MTF270: Turbulence modelling, Summary of lectures

Lecture 1

▶Boussinesq approximation: density variation only in gravitation (buoyancy) term

$$\frac{\partial \rho_0 \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho_0 \bar{v}_i \bar{v}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \mu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} - \rho_0 \beta(\bar{\theta} - \theta_0) g_i$$

$\bar{p}$ is hydrodynamic pressure: $\rho_0 f_i \to (\rho - \rho_0) g_i$

If we let density depend on pressure and temperature, differentiation gives

$$d\rho = \left(\frac{\partial \rho}{\partial \theta}\right)_p d\theta + \left(\frac{\partial \rho}{\partial p}\right)_\theta dp$$

Incompressible $\Rightarrow \partial\rho/\partial p = 0$

$$\beta = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial \theta}\right)_p \Rightarrow$$

$$d\rho = -\rho_0 \beta d\theta \Rightarrow \rho - \rho_0 = -\beta\rho_0(\theta - \theta_0)$$

$$\rho_0 f_i = (\rho - \rho_0) g_i = -\rho_0 \beta(\bar{\theta} - \theta_0) g_i$$

▶Temperature equation

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

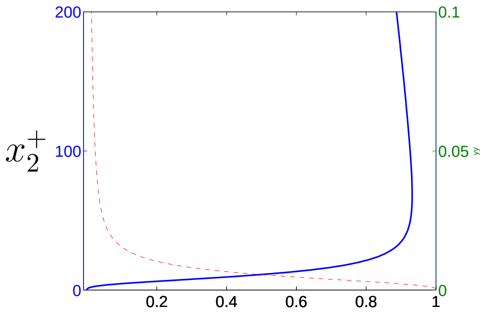
where $\alpha = k/(\rho c_p)$. Introducing $\theta = \bar{\theta} + \theta'$ we get the mean temperature equation

$$\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} = \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}$$

▶Total (viscous plus turbulent) flux: momentum and temperature equation

$$\frac{q_{2,tot}}{\rho c_p} = \frac{q_2}{\rho c_p} + \frac{q_{2,turb}}{\rho c_p} = \alpha \frac{\partial \bar{\theta}}{\partial x_2} - \overline{v_2'\theta'}, \quad \alpha = \frac{k}{\rho c_p}$$

$$\tau_{tot} = \tau_{visc} + \tau_{turb} = \mu \frac{\partial \bar{v}_1}{\partial x_2} - \rho \overline{v_1' v_2'}$$



Reynolds shear stress. —— : $-\rho\overline{v_1'v_2'}/\tau_w$; - - - : $\mu(\partial \bar{v}_1/\partial x_2)/\tau_w$.

▶$\overline{v_i'v_j'}$-equation

$$\underbrace{\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)}}_{\Pi_{ij}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_i'v_j'v_k'} + \frac{\overline{p'v_j'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu \frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}}_{G_{ij}} - \underbrace{2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

- Unkown terms

Π_{ij} Pressure-strain
$D_{ij,t}$ Turbulent diffusion
ε_{ij} Dissipation

▶ $\overline{v_i'\theta'}$ equation

$$\frac{\partial \theta'}{\partial t}+\frac{\partial}{\partial x_k}\left(v_k'\bar{\theta}+\bar{v}_k\theta'+v_k'\theta'\right)=\alpha\frac{\partial^2\theta'}{\partial x_k\partial x_k}+\frac{\partial \overline{v_k'\theta'}}{\partial x_k} \tag{1}$$

$$\frac{\partial v_i'}{\partial t}+\frac{\partial}{\partial x_k}\left(v_k'\bar{v}_i+\bar{v}_k v_i'+v_k'v_i'\right)=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 v_i'}{\partial x_k\partial x_k}+\frac{\partial \overline{v_i'v_j'}}{\partial x_k}-g_i\beta\theta' \tag{2}$$

Multiply Eq. 1 with v_i' and multiply Eq. 2 with θ', add them together and time average

$$\begin{aligned}&\overline{v_i'\frac{\partial}{\partial x_k}\left(v_k'\bar{\theta}+\bar{v}_k\theta'+v_k'\theta'\right)+\theta'\frac{\partial}{\partial x_k}\left(\bar{v}_i v_k'+\bar{v}_k v_i'+v_i'v_k'\right)}\\&\quad=-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}+\alpha\overline{v_i'\frac{\partial^2\theta'}{\partial x_k\partial x_k}}+\nu\overline{\theta'\frac{\partial^2 v_i'}{\partial x_k\partial x_k}}-g_i\beta\overline{\theta'\theta'}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'}=&\underbrace{-\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k}-\overline{v_k'\theta'}\frac{\partial\bar{v}_i}{\partial x_k}}_{P_{i\theta}}\underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}}\underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}\\&\underbrace{+(\nu+\alpha)\frac{\partial^2\overline{v_i'\theta'}}{\partial x_k\partial x_k}}_{D_{i\theta,\nu}}\underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial\theta'}{\partial x_k}}}_{\varepsilon_{i\theta}}\underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}\end{aligned}$$

▶The k equation: take the trace of the $\overline{v_i'v_j'}$ equation and divide by two

$$\frac{\partial k}{\partial t} + \underbrace{\bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = \underbrace{-\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{P^k} \underbrace{- \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\varepsilon}$$

$$\underbrace{-\frac{\partial}{\partial x_j}\left\{\overline{v_j'\left(\frac{p'}{\rho}+\frac{1}{2}v_i'v_i'\right)}\right\}}_{D_t^k} + \underbrace{\nu \frac{\partial^2 k}{\partial x_j \partial x_j}}_{D_\nu^k} \underbrace{-g_i \beta \overline{v_i' t}}_{G^k}$$

- Unkown terms

D_t^k Turbulent diffusion

ε Dissipation

▶The Boussinesq assumption

The diffusion term of time-averaged Navier-Stokes

$$\frac{\partial}{\partial x_j}\left\{\nu\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)-\overline{v_i'v_j'}\right\} \Rightarrow \frac{\partial}{\partial x_j}\left\{(\nu+\nu_t)\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)\right\}$$

$$\overline{v_i'v_j'} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)+\boxed{\tfrac{2}{3}\delta_{ij}k}$$

$$\nu_t \propto v'\ell = k^{1/2}\frac{k^{3/2}}{\varepsilon} = c_\mu\frac{k^2}{\varepsilon}$$

Lecture 2

▶Production term in k equation

$$P^k = -\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j} = \nu_t\left[\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \bar{v}_k}{\partial x_k}\right]\frac{\partial \bar{v}_i}{\partial x_j} = 2\nu_t \bar{s}_{ij}\bar{s}_{ij}$$

▶Modelled k equation

$$\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} = 2\nu_t \bar{s}_{ij}\bar{s}_{ij} + \frac{\partial}{\partial x_j}\left\{\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right\} - \varepsilon$$

▶ε equation

$$C^\varepsilon = P^\varepsilon + D^\varepsilon + G^\varepsilon - \Psi^\varepsilon$$

Use same source terms as in k equation and add turbulent time-scale ε/k to get the right dimensions:

$$P^\varepsilon - \Psi^\varepsilon + G^\varepsilon = \frac{\varepsilon}{k}(c_{\varepsilon 1}P^k - c_{\varepsilon 2}\varepsilon + c_{\varepsilon 3}G^k) \tag{3}$$

▶The final form modelled ε equation

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}(c_{\varepsilon 1}P^k - c_{\varepsilon 2}\varepsilon + c_{\varepsilon 3}G^k) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

▶The heat flux:

$$-\overline{v_i'\theta'} = -\alpha_t \frac{\partial \bar{\theta}}{\partial x_i},\ \alpha_t = \frac{\nu_t}{\sigma_t}$$

▶Diffusion term in k eq: compare with heat conduction:

$$q_i = -k\frac{\partial \bar{\theta}}{\partial x_i}.$$

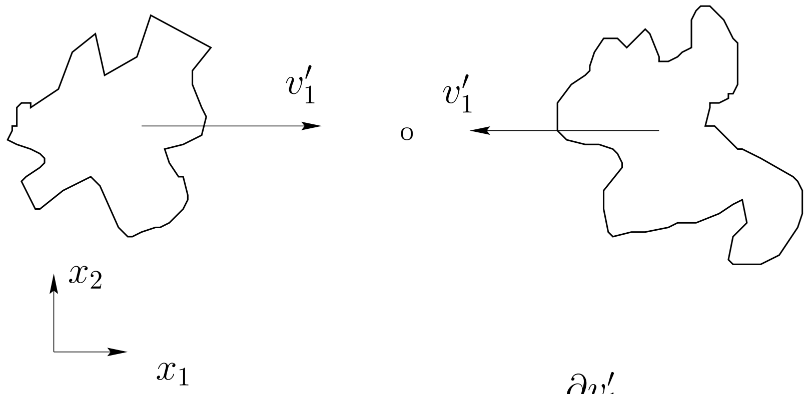
Flux of k:

$$d_{j,t}^k = \frac{1}{2}\overline{v_j'v_i'v_i'} = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j} \Rightarrow -\frac{1}{2}\frac{\partial \overline{v_j'v_i'v_i'}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\nu_t \frac{\partial k}{\partial x_j}\right)$$

