2020-10-09, Exam in

Turbulence modeling, MTF271

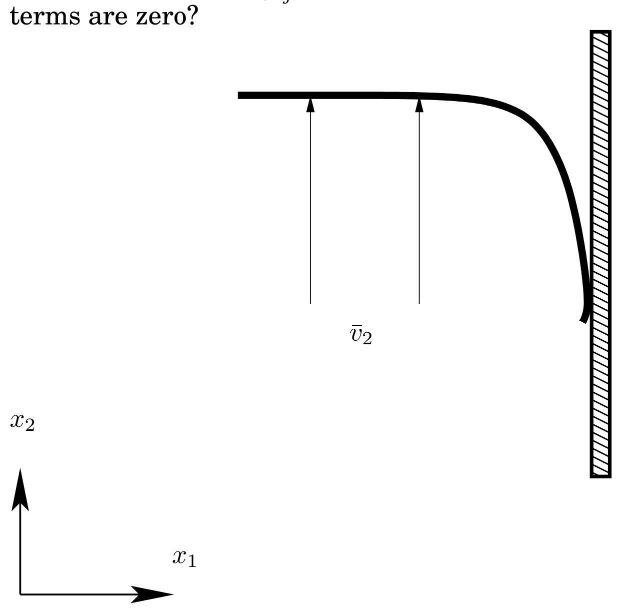
- **Time:** 14.00-18.00 **Location:** Zoom
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **The teacher is available on telephone**
- **Checking the evaluation and results of your written exam at Canvas. If you have questions on the correction of the exam, add a comment at Canvas and send me an Email.**
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Show which terms that are symmetric and/or traceless. (5p)

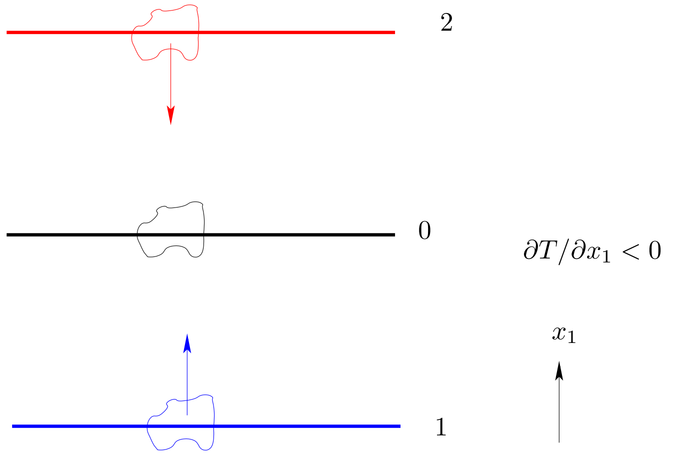
$$\bar{\Omega}_{im}\bar{\Omega}_{mk}\bar{s}_{kj} + \bar{s}_{ik}\bar{\Omega}_{km}\bar{\Omega}_{mj} - \frac{2}{3}\delta_{ij}\bar{\Omega}_{pm}\bar{\Omega}_{mk}\bar{s}_{kp}, \quad \bar{s}_{ik}\bar{s}_{km} - \bar{\Omega}_{im}, \quad \bar{\Omega}_{ji}\bar{\Omega}_{ik}$$

b) Derive the Poisson equation for the fluctuating pressure, p'. (5p)

T2. Consider the transport equation for the Reynolds stresses for fully-developed flow in a vertical channel (see figure below). Make a sketch how the four stresses ($\overline{v_1'^2}$, $\overline{v_2'^2}$, $\overline{v_3'^2}$ and $\overline{v_1'v_2'}$) vary in the boundary layer. Which are the largest source and sink terms (magnitude) in the $\overline{v_i'v_j'}$ equations? Give also the sign of the terms. Which source/sink terms are zero? (10p)



T3. a) Consider natural convection in the figure below. The convective velocity field is negligible. Is the stratification stable or unstable? Show that the Reynolds stress model faithfully can model the stratification. (5p)



b) Consider turbulent flow in the boundary layer in Question 2. Give the expression for the dissipation term in the exact $\overline{v_1'^2}$ and $\overline{v_1' v_2'}$ Reynolds stress equation. How are they modeled? (5p)

T4. Consider a $k - Z$ ($Z = k \cdot L_t$) RANS turbulence model. In this model, the k equation reads (10p)

$$\frac{dk}{dt} = P^k + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] - \varepsilon$$

How is ε expressed ? How do you express ε in a DES model?

T5. Consider the flow in a boundary layer along a plane located at $x_2 = 0$. You can download files either from the course home page (10p)

http://www.tfd.chalmers.se/~lada/comp_turb_model/exam-oct

or at Canvas. Use the Python script `read_oct.py` or the Matlab/Octave script `read_oct.m` to read the six data files `w_k2_time.dat`, ...`w_k22_time.dat`. The data files include time history of $\bar{w}$ ($\Delta t = 0.04$) taken at the same x_1 and x_2 locations; in the spanwise direction, the time histories are separated by $4\Delta z$ ($\Delta z = 0.025$).
Compute and plot the two-point correlation $B_{33}(\hat{z})$ and compute the integral lengthscale L_z. Compute also the auto correlation and the integral timescale for one of the time histories. Why is it useful to know the integral lengthscale and timescale?
<u>Note:</u> your Python/Matlab/Octave code must be uploaded to Canvas.

