2017-10-06, Exam in

Turbulence modeling, MTF270

- **Time:** 14.00-18.00 **Location:** M
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 2 pages, appended to the exam
- **Checking the evaluation and results of your written exam**: at 12-13 on 23 and 24 Octobet in room Euler
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Show how the turbulent diffusion (i.e. the term which includes the triple correlation) in the k equation is modeled. (5p)

b) Using physical reasoning, the model for the pressure-strain term is formulated only for the normal strese i.e. (5p)

$$\overline{p'\frac{\partial v_1'}{\partial x_1}} = -\rho\frac{\varepsilon}{k}\left(\frac{3}{2}\overline{v_1'^2} - k\right)$$

Show that if this model is expressed in the principal directions, then a model for the shear stress is also obtained.

T2. a) Derive the algebraic stress model (ASM) (5p)

$$\overline{v_i'v_j'} = \frac{2}{3}\delta_{ij}k + \frac{k}{\varepsilon}\frac{(1-c_2)\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) + \Phi_{ij,1w} + \Phi_{ij,2w}}{c_1 + P^k/\varepsilon - 1}$$

Start with the assumption that the transport (convective and diffusive) of $\overline{v_i'v_j'}$ is related to that of k

$$C_{ij} - D_{ij} = \frac{\overline{v_i'v_j'}}{k}\left(C^k - D^k\right)$$

Multiplying with k/ε and using the IP models for $\Phi_{ij,1}$ and $\Phi_{ij,2}$ and the isotropic model for ε_{ij} give

$$\frac{k}{\varepsilon}P_{ij} - c_1\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij}k\right) - c_2\frac{k}{\varepsilon}\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) - \frac{2}{3}\delta_{ij}k + \frac{k}{\varepsilon}\left(\Phi_{ij,1w} + \Phi_{ij,2w}\right) = \frac{\overline{v_i'v_j'}}{\varepsilon}\left(P^k - \varepsilon\right)$$

b) Streamline curvature: consider a boundary layer where the streamlines are curved away from the wall (concave curvature). Show that the Reynolds stress model gives an enhanced turbulence production (as it should) because of positive feedback between the production terms. (5p)

T3. a) The transport equation for $\overline{v_2'^2}$ reads (the turbulent diffusion term is modeled) (7p)

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v} \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu + \mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] \underbrace{-2\overline{v_2' \partial p'/\partial x_2}}_{\rho\Phi_{22}} - \rho\varepsilon_{22}$$

Show how this equation is re-written in the V2F model.

b) Consider a 1D finite volume grid. Carry out a second filtering of $\bar{v}$ at node I and show that $\overline{\bar{v}}_I \neq \bar{v}_I$. (5p)

T4. a) Consider Fig. 1. A-F denotes energy transfer. Identify A-F with one of the following expression (5p)

$$-\left\langle \bar{v}_i' \bar{v}_j' \right\rangle \frac{\partial \langle \bar{v}_i \rangle}{\partial x_j}, \quad 2\langle \nu_{sgs} \rangle \langle \bar{s}_{ij} \rangle \langle \bar{s}_{ij} \rangle, \quad \nu \frac{\partial \langle \bar{v}_i \rangle}{\partial x_j}\frac{\partial \langle \bar{v}_i \rangle}{\partial x_j}, \quad \nu \left\langle \frac{\partial \bar{v}_i'}{\partial x_j}\frac{\partial \bar{v}_i'}{\partial x_j} \right\rangle, \quad \varepsilon_{sgs}', \quad \varepsilon$$