▶Dissipation term in the RSM, ε_{ij} Small-scale turbulence is isotropic

1. $\overline{v_1'^2} = \overline{v_2'^2} = \overline{v_3'^2}$.
2. All shear stresses are zero $\Rightarrow \varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$

▶Slow pressure-strain term



$\partial v_1'/\partial x_1 < 0$ and $p' > 0$ so that $p'\dfrac{\partial v_1'}{\partial x_1} < 0$

$$\frac{\partial v_2'}{\partial x_2} > 0, \quad \frac{\partial v_3'}{\partial x_3} > 0$$

If this happens then

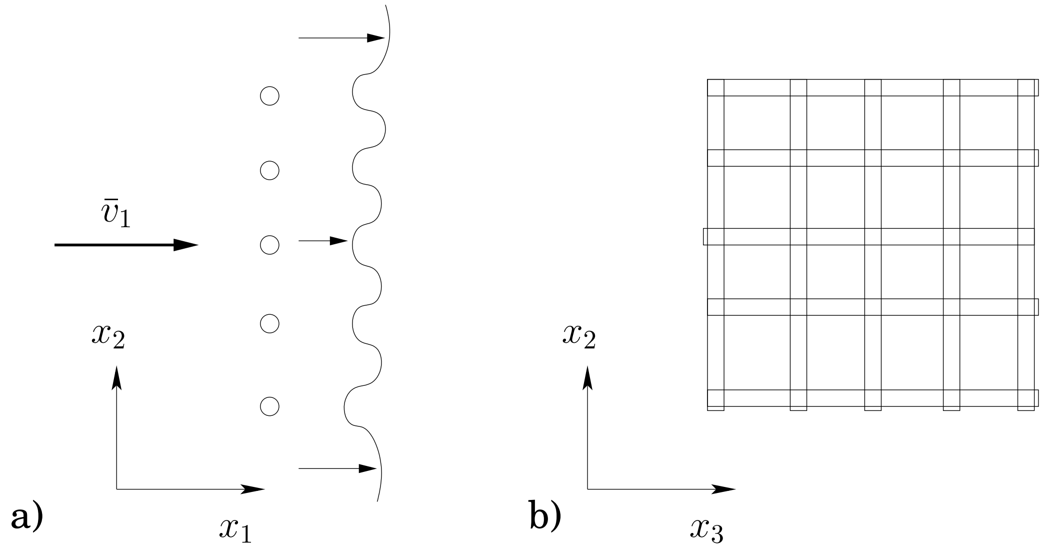
$$\overline{v_1'^2} > \overline{v_2'^2}, \overline{v_1'^2} > \overline{v_3'^2}$$

$$\begin{aligned}\frac{1}{\rho}\overline{p'\frac{\partial v_1'}{\partial x_1}} &\propto -\frac{\rho}{2t}\left[\left(\overline{v_1'^2}-\overline{v_2'^2}\right)+\left(\overline{v_1'^2}-\overline{v_3'^2}\right)\right] = -\frac{\rho}{t}\left[\overline{v_1'^2}-\frac{1}{2}\left(\overline{v_2'^2}+\overline{v_3'^2}\right)\right]\\ &= -\frac{\rho}{t}\left[\frac{3}{2}\overline{v_1'^2}-\frac{1}{2}\left(\overline{v_1'^2}+\overline{v_2'^2}+\overline{v_3'^2}\right)\right] = -\frac{\rho}{t}\left(\frac{3}{2}\overline{v_1'^2}-k\right)\end{aligned}$$

$$\Phi_{ij,1} \equiv \overline{p'\left(\frac{\partial v_i'}{\partial x_j}+\frac{\partial v_j'}{\partial x_i}\right)} = -c_1\rho\frac{\varepsilon}{k}\left(\overline{v_i'v_j'}-\frac{2}{3}\delta_{ij}k\right)$$

Lecture 3

▶Decaying grid turbulence



▶$\overline{v'_i v'_j}$ equation

$$\bar v_1 \frac{d\overline{v'_i v'_j}}{dx_1} = \overline{\frac{p'}{\rho}\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)} - \varepsilon_{ij} \tag{4}$$

stress tensor which is defined as

$$a_{ij} = \frac{\overline{v'_i v'_j}}{k} - \frac{2}{3}\delta_{ij} \tag{5}$$

When isotropic, $a_{ij} = 0$. We introduce a_{ij} (Eq. 5), $\phi_{ij,1}$ and $\varepsilon_{ij} = (2/3)\delta_{ij}$ into Eq. 4 so that

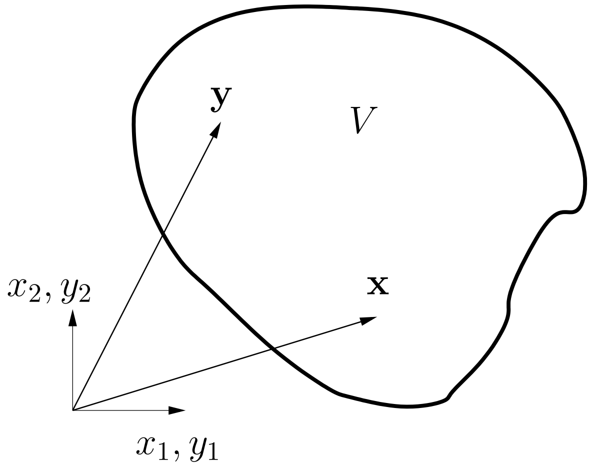
$$\bar v_1 \left(\frac{d(ka_{ij})}{dx_1} + \delta_{ij}\frac{2}{3}\frac{dk}{dx_1}\right) = \underline{-c_1 \varepsilon a_{ij} - \frac{2}{3}\delta_{ij}\varepsilon}$$

Using $\bar v_1 \frac{dk}{dx_1} = -\varepsilon$ gives

$$\bar v_1 \frac{da_{ij}}{dx_1} = \underline{-c_1 \frac{\varepsilon}{k} a_{ij} - \frac{2}{3}\delta_{ij}}\frac{\varepsilon}{k} + \frac{\varepsilon}{k}a_{ij} + \frac{2}{3}\delta_{ij}\frac{\varepsilon}{k} = \frac{\varepsilon}{k}a_{ij}(1-c_1)$$

$da_{ij}/dx < 0$ (the turbulence becomes isotropic) $\Rightarrow c_1 > 1$.

▶Pressure strain: rapid part



$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_j \partial x_j} = -\underbrace{2\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial v'_j}{\partial x_i}}_{\text{rapid term}} - \underbrace{\frac{\partial^2}{\partial x_i \partial x_j}\left(v'_i v'_j - \overline{v'_i v'_j}\right)}_{\text{slow term}}$$

$$\frac{\partial^2 \varphi}{\partial x_j \partial x_j} = f$$

▶There exists an exact analytical solution given by Green's formula (derived from Gauss divergence law)

$$\varphi(\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{f(\mathbf{y})dy_1 dy_2 dy_3}{|\mathbf{y}-\mathbf{x}|}$$

where $dy_1 dy_2 dy_3 = dV = d\mathbf{y}^3$. The integral is carried out for all points, $\mathbf{y}$, in volume V.

$$p'(\mathbf{x}) = -\frac{\rho}{4\pi}\int_V \left[\underbrace{2\frac{\partial \bar{v}_i(\mathbf{y})}{\partial y_j}\frac{\partial v'_j(\mathbf{y})}{\partial y_i}}_{\text{rapid term}} + \underbrace{\frac{\partial^2}{\partial y_i \partial y_j}\left(v'_i(\mathbf{y})v'_j(\mathbf{y}) - \overline{v'_i(\mathbf{y})v'_j(\mathbf{y})}\right)}_{\text{slow term}}\right]\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|}$$

$$\Phi_{ij,2} = -c_2\rho\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) \quad \text{IP model}$$

▶Wall models of pressure-strain:

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,1w} + \Phi_{ij,2w}$$

$$\Phi_{22,1w} = -2c_{1w}\frac{\varepsilon}{k}\overline{v_2'^2}f, \quad f = \frac{k^{\frac{3}{2}}}{2.55|n_{i,w}(x_i - x_{i,w})|\varepsilon}$$