MTF271 Turbulence modeling: Formula sheet

October 4, 2020

The continuity and Navier-Stokes equations for compressible flow read

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_i}{\partial x_i}&=0\\
\frac{\partial \rho v_i}{\partial t}+\frac{\partial \rho v_i v_j}{\partial x_j}&=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\}+\rho g_i
\end{aligned}
$$

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$
\begin{aligned}
\frac{\partial v_i}{\partial x_i}&=0\\
\rho_0\frac{\partial v_i}{\partial t}+\rho_0\frac{\partial v_i v_j}{\partial x_j}&=-\frac{\partial P}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}-\rho_0\beta(\theta-\theta_0)g_i\\
\frac{\partial \theta}{\partial t}+\frac{\partial v_i\theta}{\partial x_i}&=\alpha\frac{\partial^2\theta}{\partial x_i\partial x_i}
\end{aligned}
$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p=0$ if $v_i=0$)

$$
\rho_0\frac{\partial v_i}{\partial t}+\rho_0 v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}
$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$
\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i}&=0\\
\rho_0\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j}&=-\frac{\partial \bar{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j}-\rho_0\overline{v_i'v_j'}\right)-\rho_0\beta(\bar{\theta}-\theta_0)g_i\\
\frac{\partial \bar{v}_i\bar{\theta}}{\partial x_i}&=\alpha\frac{\partial^2\bar{\theta}}{\partial x_i\partial x_i}-\frac{\partial\overline{v_i'\theta'}}{\partial x_i}
\end{aligned}
$$

The Boussinesq assumption reads

$$
\overline{v_i'v_j'}=-\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)+\frac{2}{3}\delta_{ij}k=-2\nu_t\bar{s}_{ij}+\frac{2}{3}\delta_{ij}k
$$

The modeled $\overline{v_i'v_j'}$ equation with IP model reads

$$
\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k} = & \quad \text{(convection)} \\
-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij}k\right) & \quad \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) & \quad \text{(rapid part)} \\
+c_{1w}\rho_0 \frac{\varepsilon}{k}\ [\ \overline{v_k'v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i'v_k'} n_k n_j & \\
-\frac{3}{2}\overline{v_j'v_k'} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, slow part)} \\
+c_{2w}\ [\ \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j & \\
-\frac{3}{2}\Phi_{jk,2} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k} & \quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\ \overline{v_k'v_m'}\ \frac{k}{\varepsilon}\frac{\partial \overline{v_i'v_j'}}{\partial x_m}\right] & \quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} & \quad \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & \quad \text{(dissipation)}
\end{aligned}
$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2} \frac{\partial A_i A_i}{\partial x_j}$$

► The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k} \bar{v}_k \overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'} \frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'} \frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{- \overline{\frac{\theta'}{\rho} \frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{- \frac{\partial}{\partial x_k} \overline{v_k' v_i' \theta'}}_{D_{i\theta,t}}$$
$$\underbrace{+ \alpha \overline{\frac{\partial}{\partial x_k}\left(v_i' \frac{\partial \theta'}{\partial x_k}\right)} + \nu \overline{\frac{\partial}{\partial x_k}\left(\theta' \frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{- (\nu + \alpha) \overline{\frac{\partial v_i'}{\partial x_k} \frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{- g_i \beta \overline{\theta'^2}}_{G_{i\theta}}$$

► The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'} \frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu \frac{\partial k}{\partial x_j}\right] - \nu \overline{\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j}} - g_i \beta \overline{v_i'\theta'}$$

► The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}
\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k \overline{v_i'v_j'}) &= -\overline{v_j'v_k'} \frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'} \frac{\partial \bar{v}_j}{\partial x_k} \\
-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu \frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right) \\
+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} - 2\nu \overline{\frac{\partial v_i'}{\partial x_k} \frac{\partial v_j'}{\partial x_k}}
\end{aligned}$$

► The modelled k and ε equations

$$\begin{aligned}
\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} &= \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) \frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} \\
&\quad -\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right) \frac{\partial k}{\partial x_j}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k} c_{\varepsilon 1} \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) \frac{\partial \bar{v}_i}{\partial x_j} \\
&\quad + c_{\varepsilon 1} g_i \frac{\varepsilon}{k} \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right) \frac{\partial \varepsilon}{\partial x_j}\right]
\end{aligned}$$

► DES

$$L_t = \frac{k^{3/2}}{\varepsilon} = \frac{k^{1/2}}{\beta^* \omega} : \text{RANS lengthscale}$$
$$C_{DES}\Delta, \quad \Delta = \max(\Delta x, \Delta y, \Delta z) : \text{LES lengthscale}$$