2020-08-18, Exam in

Turbulence modeling, MTF270

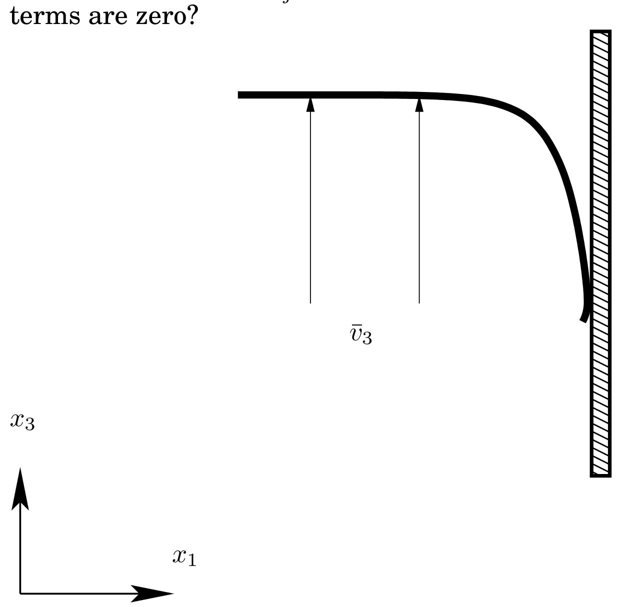
- **Time:** 8.30-12.30 **Location:** Zoom
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **The teacher is available on telephone**
- **Checking the evaluation and results of your written exam at Canvas. If you have questions on the correction of the exam, add a comment at Canvas and send me an Email.**
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Consider the Explicit Algebraic Reynolds Stress Model (EARSM). Show that the term below is symmetric, dimension-less and trace-less. (5p)

$$T_{ij}^5 = \frac{k^3}{\varepsilon^3}\left(\bar{\Omega}_{ik}\bar{s}_{km}\bar{s}_{mj} - \bar{s}_{im}\bar{s}_{mk}\bar{\Omega}_{kj}\right)$$

b) The $k-\omega$ SST model includes two transport equations, the k and ω equations. The model is a combination of the $k-\omega$ model (near the wall) and the $k-\varepsilon$ model (far from the wall). Explain why the $k-\omega$ SST model corresponds to a $k-\varepsilon$ model far from the wall although the model uses the k and ω equations. (5p)

T2. Consider the transport equation for the Reynolds stresses for fully-developed flow in a vertical channel (see figure below). Make a sketch how the four stresses ($\overline{v_1'^2}$, $\overline{v_2'^2}$, $\overline{v_3'^2}$ and $\overline{v_1'v_3'}$) vary in the boundary layer. Which are the largest source and sink terms (magnitude) in the $\overline{v_i'v_j'}$ equations? Give also the sign of the terms. Which source/sink terms are zero? (10p)