Traceless $\Rightarrow$

$$\Phi_{11,1w} = \Phi_{33,1w} = c_{1w}\frac{\varepsilon}{k}\overline{v_2'^2}f$$

The wall model for the shear stress is set as

$$\Phi_{12,1w} = -\frac{3}{2}c_{1w}\frac{\varepsilon}{k}\overline{v_1'v_2'}f$$

The general form reads:

$$\Phi_{ij,1w} = c_{1w}\frac{\varepsilon}{k}\left(\overline{v_k'v_m'}n_{k,w}n_{m,w}\delta_{ij} - \frac{3}{2}\overline{v_k'v_i'}n_{k,w}n_{j,w} - \frac{3}{2}\overline{v_k'v_j'}n_{i,w}n_{k,w}\right)f$$

The analogous wall model for the rapid part reads

$$\Phi_{ij,2w} = c_{2w}\left(\Phi_{km,2}n_{k,w}n_{m,w}\delta_{ij} - \frac{3}{2}\Phi_{ki,2}n_{k,w}n_{j,w} - \frac{3}{2}\Phi_{kj,2}n_{i,w}n_{k,w}\right)f$$

▶The **modelled** $\overline{v_i'v_j'}$ equation with IP model

$$
\begin{aligned}
\frac{\partial \overline{v_i'v_j'}}{\partial t} &= \quad \text{(unsteady term)}\\
\bar{v}_k\frac{\partial \overline{v_i'v_j'}}{\partial x_k} &= \quad \text{(convection)}\\
-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)}\\
-c_1\frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij}k\right) & \quad \text{(slow part)}\\
-c_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) & \quad \text{(rapid part)}\\
+c_{1w}\rho\frac{\varepsilon}{k}\;[\;\overline{v_k'v_m'}n_k n_m\delta_{ij} - \frac{3}{2}\overline{v_i'v_k'}n_k n_j & \\
-\frac{3}{2}\overline{v_j'v_k'}n_k n_i\;]f & \quad \text{(wall, slow part)}\\
+c_{2w}\;[\;\Phi_{km,2}n_k n_m\delta_{ij} - \frac{3}{2}\Phi_{ik,2}n_k n_j & \\
-\frac{3}{2}\Phi_{jk,2}n_k n_i\;]f & \quad \text{(wall, rapid part)}\\
+\nu\frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k\partial x_k} & \quad \text{(viscous diffusion)}\\
+\frac{\partial}{\partial x_k}\left[\frac{\nu_t}{\sigma_k}\frac{\partial \overline{v_i'v_j'}}{\partial x_m}\right] & \quad \text{(turbulent diffusion)}\\
-g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'} & \quad \text{(buoyancy production)}\\
-\frac{2}{3}\varepsilon\delta_{ij} & \quad \text{(dissipation)}
\end{aligned}
$$

Lecture 4

▶Boundary layer flow where $\bar{v}_2 = 0,\ \bar{v}_1 = \bar{v}_1(x_2)$.

$$P_{ij} = -\overline{v'_i v'_k}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v'_j v'_k}\frac{\partial \bar{v}_i}{\partial x_k}$$

In this special case we get:

$$P_{11} = -2\overline{v'_1 v'_2}\frac{\partial \bar{v}_1}{\partial x_2}$$

$$P_{12} = -\overline{v'^2_2}\frac{\partial \bar{v}_1}{\partial x_2}$$

$$P_{22} = 0$$

$$\Phi_{22,1} = c_1\frac{\varepsilon}{k}\left(\frac{2}{3}k - \overline{v'^2_2}\right) > 0$$

$$\Phi_{22,2} = c_2\frac{1}{3}P_{11} = -c_2\frac{2}{3}\overline{v'_1 v'_2}\frac{\partial \bar{v}_1}{\partial x_2} > 0$$

$\varepsilon_{12} = 0$: No sink term in $\overline{v'_1 v'_2}$ eq? Answer: the pressure strain term $\Phi_{12,1}$ and $\Phi_{12,2}$.

▶The Algebraic Reynolds Stress Model (ASM)

$$\mathrm{RSM}: C_{ij} - D_{ij} = P_{ij} + \Phi_{ij} - \varepsilon_{ij}$$
$$k-\varepsilon: \; C^k - D^k = P^k - \varepsilon$$

Assumption in ASM:

$$C_{ij} - D_{ij} = \frac{\overline{v_i'v_j'}}{k}\left(C^k - D^k\right)$$

$$\Rightarrow P_{ij} + \Phi_{ij} - \varepsilon_{ij} = \frac{\overline{v_i'v_j'}}{k}\left(P^k - \varepsilon\right)$$

This gives

$$\overline{v_i'v_j'} = \frac{2}{3}\delta_{ij}k + \frac{k}{\varepsilon}\frac{(1-c_2)\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) + \Phi_{ij,1w} + \Phi_{ij,2w}}{c_1 + P^k/\varepsilon - 1}$$

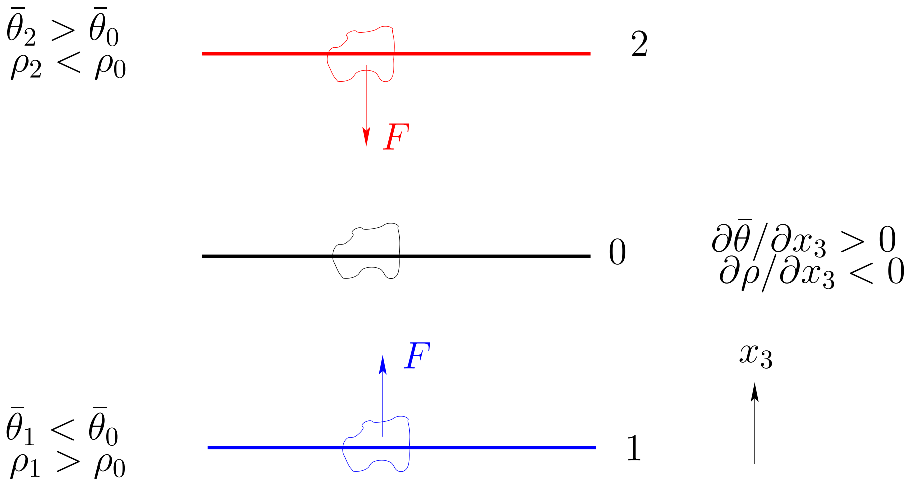
▶Φ_{ij} in boundary layer flow

$$P_{11} = -2\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2}, \quad P_{12} = -\overline{v_2'^2}\frac{\partial \bar{v}_1}{\partial x_2}, \quad P_{22} = 0$$

$$\Phi_{22,1} = c_1\frac{\varepsilon}{k}\left(\frac{2}{3}k - \overline{v_2'^2}\right) > 0$$

$$\varepsilon_{12} = \delta_{12}\varepsilon = 0, \quad \Phi_{12,1} = -c_1\frac{\varepsilon}{k}\overline{v_1'v_2'} < 0 \;\; (\text{if } \overline{v_1'v_2'} > 0)$$

▶Stable stratification because $\partial\bar{\theta}/\partial x_3 > 0$.



$$G_{ij} = -g_i\beta\overline{v'_j\theta'} - g_j\beta\overline{v'_i\theta'} \quad \Rightarrow \overline{v'^2_3}\text{ eq.:} \quad G_{33} = 2g\beta\overline{v'_3\theta'}$$

which is the source term in the $\overline{v'^2_3}$ eq due to buoyancy. Now we need $\overline{v'_3\theta'}$. Its main source term reads

$$P_{3\theta} = -\overline{v'_3v'_k}\frac{\partial\bar{\theta}}{\partial x_k} - \cancel{\overline{v'_k\theta'}\frac{\partial\bar{v}_3}{\partial x_k}} = -\overline{v'^2_3}\frac{\partial\bar{\theta}}{\partial x_3} < 0$$

$P_{3\theta}$ makes $\overline{v'_3\theta'} < 0$ so that $G_{33} < 0$ which dampens $\overline{v'^2_3}$ (but not $\overline{v'^2_1}, \overline{v'^2_2}$) as it should.

