2020-06-02, Exam in

Turbulence modeling, MTF270

- **Time:** 14.00-18.00 **Location:** Zoom
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **The teacher is available on telephone**
- **Checking the evaluation and results of your written exam**: at 12-13 on 28 and 29 October in my office
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Show that the term below in the non-linear model is symmetric and trace-less. (5p)

$$c_5\tau^3\left(\bar{\Omega}_{i\ell}\bar{\Omega}_{\ell m}\bar{s}_{mj}+\bar{s}_{i\ell}\bar{\Omega}_{\ell m}\bar{\Omega}_{mj}-\frac{2}{3}\bar{\Omega}_{mn}\bar{\Omega}_{n\ell}\bar{s}_{\ell m}\delta_{ij}\right)$$

b) Consider the flow in the upper half of a fully-developed channel flow (see Fig. 1). Make a sketch how the three terms in the streamwise momentum equation vary vs. x_2. Which terns are non-zero at the wall? (5p)

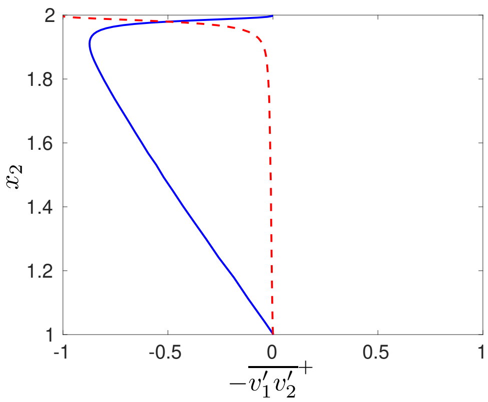

Figure 1. Resolved (blue solid line) and viscus (dashed red line) shear stress

T2. a) Consider the transport equation for the Reynolds stresses in the upper half of a fully-developed channel flow (see Fig. 1). Make a sketch how the largest source terms in the $\overline{v_1'v_2'}$ equation vary vs. x_2 in the log-region. Which terns are non-zero at the wall? (5p)

b) Streamline curvature: consider a boundary layer, $\delta(x_1)$, on the upper surface of an airfoil (see figure below) where the streamlines are curved along the wall. Show that the Reynolds stress model gives an increased or decreased (which is it?) turbulence production, i.e. the turbulence is decreased or increased. (5p)

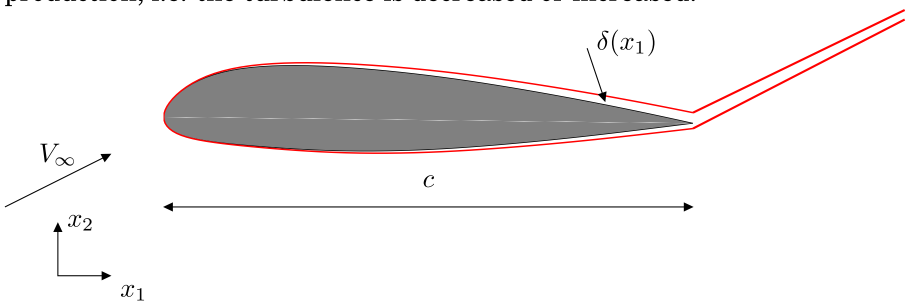

Figure 2. Airfoil. The boundary layers, $\delta(x_1)$, and the wake illustrated in red. $x_1 = 0$ and $x_1 = c$ at leading and trailing edge, respectively.

T3. a) Consider the V2F model. Explain the difference between v^2 and $\overline{v_2'^2}$. (5p)

b) Consider a 1D finite volume grid. In SGS models based on the dynamic procedure we use test filtering. Compute the test filtered velocity using a test filter which is four times larger than the grid filter, i.e. $\widehat{\Delta} = 4\Delta$ (5p)

T4. a) Explain the energy transfer in the figure below. (5p)