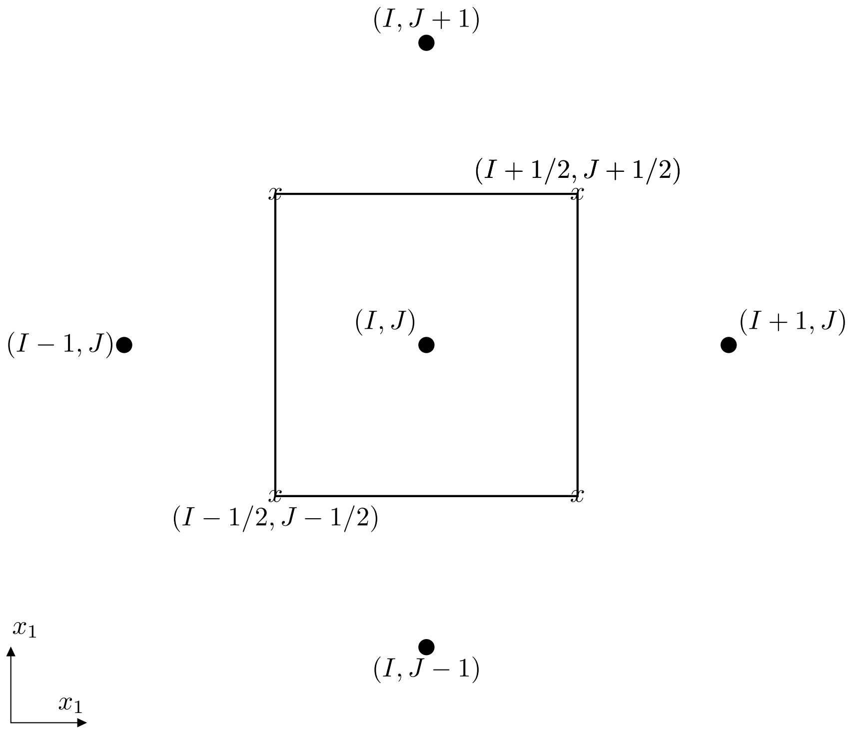


T3. a) Consider the Reynolds stress model. The slow pressure strain model for the wall effect reads (5p)

$$\Phi_{ij,1w} = c_{1w}\frac{\varepsilon}{k}\left(\overline{v'_k v'_m} n_{k,w} n_{m,w}\delta_{ij} - \frac{3}{2}\overline{v'_k v'_i} n_{k,w} n_{j,w} - \frac{3}{2}\overline{v'_k v'_j} n_{i,w} n_{k,w}\right) f$$

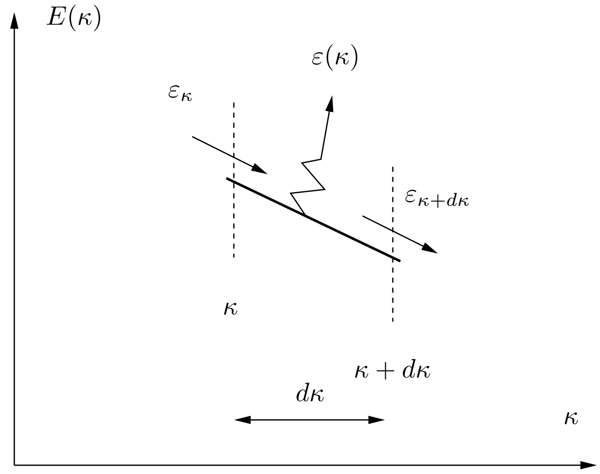
Simplify this expression for a plane wall in the $x_2 - x_3$ plane.

b) Consider a 2D finite volume grid. In scale-similarity we filter twice, i.e. $\overline{\overline{v}}$. Given $\bar{v}$, compute $\overline{\overline{v}}$ on a square control volume. (5p)



A 2D control volume.

T4. a) Explain the difference between ε_κ, $\varepsilon(\kappa)$ and $\varepsilon(\kappa = \infty)$, see figure below. (5p)



Zoom of the energy spectrum for a wavenumber located in Region II or III

b) Consider a $k - \tau$ turbulence model. Using the k and ε equations, derive the form of the production and destruction terms in the τ equation. How is the turbulent viscosity computed in the $k - \tau$ model? (5p)

T5. Consider the flow in a boundary layer along a plane located at $x_2 = 0$. You can download files either from the course home page (10p)

http://www.tfd.chalmers.se/~lada/comp_turb_model/exam-august

or at Canvas. Use the Python script `read_august.py` or the Matlab/Octave script `read_august.m` to read the six data files `w_k2_time.dat`, ...`w_k22_time.dat`. The data files include time history of $\bar{w}$ ($\Delta t = 0.04$) taken at the same x_1 and x_2 locations; in the spanwise direction, the time histories are separated by $4\Delta z$ ($\Delta z = 0.025$). Compute and plot the two-point correlation $B_{33}(\hat{z})$ and compute the integral lengthscale L_z. Compute also the auto correlation and the integral timescale for one of the time histories. Why is it useful to know the integral lengthscale and timescale?
<u>Note:</u> your Python/Matlab/Octave code must be uploaded to Canvas.