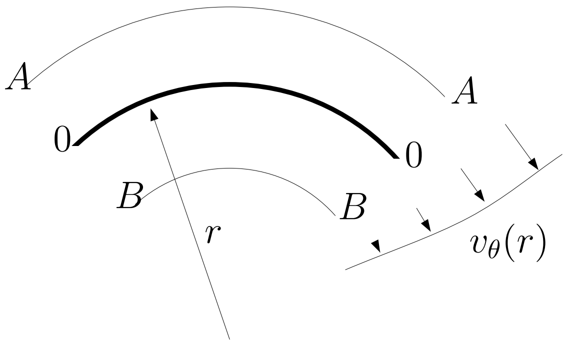
▶$k - \varepsilon$ model

$$G^k = 0.5G_{ii} = -g_i\beta\overline{v'_i\theta'} = g\beta\overline{v'_3\theta'}, \quad \overline{v'_i\theta'} = -\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i}$$

For $g_i = (0, 0, -g)$ it reads $G^k = g\beta\overline{v'_3\theta'}$ which gives

$$G^k = -g\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_3}$$

which dampens k (i.e. $\overline{v'^2_1}, \overline{v'^2_2}, \overline{v'^2_3}$).

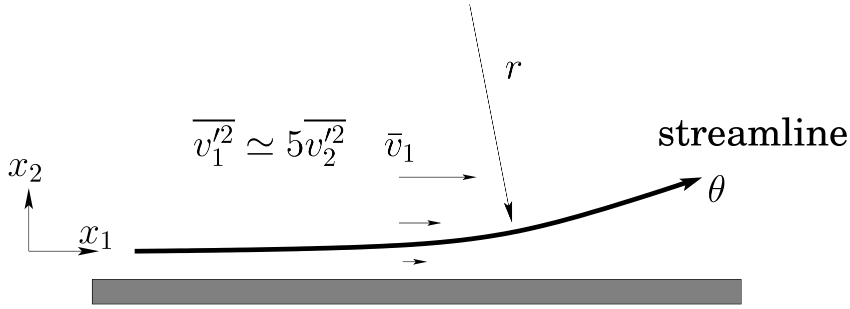


►Streamline curvature. Streamline aligned with the θ axis.

$$v_r \text{ eq. with } \mu = 0: \quad \frac{\rho v_\theta^2}{r} - \frac{\partial p}{\partial r} = 0 \tag{6}$$

$\partial v_\theta/\partial r > 0$. Hence $(v_\theta)_A > (v_\theta)_0$, which from Eq. 6 gives $(\partial p/\partial r)_A > (\partial p/\partial r)_0$.

Lecture 5



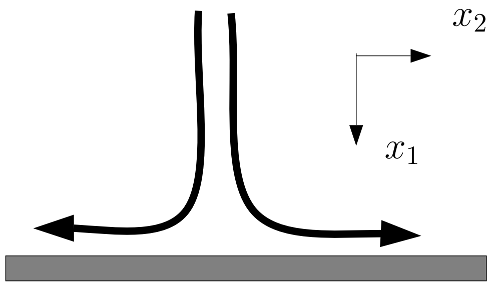
$$\text{RSM, } \overline{v_1'^2} - \text{eq.}: \quad P_{11} = -2\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2} \tag{7a}$$

$$\text{RSM, } \overline{v_1'v_2'} - \text{eq.}: \quad P_{12} = \boxed{-\overline{v_1'^2}\frac{\partial \bar{v}_2}{\partial x_1}} - \overline{v_2'^2}\frac{\partial \bar{v}_1}{\partial x_2} \tag{7b}$$

$$\text{RSM, } \overline{v_2'^2} - \text{eq.}: \quad P_{22} = \boxed{-2\overline{v_1'v_2'}\frac{\partial \bar{v}_2}{\partial x_1}} \tag{7c}$$

$$k - \varepsilon \quad P^k = \nu_t \left\{ \left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2 + \boxed{\left(\frac{\partial \bar{v}_2}{\partial x_1}\right)^2} \right\} \tag{7d}$$

▶Stagnation 2D flow



The flow pattern for stagnation flow.

$$RSM: 0.5\left(P_{11}+P_{22}\right) = -\overline{v_1'^2}\frac{\partial \bar{v}_1}{\partial x_1} - \overline{v_2'^2}\frac{\partial \bar{v}_2}{\partial x_2} = -\frac{\partial \bar{v}_1}{\partial x_1}(\overline{v_1'^2} - \overline{v_2'^2})$$

$$k-\varepsilon: P^k = 2\nu_t\left\{\left(\frac{\partial \bar{v}_1}{\partial x_1}\right)^2 + \left(\frac{\partial \bar{v}_2}{\partial x_2}\right)^2\right\}$$

▶Realizability

$$\overline{v_i'^2} \geq 0 \text{ for all } i$$

$$\frac{\left|\overline{v_i' v_j'}\right|}{\left(\overline{v_i'^2}\,\overline{v_j'^2}\right)^{1/2}} \leq 1 \text{ no summation over } i \text{ and } j,\ i \neq j$$

$$\overline{v_1'^2} = \frac{2}{3}k - 2\nu_t \frac{\partial \bar{v}_1}{\partial x_1} = \frac{2}{3}k - 2\nu_t \bar{s}_{11}$$

$\bar{s}_{11}$ largest in the principal coordinate directions

$$|\bar{s}_{ij} - \delta_{ij}\lambda| = 0$$

which gives in 2D

$$\begin{vmatrix} \bar{s}_{11} - \lambda & \bar{s}_{12} \\ \bar{s}_{21} & \bar{s}_{22} - \lambda \end{vmatrix} = 0$$

The resulting equation is

$$\lambda^2 - I_1^{2D}\lambda + I_2^{2D} = 0$$

$$I_1^{2D} = \bar{s}_{ii} = 0 \quad \textbf{continuity}$$

$$I_2^{2D} = \frac{1}{2}(\bar{s}_{ii}\bar{s}_{jj} - \bar{s}_{ij}\bar{s}_{ij}) = \det(C_{ij}) = -\bar{s}_{ij}\bar{s}_{ij}/2$$

$$\lambda_{1,2} = \pm\left(-I_2^{2D}\right)^{1/2} = \pm\left(\frac{\bar{s}_{ij}\bar{s}_{ij}}{2}\right)^{1/2}$$

$$\overline{v_1'^2} = \frac{2}{3}k - 2\nu_t\lambda_1 = \frac{2}{3}k - 2\nu_t\left(\frac{\bar{s}_{ij}\bar{s}_{ij}}{2}\right)^{1/2} \Rightarrow \nu_t \leq \frac{k}{3|\lambda_1|} = \frac{k}{3}\left(\frac{2}{\bar{s}_{ij}\bar{s}_{ij}}\right)^{1/2}$$

In 3D

$$|\lambda_k| = k\left(\frac{2\bar{s}_{ij}\bar{s}_{ij}}{3}\right)^{1/2}$$

►Non-linear eddy-viscosity models

The anisotropy tensor is defined as

$$a_{ij} \equiv \frac{\overline{v'_i v'_j}}{k} - \frac{2}{3}\delta_{ij}$$

$$\begin{aligned}
a_{ij} = & \boxed{-2c_\mu \tau \bar{s}_{ij}} \\
& + c_1 \tau^2 \left(\bar{s}_{ik}\bar{s}_{kj} - \frac{1}{3}\bar{s}_{\ell k}\bar{s}_{\ell k}\delta_{ij}\right) + c_2\tau^2\left(\bar{\Omega}_{ik}\bar{s}_{kj} - \bar{s}_{ik}\bar{\Omega}_{kj}\right) \\
& + c_3\tau^2\left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{\ell k}\bar{\Omega}_{\ell k}\delta_{ij}\right) + c_4\tau^3\left(\bar{s}_{ik}\bar{s}_{k\ell}\bar{\Omega}_{\ell j} - \bar{\Omega}_{i\ell}\bar{s}_{\ell k}\bar{s}_{kj}\right) \\
& + c_5\tau^3\left(\bar{\Omega}_{i\ell}\bar{\Omega}_{\ell m}\bar{s}_{mj} + \bar{s}_{i\ell}\bar{\Omega}_{\ell m}\bar{\Omega}_{mj} - \frac{2}{3}\bar{\Omega}_{mn}\bar{\Omega}_{n\ell}\bar{s}_{\ell m}\delta_{ij}\right) \\
& + c_6\tau^3\bar{s}_{k\ell}\bar{s}_{k\ell}\bar{s}_{ij} + c_7\tau^3\bar{\Omega}_{k\ell}\bar{\Omega}_{k\ell}\bar{s}_{ij}
\end{aligned}$$

$$\bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial \bar{v}_j}{\partial x_i}\right)$$

The advantage is better normal stresses

$$\overline{v_1'^2} = \frac{2}{3}k + \frac{0.82}{12}k\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2$$

$$\overline{v_2'^2} = \frac{2}{3}k - \frac{0.5}{12}k\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2$$

$$\overline{v_3'^2} = \frac{2}{3}k - \frac{0.16}{12}k\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2$$

