mu u^3} \) transport equation solver using old values of the correlation \( \overline{\mu \varepsilon} \) and new values of \( k \), \( \varepsilon \), \( \overline{\mu u^3} \) and the averaged flow variables.} \]

\[ * \text{Perform an iteration of the steady \( \overline{\mu \varepsilon} \) transport equation solver using new values for all quantities.} \]

UNTIL CONVERGENCE

In the above solution method description, old values are values that are obtained from the previous iteration and new values are values that are obtained earlier in the iteration.

3.2 Space discretization

3.2.1 The finite-volume approximation

As seen in equation (5) the Navier-Stokes equations for the averaged flow variables are on integral form, valid for an