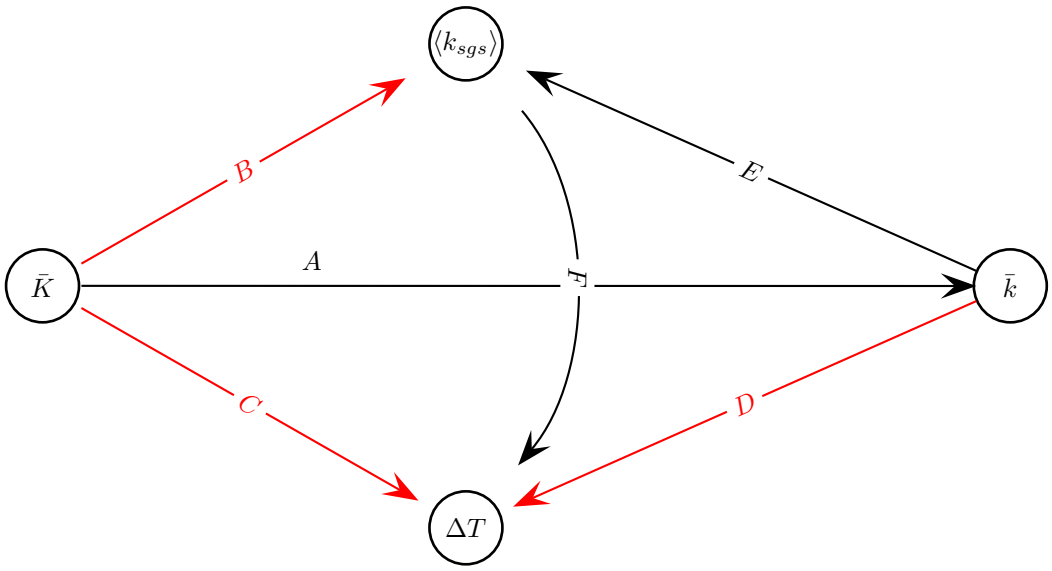

Figure 1: Transfer of kinetic turbulent energy. $\bar{K} = \frac{1}{2}\langle \bar{v}_i \rangle \langle \bar{v}_i \rangle$ and $\bar{k} = \frac{1}{2}\langle \bar{v}_i' \bar{v}_i' \rangle$ denote time-averaged kinetic and resolved turbulent kinetic energy, respectively. ΔT denotes increase in internal energy, i.e. dissipation. The cascade process assumes that the terms in red are negligible.

b) When deriving the dynamic Smagorinsky model, we arrive at the relation (5p)

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C\left(\widehat{\Delta}^2 |\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2\widehat{|\bar{s}|\bar{s}_{ij}}\right)$$

This equation is a tensor equation for C. Use this relation and derive the final expression for the dynamic coefficient, C, i.e.

$$C = -\frac{\mathcal{L}_{ij}M_{ij}}{2M_{ij}M_{ij}}$$

Hint:

$$M_{ij} = \left(\widehat{\Delta}^2 |\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2\widehat{|\bar{s}|\bar{s}_{ij}}\right)$$

T5. a) Derive the Smagorinsky model from the k_{sgs} equation. (5p)

b) Describe hybrid LES-RANS based on a one-equation model. (5p)

MTF270 Turbulence modeling: Formula sheet

October 4, 2017

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$
$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$
$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

▶The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v_i' v_j'}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v_i' v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} = & \quad \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i' v_j'} - \frac{2}{3}\delta_{ij} k\right) & \quad \text{(slow part)} \\
-c_2\left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & \quad \text{(rapid part)} \\
+c_{1w}\rho_0 \frac{\varepsilon}{k}\,[\,\overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i' v_k'} n_k n_j & \\
-\frac{3}{2}\overline{v_j' v_k'} n_k n_i\,]\, f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, slow part)} \\
+c_{2w}\,[\,\Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j & \\
-\frac{3}{2}\Phi_{jk,2} n_k n_i\,]\, f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} & \quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\, \overline{v_k' v_m'}\, \frac{k}{\varepsilon} \frac{\partial \overline{v_i' v_j'}}{\partial x_m}\right] & \quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} & \quad \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & \quad \text{(dissipation)}
\end{aligned}$$

▶The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$
$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial \theta'}{\partial x_k}\right)} + \nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

▶The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶The exact $\overline{v_i'v_j'}$ equation reads

$$\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) = -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right)$$
$$+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}$$

▶The modelled k and ε equations

$$\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} = \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i}$$
$$-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j}$$
$$+ c_{\varepsilon 1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$