Lecture 6

▶The V2F Model

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v} \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu+\mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] \underbrace{-2\overline{v_2'\partial p'/\partial x_2}}_{\Phi_{22}} - \rho\varepsilon_{22}$$

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v}_2 \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu+\mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] + \underbrace{\Phi_{22} - \rho\varepsilon_{22} + \rho\frac{\overline{v_2'^2}}{k}\varepsilon}_{fk} - \rho\frac{\overline{v_2'^2}}{k}\varepsilon$$

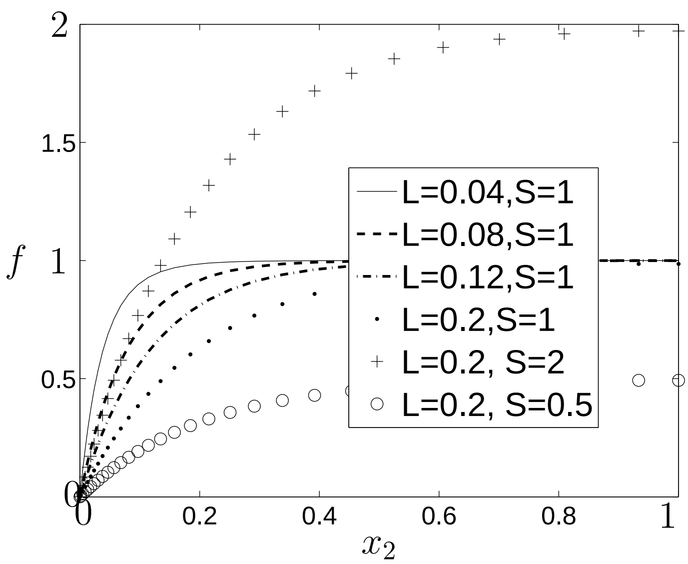
An equation is solved for $fk \propto \Phi_{22}$ (IP & Rotta model).

$$L^2\frac{\partial^2 f}{\partial x_2^2} - f = -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right), \quad T \propto \frac{k}{\varepsilon}, \quad L \propto \frac{k^{3/2}}{\varepsilon}$$

Far from wall $\frac{\partial^2 f}{\partial x_2^2} \simeq 0$ so that $-f \to -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right)$

$\Rightarrow$

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v} \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu+\mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] + \Phi_{22} - \rho\varepsilon_{22}$$



▶V2F model. Wall boundary conditions

Near the wall, the $\overline{v_2'^2}$ equation reads

$$0 = \nu\frac{\partial^2\overline{v_2'^2}}{\partial x_2^2} + fk - \frac{\overline{v_2'^2}}{k}\varepsilon$$

$$f = -\frac{20\nu^2}{\varepsilon}\frac{\overline{v_2'^2}}{x_2^4}$$

▶The SST (<u>S</u>hear <u>S</u>tress <u>T</u>ransport) model

1. Combination of a $k-\omega$ model (in the inner boundary layer) and $k-\varepsilon$ model (in the outer region of the boundary layer as well as outside of it);
2. A limitation of the shear stress in adverse pressure gradient regions.

▶$\omega = \varepsilon/(\beta^* k) = \varepsilon/(c_\mu k)$. Use this to obtain an eq. for ω

$$\frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{\varepsilon}{\beta^* k}\right) = \frac{1}{\beta^* k}\frac{d\varepsilon}{dt} - \frac{\varepsilon}{\beta^* k^2}\frac{dk}{dt} = \frac{1}{\beta^* k}\frac{d\varepsilon}{dt} - \frac{\omega}{k}\frac{dk}{dt}$$

▶Production term

$$P_\omega = \frac{1}{\beta^* k}P_\varepsilon - \frac{\omega}{k}P^k = C_{\varepsilon 1}\frac{\varepsilon}{\beta^* k^2}P^k - \frac{\omega}{k}P^k = (C_{\varepsilon 1} - 1)\frac{\omega}{k}P^k$$

▶Destruction term

$$\Psi_\omega = \frac{1}{\beta^* k}\Psi_\varepsilon - \frac{\omega}{k}\Psi_k = C_{\varepsilon 2}\frac{\varepsilon^2}{k} - \frac{\omega}{k}\varepsilon = (C_{\varepsilon 2}-1)\beta^*\omega^2$$

▶Viscous diffusion term

$$\begin{aligned}
D_\omega^\nu =& \frac{\nu}{\beta^* k}\frac{\partial^2\varepsilon}{\partial x_j^2} - \frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2} = \frac{\nu}{k}\frac{\partial^2 \omega k}{\partial x_j^2} - \frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2}\\
=& \frac{\nu}{k}\left[\frac{\partial}{\partial x_j}\left(\omega\frac{\partial k}{\partial x_j} + k\frac{\partial \omega}{\partial x_j}\right)\right] - \nu\frac{\omega}{k}\frac{\partial^2 k}{\partial x_j^2} = \frac{2\nu}{k}\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \omega}{\partial x_j}\right)
\end{aligned}$$

The ω eq. (which really is an ε eq. when the $k-\varepsilon$ constants are used)

$$\frac{\partial}{\partial x_j}(\bar{u}_j\omega) = \frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \alpha\frac{\omega}{k}P^k - \beta\omega^2 + \frac{2}{k}\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i}$$

$$\alpha = C_{\varepsilon 1} - 1 = 0.44, \beta = (C_{\varepsilon 2}-1)\beta^* = 0.0828$$

Inner region: $k-\omega$ coeff; outer region: $k-\varepsilon$ coeff. Blending function

$$F_1 = \tanh(\xi^4), \quad \xi \propto \frac{L_t}{x_n} = \frac{k^{1/2}}{\omega x_n} \tag{8}$$

$F_1 = 1$ in the near-wall region and $F_1 = 0$ in the outer region. The β-coefficient, for example, is computed as

$$\beta_{SST} = F_1\beta_{k-\omega} + (1-F_1)\beta_{k-\varepsilon} \tag{9}$$

►SST model. Limitation of shear stress.

The $k-\omega$ gives too high shear stress. The JK model based on $-\overline{v_1'v_2'} = a_1 k$ ($a_1 = c_\mu^{1/2}$) gives good results.

►Two formulas for ν_t. $\Omega = \partial\bar{v}_1/\partial x_2$. Formulate JK model with Boussinesq.

$$\left.\begin{array}{ll} \text{JK Model:} & \nu_t = \frac{-\overline{v_1'v_2'}}{\Omega} = \frac{a_1 k}{\Omega} \\ k-\omega \text{ model:} & \nu_t = \frac{k}{\omega} = \frac{a_1 k}{a_1\omega} \end{array}\right\} \nu_t = \frac{a_1 k}{\max(a_1\omega, F_2\Omega)}$$

F_2 is one near walls and zero elsewhere

►The idea is that

- the second part, $a_1 k/\Omega$ (which mimics the Johnson-King model), should be used in APG flow (where $P^k > \varepsilon$)
- the first part, k/ω (which corresponds to the usual Boussinesq model), should be in the remaining of the flow

Lecture 7

▶LES

in RANS:

$$\langle \Phi \rangle = \frac{1}{2T}\int_{-T}^{T}\Phi(t)dt, \quad \Phi = \langle \Phi \rangle + \Phi', \quad \langle \Phi' \rangle = 0$$

in LES:

$$\bar{\Phi}(x,t) = \frac{1}{\Delta x}\int_{x-0.5\Delta x}^{x+0.5\Delta x}\Phi(\xi,t)d\xi, \quad \Phi = \bar{\Phi} + \Phi'', \quad \overline{\Phi''} \neq 0$$

$$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{v}_i\bar{v}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}, \quad \tau_{ij} = \overline{v_i v_j} - \bar{v}_i\bar{v}_j$$

▶Filter and derivative

$$\overline{\frac{\partial p}{\partial x_i}} = \frac{1}{V}\int_V \frac{\partial p}{\partial x_i}dV \stackrel{?}{=} \frac{\partial}{\partial x_i}\left(\frac{1}{V}\int_V p dV\right) = \frac{\partial \bar{p}}{\partial x_i}$$

$$\overline{\frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i}\left(\frac{1}{V}\int_V p dV\right) + \mathcal{O}\left((\Delta x)^2\right) = \frac{\partial \bar{p}}{\partial x_i} + \mathcal{O}\left((\Delta x)^2\right)$$