MTF270 Turbulence modeling: Formula sheet

August 13, 2020

The continuity and Navier-Stokes equations for compressible flow read

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \\
\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\} + \rho g_i
\end{aligned}
$$

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$
\begin{aligned}
\frac{\partial v_i}{\partial x_i} = 0 \\
\rho_0\frac{\partial v_i}{\partial t} + \rho_0\frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0\beta(\theta - \theta_0)g_i \\
\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha\frac{\partial^2 \theta}{\partial x_i \partial x_i}
\end{aligned}
$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$
\rho_0\frac{\partial v_i}{\partial t} + \rho_0 v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}
$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$
\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0\frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j} - \rho_0\overline{v'_i v'_j}\right) - \rho_0\beta(\bar{\theta} - \theta_0)g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha\frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v'_i \theta'}}{\partial x_i}
\end{aligned}
$$

The Boussinesq assumption reads

$$
\overline{v'_i v'_j} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}k = -2\nu_t\bar{s}_{ij} + \frac{2}{3}\delta_{ij}k
$$

The modeled $\overline{v_i'v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k} = &\quad \text{(convection)} \\
-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} &\quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij} k\right) &\quad \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) &\quad \text{(rapid part)} \\
+c_{1w}\rho_0 \frac{\varepsilon}{k}\ [\ \overline{v_k'v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i'v_k'} n_k n_j & \\
-\frac{3}{2}\overline{v_j'v_k'} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] &\quad \text{(wall, slow part)} \\
+c_{2w}\ [\ \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j & \\
-\frac{3}{2}\Phi_{jk,2} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] &\quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k} &\quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\ \overline{v_k'v_m'}\ \frac{k}{\varepsilon}\frac{\partial \overline{v_i'v_j'}}{\partial x_m}\right] &\quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} &\quad \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} &\quad \text{(dissipation)}
\end{aligned}$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2}\frac{\partial A_i A_i}{\partial x_j}$$

▶The exact transport equation for turbulent heat heat flux vector $\overline{v'_i \theta'}$ reads

$$\frac{\partial \overline{v'_i \theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k \overline{v'_i \theta'} = \underbrace{-\overline{v'_i v'_k}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v'_k \theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v'_k v'_i \theta'}}_{D_{i\theta,t}}$$
$$\underbrace{+\alpha \overline{\frac{\partial}{\partial x_k}\left(v'_i \frac{\partial \theta'}{\partial x_k}\right)} + \nu \overline{\frac{\partial}{\partial x_k}\left(\theta' \frac{\partial v'_i}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v'_i}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i \beta \overline{\theta'^2}}_{G_{i\theta}}$$

▶The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v'_i v'_j}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v'_j p'} + \frac{1}{2}\overline{v'_j v'_i v'_i} - \nu \frac{\partial k}{\partial x_j}\right] - \nu \overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}} - g_i \beta \overline{v'_i \theta'}$$

▶The exact $\overline{v'_i v'_j}$ equation reads

$$\begin{aligned}\frac{\partial \overline{v'_i v'_j}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k \overline{v'_i v'_j}) &= -\overline{v'_j v'_k}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v'_i v'_k}\frac{\partial \bar{v}_j}{\partial x_k} \\ -\frac{\partial}{\partial x_k}&\left(\overline{v'_i v'_j v'_k} + \frac{1}{\rho}\delta_{jk}\overline{v'_i p'} + \frac{1}{\rho}\delta_{ik}\overline{v'_j p'} - \nu \frac{\partial \overline{v'_i v'_j}}{\partial x_k}\right) \\ +\frac{1}{\rho}\overline{p'\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)} &- g_i \beta \overline{v'_j \theta'} - g_j \beta \overline{v'_i \theta'} - 2\nu \overline{\frac{\partial v'_i}{\partial x_k}\frac{\partial v'_j}{\partial x_k}}\end{aligned}$$

▶The modelled k and ε equations

$$\begin{aligned}\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} &= \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} \\ &\quad -\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k} c_{\varepsilon 1}\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} \\ &\quad + c_{\varepsilon 1} g_i \frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]\end{aligned}$$

▶DES

$$L_t = \frac{k^{3/2}}{\varepsilon} = \frac{k^{1/2}}{\beta^* \omega} : \text{RANS lengthscale}$$
$$C_{DES}\Delta, \quad \Delta = \max(\Delta x, \Delta y, \Delta z) : \text{LES lengthscale}$$