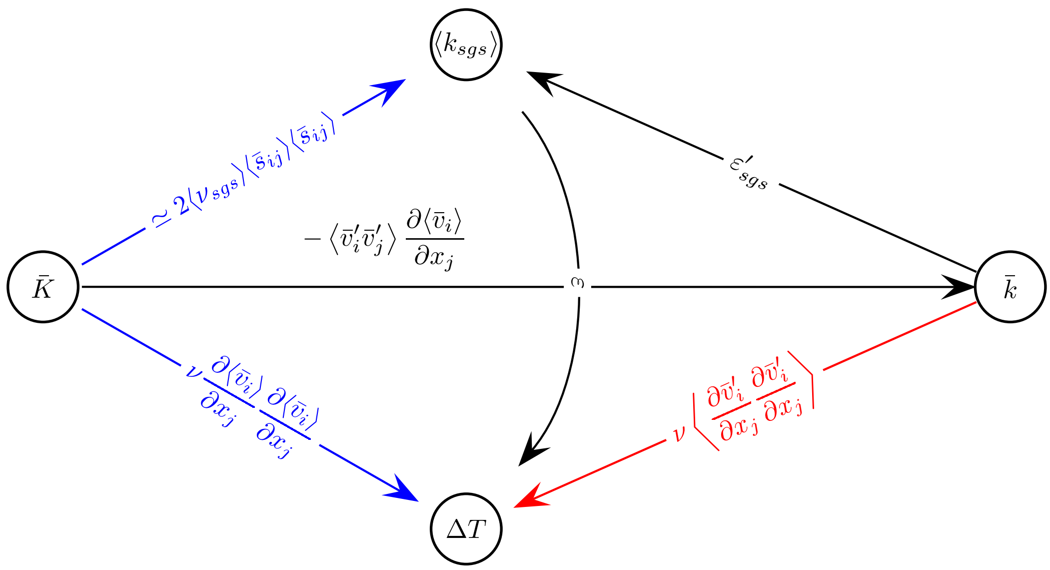

Figure 1: Transfer of kinetic turbulent energy.

b) Consider a $k - \nu_t$ turbulence model. In this model, the k equation reads (5p)

$$\frac{Dk}{Dt} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - C_k\frac{k^2}{\tilde{\nu}_t} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

How do you modify this model into a $k - \nu_t$ DES model?

T5. Consider the flow in a boundary later along a plane located at $x_2 = 0$. You can download files either from the course home page (10p)

http://www.tfd.chalmers.se/~lada/comp_turb_model/exam

or at Canvas. Use the Python script `read.py` or the Matlab/Octave script `read.m` to read the data `exam_data.txt`. The data file has five columns with $y, U, k, \omega, \overline{v_1'v_2'}$ at one x station far downstream.

- Plot $\bar{v}_1^+$ vs. x_2^+.
- There are requirements on LES that $\delta/\Delta x$ and $\delta/\Delta z$ should stay within certain limits. Are those requirements satisfied?
- Compute the location of the RANS-LES interface defined by the $k-\omega$ DES model.
- Compute and plot the ratio of the modelled to the resolved shear stress. Does this ratio show that the flow is well-resolved or not?

<u>Note:</u> your Python/Matlab/Octave code must be uploaded to Canvas.

MTF270 Turbulence modeling: Formula sheet

The continuity and Navier-Stokes equations for compressible flow read

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_i}{\partial x_i}=0 \\
\frac{\partial \rho v_i}{\partial t}+\frac{\partial \rho v_i v_j}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\}+\rho g_i
\end{aligned}
$$

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$
\begin{aligned}
\frac{\partial v_i}{\partial x_i}=0 \\
\rho_0\frac{\partial v_i}{\partial t}+\rho_0\frac{\partial v_i v_j}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}-\rho_0\beta(\theta-\theta_0)g_i \\
\frac{\partial \theta}{\partial t}+\frac{\partial v_i\theta}{\partial x_i}=\alpha\frac{\partial^2\theta}{\partial x_i\partial x_i}
\end{aligned}
$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p=0$ if $v_i=0$)

$$
\rho_0\frac{\partial v_i}{\partial t}+\rho_0 v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}
$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$
\begin{aligned}
\frac{\partial \bar v_i}{\partial x_i} &= 0 \\
\rho_0\frac{\partial \bar v_i\bar v_j}{\partial x_j} &= -\frac{\partial \bar P}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar v_i}{\partial x_j}-\rho_0\overline{v_i'v_j'}\right)-\rho_0\beta(\bar\theta-\theta_0)g_i \\
\frac{\partial \bar v_i\bar\theta}{\partial x_i} &= \alpha\frac{\partial^2\bar\theta}{\partial x_i\partial x_i}-\frac{\partial\overline{v_i'\theta'}}{\partial x_i}
\end{aligned}
$$