▶Non-linear term

$$\overline{\frac{\partial v_i v_j}{\partial x_j}} = \frac{\partial}{\partial x_j}(\overline{v_i v_j}) + \mathcal{O}\left((\Delta x)^2\right)$$

$$\text{Left side}: \frac{\partial}{\partial x_j}(\overline{v_i v_j}) \underbrace{-\frac{\partial}{\partial x_j}(\overline{v_i v_j}) + \frac{\partial}{\partial x_j}(\bar{v}_i\bar{v}_j)} = \frac{\partial}{\partial x_j}(\bar{v}_i\bar{v}_j)$$

$$\text{Right side}: \underbrace{-\frac{\partial}{\partial x_j}(\overline{v_i v_j}) + \frac{\partial}{\partial x_j}(\bar{v}_i\bar{v}_j)} = -\frac{\partial \tau_{ij}}{\partial x_j}$$

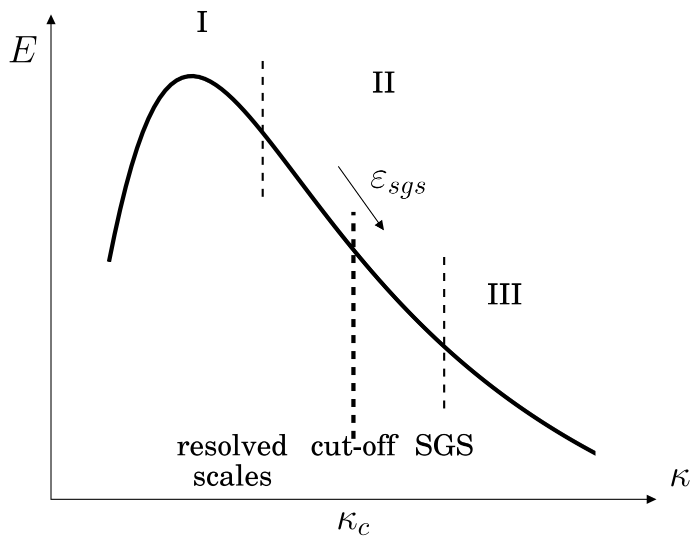
▶ $$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{v}_i\bar{v}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\tau_{ij} = \overline{v_i v_j} - \bar{v}_i\bar{v}_j$$

▶Filtering (used for turbulence modelling)

$$
\begin{aligned}
\bar{\bar{v}}_I &= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \bar{v}(\xi)d\xi = \frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{0} \bar{v}(\xi)d\xi + \int_{0}^{\Delta x/2} \bar{v}(\xi)d\xi\right) = \\
&= \frac{1}{\Delta x}\left(\frac{\Delta x}{2}\bar{v}_A + \frac{\Delta x}{2}\bar{v}_B\right). = \frac{1}{2}\left[\left(\frac{1}{4}\bar{v}_{I-1} + \frac{3}{4}\bar{v}_I\right) + \left(\frac{3}{4}\bar{v}_I + \frac{1}{4}\bar{v}_{I+1}\right)\right] \\
&= \frac{1}{8}\left(\bar{v}_{I-1} + 6\bar{v}_I + \bar{v}_{I+1}\right) \neq \bar{v}_I
\end{aligned}
$$

▶Resolved & SGS scales (GS & SGS)



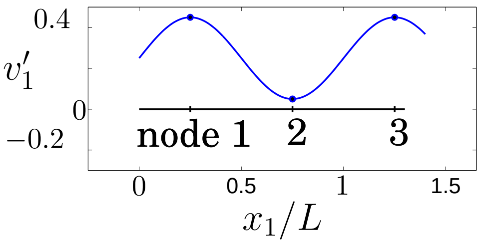
I=large scales
II=$-5/3$ range
III=dissipation range
$\kappa \leq \kappa_c$: Grid Scales
$\kappa > \kappa_c$: Sub-Grid Scales

▶Physical and wavenumber space.

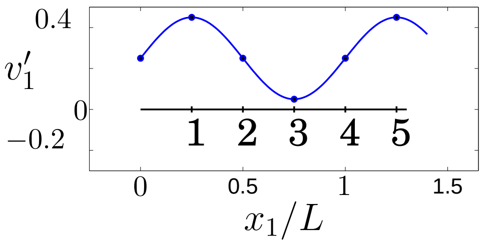
A Fourier series (see Appendix D)

$$v_1'(x) = \sum_{n=-\infty}^{\infty} c_n \exp(\imath \kappa_n x_1))$$

One period=two cells



One period=four cells



$v_1' = \sin(\kappa_c x_1)$

two cells : $\kappa_c 2\Delta x_1 = 2\pi \Rightarrow \kappa_c = 2\pi/(2\Delta x_1) = \pi/\Delta x_1$
four cells : $\kappa_c 4\Delta x_1 = 2\pi \Rightarrow \kappa_c = 2\pi/(4\Delta x_1) = \pi/(2\Delta x_1)$

▶Smagorinsky Subgrid model

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -\nu_{sgs}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) = -2\nu_{sgs}\bar{s}_{ij}$$

$$\nu_{sgs} = (C_S\Delta)^2\sqrt{2\bar{s}_{ij}\bar{s}_{ij}} \equiv (C_S\Delta)^2|\bar{s}|$$

$$\Delta = (\Delta V_{IJK})^{1/3}$$

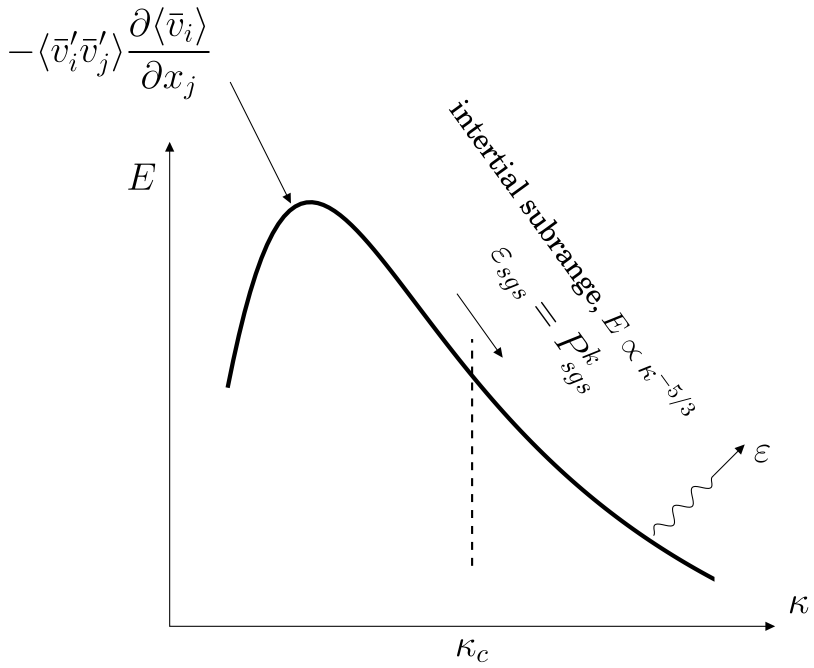
$|\bar{s}|$ stems from the production term in the k eq., $|\bar{s}^2| = 2\bar{s}_{ij}\bar{s}_{ij}$

▶One-equation model

$$\frac{\partial k_{sgs}}{\partial t}+\frac{\partial}{\partial x_j}(\bar{v}_j k_{sgs})=\frac{\partial}{\partial x_j}\left[(\nu+\nu_{sgs})\frac{\partial k_{sgs}}{\partial x_j}\right]+P_{k_{sgs}}-\varepsilon$$

$$\nu_{sgs}=c_k\Delta k_{sgs}^{1/2},\quad P_{k_{sgs}}=2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij},\quad \varepsilon=C_\varepsilon\frac{k_{sgs}^{3/2}}{\Delta}$$

▶Energy path



$$v_i=\langle v_i\rangle+v_i',\quad v_i=\bar{v}_i+v_i''=\langle\bar{v}_i\rangle+\bar{v}_i'+v_i''$$

$$k\equiv\frac{1}{2}\langle v_i'v_i'\rangle=\int_0^\infty E(\kappa)d\kappa,\quad k_{sgs}\equiv\frac{1}{2}\langle v_i''v_i''\rangle=\int_{\kappa_c}^\infty E(\kappa)d\kappa$$

$$\bar{k}\equiv\frac{1}{2}\langle\bar{v}_i'\bar{v}_i'\rangle=\int_0^{\kappa_c} E(\kappa)d\kappa,\quad \bar{K}\equiv\frac{1}{2}\langle\bar{v}_i\rangle\langle\bar{v}_i\rangle$$

$$K=\frac{1}{2}\langle v_i\rangle\langle v_i\rangle$$

Lecture 8

▶The dynamic model. C is computed. Test filter, $\widehat{\Delta}=2\Delta$

$$\frac{\partial \widehat{\bar{v}}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\widehat{\bar{v}}_i\widehat{\bar{v}}_j\right)=-\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i}+\nu\frac{\partial^2\widehat{\bar{v}}_i}{\partial x_j\partial x_j}-\frac{\partial T_{ij}}{\partial x_j} \tag{10}$$

$$T_{ij}=\widehat{\overline{v_i v_j}}-\widehat{\bar{v}}_i\widehat{\bar{v}}_j$$

$$\begin{aligned}\frac{\partial \widehat{\bar{v}}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\widehat{\bar{v}}_i\widehat{\bar{v}}_j\right)=&-\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i}+\nu\frac{\partial^2\widehat{\bar{v}}_i}{\partial x_j\partial x_j}-\frac{\partial \widehat{\tau}_{ij}}{\partial x_j}\\ &-\frac{\partial}{\partial x_j}\underbrace{\left(\widehat{\bar{v}_i\bar{v}_j}-\widehat{\bar{v}}_i\widehat{\bar{v}}_j\right)}_{\mathcal{L}_{ij}}\end{aligned} \tag{11}$$

Identification of Eqs. 10 and 11 gives

$$T_{ij}=\widehat{\bar{v}_i\bar{v}_j}-\widehat{\bar{v}}_i\widehat{\bar{v}}_j+\widehat{\tau}_{ij}=\mathcal{L}_{ij}+\widehat{\tau}_{ij},\quad \frac{1}{3}\delta_{ij}T_{kk}=\frac{1}{3}\delta_{ij}\mathcal{L}_{kk}+\frac{1}{3}\delta_{ij}\widehat{\tau}_{kk} \tag{12}$$

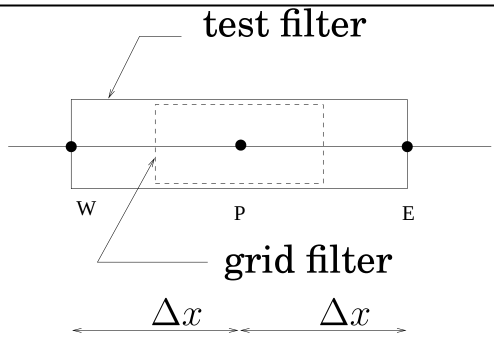
Smag.model for both grid and test level SGS stresses:

$$\tau_{ij}-\frac{1}{3}\delta_{ij}\tau_{kk}=-2C\Delta^2|\bar{s}|\bar{s}_{ij} \tag{13}$$

$$T_{ij}-\frac{1}{3}\delta_{ij}T_{kk}=-2C\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} \tag{14}$$

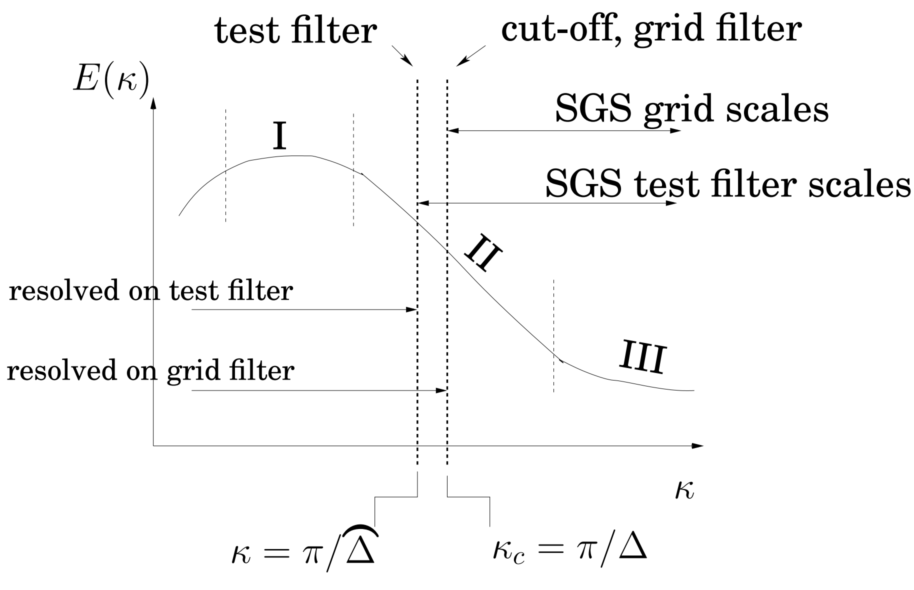
where

$$\widehat{\bar{s}}_{ij}=\frac{1}{2}\left(\frac{\partial\widehat{\bar{v}}_i}{\partial x_j}+\frac{\partial\widehat{\bar{v}}_j}{\partial x_i}\right),\quad |\widehat{\bar{s}}|=\left(2\widehat{\bar{s}}_{ij}\widehat{\bar{s}}_{ij}\right)^{1/2}$$



$\widehat{\bar{v}}$ is computed as ($\widehat{\Delta x} = 2\Delta x$)

$$
\begin{aligned}
\widehat{\bar{v}} &= \frac{1}{2\Delta x}\int_W^E \bar{v}dx = \frac{1}{2\Delta x}\left(\int_W^P \bar{v}dx + \int_P^E \bar{v}dx\right)\\
&= \frac{1}{2\Delta x}\left(\bar{v}_w \Delta x + \bar{v}_e \Delta x\right) = \frac{1}{2}\left(\frac{\bar{v}_W + \bar{v}_P}{2} + \frac{\bar{v}_P + \bar{v}_E}{2}\right)\\
&= \frac{1}{4}\left(\bar{v}_W + 2\bar{v}_P + \bar{v}_E\right)
\end{aligned}
$$



$$\tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j \quad \textbf{stresses with } \ell < \Delta$$

$$T_{ij} = \widehat{\overline{v_i v_j}} - \widehat{\bar{v}}_i \widehat{\bar{v}}_j \quad \textbf{stresses with } \ell < \widehat{\Delta}$$

$$\mathcal{L}_{ij} = T_{ij} - \widehat{\tau}_{ij} \quad \textbf{stresses with } \Delta < \ell < \widehat{\Delta}$$

Eqs. 13. 14 into Eq. 12 gives

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C\left(\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2\widehat{|\bar{s}|\bar{s}_{ij}}\right) \tag{15}$$

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C\left(\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2\widehat{|\bar{s}|\bar{s}_{ij}}\right) \tag{16}$$

$$Q = \left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij}\right)^2 \tag{17a}$$

$$M_{ij} = \left(\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2\widehat{|\bar{s}|\bar{s}_{ij}}\right) \tag{17b}$$

Find a Q which best satisfies Eq. 16 for all i, j

$$\frac{\partial Q}{\partial C} = 4M_{ij}\left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij}\right) = 0, \quad \partial^2 Q/\partial C^2 = 8M_{ij}M_{ij} > 0$$

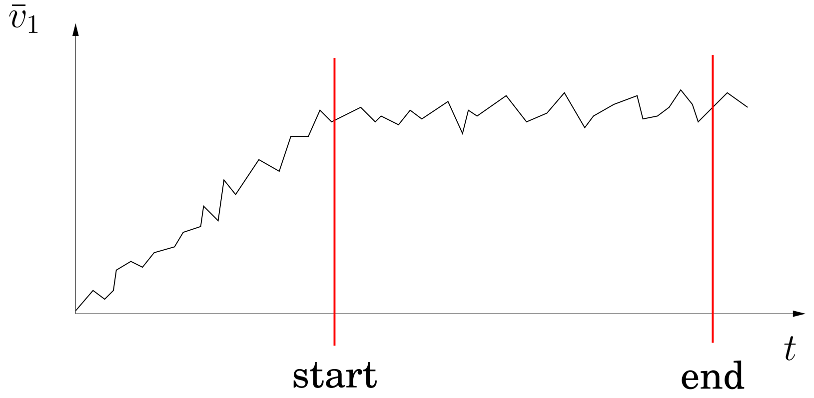
We get

$$C = -\frac{\mathcal{L}_{ij}M_{ij}}{2M_{ij}M_{ij}}, \quad \text{stability problems: needs smoothing}$$

▶Numerical method: RANS vs. LES

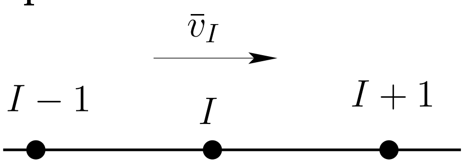
	RANS	**LES**
Domain	2D or 3D	always 3D
Time domain	steady or unsteady	always unsteady
Space discretization	2nd order upwind	central differencing
Time discretization	1st order	2nd order (e.g. C-N)
Turbulence model	$>$ two-equations	zero- or one-equation