The Boussinesq assumption reads

$$
\overline{v_i'v_j'}=-\nu_t\left(\frac{\partial \bar v_i}{\partial x_j}+\frac{\partial \bar v_j}{\partial x_i}\right)+\frac{2}{3}\delta_{ij}k=-2\nu_t\bar s_{ij}+\frac{2}{3}\delta_{ij}k
$$

The modeled $\overline{v_i'v_j'}$ equation with IP model reads

$$
\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k} = & \quad \text{(convection)} \\
-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij} k\right) & \quad \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & \quad \text{(rapid part)} \\
+c_{1w}\rho_0 \frac{\varepsilon}{k}\ [\ \overline{v_k'v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i'v_k'} n_k n_j & \\
-\frac{3}{2}\overline{v_j'v_k'} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, slow part)} \\
+c_{2w}\ [\ \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j & \\
-\frac{3}{2}\Phi_{jk,2} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k} & \quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\ \overline{v_k'v_m'}\ \frac{k}{\varepsilon}\frac{\partial \overline{v_i'v_j'}}{\partial x_m}\right] & \quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} & \quad \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & \quad \text{(dissipation)}
\end{aligned}
$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2} \frac{\partial A_i A_i}{\partial x_j}$$

► The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k} \bar{v}_k \overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'} \frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'} \frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho} \frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k} \overline{v_k' v_i' \theta'}}_{D_{i\theta,t}}$$

$$\underbrace{+\alpha \overline{\frac{\partial}{\partial x_k} \left(v_i' \frac{\partial \theta'}{\partial x_k} \right)} + \nu \overline{\frac{\partial}{\partial x_k} \left(\theta' \frac{\partial v_i'}{\partial x_k} \right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha) \overline{\frac{\partial v_i'}{\partial x_k} \frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i \beta \overline{\theta'^2}}_{G_{i\theta}}$$

► The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'} \frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \overline{v_j' p'} + \frac{1}{2} \overline{v_j' v_i' v_i'} - \nu \frac{\partial k}{\partial x_j} \right] - \nu \overline{\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j}} - g_i \beta \overline{v_i'\theta'}$$

► The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}
\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{v}_k \overline{v_i'v_j'}) &= -\overline{v_j'v_k'} \frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'} \frac{\partial \bar{v}_j}{\partial x_k} \\
-\frac{\partial}{\partial x_k} &\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho} \delta_{jk} \overline{v_i'p'} + \frac{1}{\rho} \delta_{ik} \overline{v_j'p'} - \nu \frac{\partial \overline{v_i'v_j'}}{\partial x_k} \right) \\
+\frac{1}{\rho} \overline{p' \left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i} \right)} &- g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} - 2\nu \overline{\frac{\partial v_i'}{\partial x_k} \frac{\partial v_j'}{\partial x_k}}
\end{aligned}$$

► The modelled k and ε equations

$$\begin{aligned}
\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} &= \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} \\
&\quad -\varepsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k} c_{\varepsilon 1} \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} \\
&\quad + c_{\varepsilon 1} g_i \frac{\varepsilon}{k} \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right]
\end{aligned}$$

► DES

$$L_t = \frac{k^{3/2}}{\varepsilon} = \frac{k^{1/2}}{\beta^* \omega} : \text{RANS lengthscale}$$

$$C_{DES}\Delta, \quad \Delta = \max(\Delta x, \Delta y, \Delta z) : \text{LES lengthscale}$$