▶**Start** and **end** time averaging. $t_{end} - t_{start} \simeq 100H/\langle\bar{v}\rangle_{center}$


$\bar{v}_1$
t
start
end

Lecture 9

▶Numerical dissipation



$$\bar{v}_I\left(\frac{\partial\bar{v}}{\partial x}\right)_{exact} = \bar{v}_I\left(\frac{\bar{v}_I-\bar{v}_{I-1}}{\Delta x}+\mathcal{O}\left(\Delta x\right)\right)$$

$$\bar{v}_{I-1} = \bar{v}_I - \Delta x\left(\frac{\partial\bar{v}}{\partial x}\right)_I + \frac{1}{2}(\Delta x)^2\left(\frac{\partial^2\bar{v}}{\partial x^2}\right)_I + \mathcal{O}\left((\Delta x)^3\right)$$

$$\text{diffusion term} = \frac{\partial}{\partial x}\left\{(\nu+\nu_{sgs}+\nu_{num})\frac{\partial\bar{v}}{\partial x}\right\}, \quad \varepsilon_{sgs+num} = 2(\nu_{sgs}+\nu_{num})\bar{s}_{ij}\bar{s}_{ij}$$

►Smagorinsky model derived from the one-equation model

- Small isotropic scales: production = dissipation

$P_{k_{sgs}} = 2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij} = \varepsilon$

Replace ε by ν_{sgs} and Δ.

$\nu_{sgs} = \varepsilon^a (C_S\Delta)^b \Rightarrow a = 1/3, b = 4/3 \Rightarrow \nu_{sgs} = (C_S\Delta)^{4/3}\varepsilon^{1/3}$

which gives

$\nu_{sgs} = (C_s\Delta)^2|\bar{s}|$

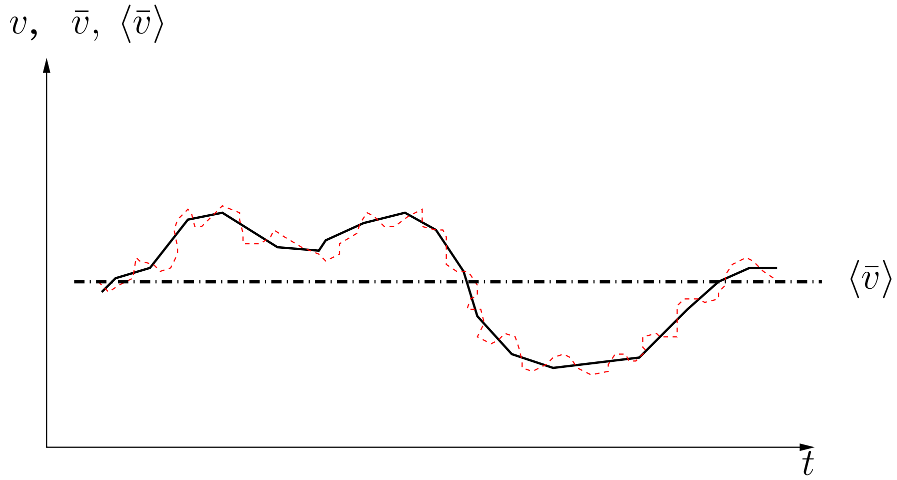
►In LES we resolve *large* scales. Near the wall, the "large" scales are not that large $\Rightarrow$ very expensive to resolve these "large" scales.

$\Delta x_1^+ \simeq 100$, $\Delta x_{2,min}^+ \simeq 1$, $\Delta x_3^+ \simeq 30$ $\quad\Rightarrow$ VERY expensive

►URANS. The usual Reynolds decomposition is employed

$$\bar{v}(t) = \frac{1}{2T}\int_{t-T}^{t+T} v(t)dt,\ v = \bar{v} + v''$$

URANS eqns=RANS, but with the unsteady term retained



- Decomposition of velocities: $v = \bar{v} + v'' = \langle\bar{v}\rangle + \bar{v}' + v''$ (note that v'' is not shown in the figure above)
- In theory, T should be $\ll$ resolved time scale= *scale separation*. In practice, it is seldom satisfied.
- RANS turbulence models are used: one should choose a model with small dissipation (i.e. small ν_t)
- Modelled dissipation (turbulence model) and numerical dissipation (discretization scheme) may be of equal importantance

▶DES=<u>D</u>etached <u>E</u>ddy <u>S</u>imulations: Use RANS near walls and LES away from walls

The S-A one-equation model (RANS)

$$\frac{d\rho\tilde{\nu}_t}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu+\mu_t}{\sigma_{\tilde{\nu}_t}}\frac{\partial\tilde{\nu}_t}{\partial x_j}\right) + \text{cr. term} + P - C_{w1}\rho f_w\left(\frac{\tilde{\nu}_t}{d}\right)^2, \quad d = x_n$$

$$\left(\frac{\tilde{\nu}_t}{d}\right)^2 \Rightarrow \left(\frac{\tilde{\nu}_t}{\tilde{d}}\right)^2, \quad \tilde{d} = \min\{C_{DES}\Delta, d\}, \quad \Delta = \max\{\Delta x_1, \Delta x_3, \Delta x_3\}$$

Lecture 10

The S-A one-equation model (DES)

▶Two-equation DES models

$k-\varepsilon$ RANS

$$C^k = D^k + P^k - \varepsilon, \quad C^\varepsilon = D^\varepsilon + P^\varepsilon - \Psi$$

$k-\varepsilon$ DES (modify ε_T)

$$C^k = D^k + P^k - \varepsilon \Rightarrow C^k = D^k + P^k - \varepsilon_T, \quad \varepsilon_T = \max\left(\varepsilon, C_\varepsilon \frac{k^{3/2}}{\Delta}\right)$$

$k-\varepsilon$ DES (modify ν_T and ε_T)

$$C^k = D^k + P^k - \varepsilon_T, \quad C^\varepsilon = D^\varepsilon + P^\varepsilon - \Psi^\varepsilon \quad \nu_t = k^{1/2}\ell_t$$

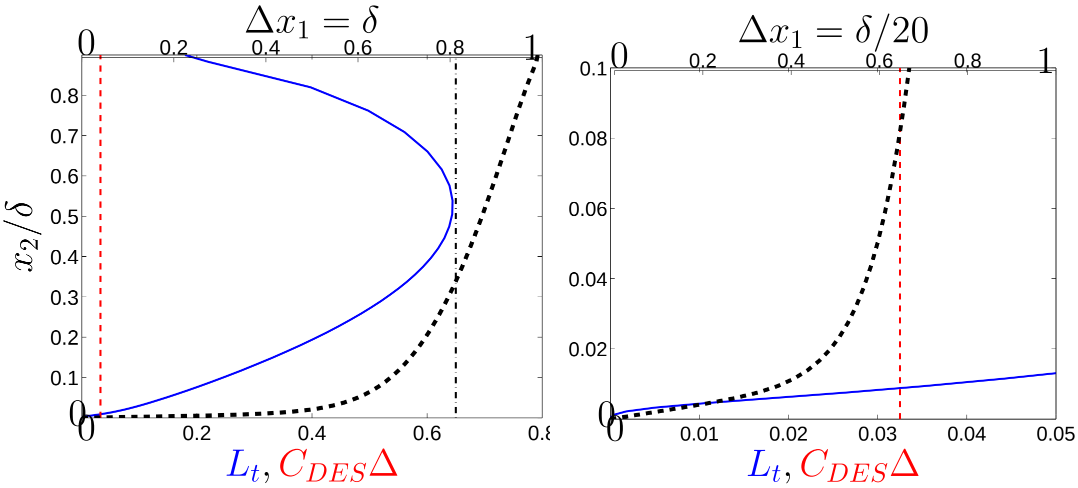
1. $\varepsilon_T = \max\left(\varepsilon, C_\varepsilon \frac{k^{3/2}}{\Delta}\right)$: $\varepsilon_T \uparrow \Rightarrow \nu_t$, $k \downarrow$ in LES region
2. $\ell_t = \min\left(C_\mu \frac{k^{3/2}}{\varepsilon}, C_k\Delta\right) \Rightarrow \nu_t$ decreases in LES region

$k-\omega$ SST DES (modify $\beta^* k\omega$)

$$C^k = D^k + P^k - F_{DES}\beta^* k\omega, F_{DES} = \max\left\{\frac{L_t}{C_{DES}\Delta}, 1\right\}$$

Dissip. term $\beta^* k\omega \propto \dfrac{k^{3/2}}{L_t} \Rightarrow \beta^* k\omega \propto \dfrac{k^{3/2}}{C_{DES}\Delta}$ in LES region

►The grid size is critical to the switching position. Two examples below: $\Delta x_1 = \delta$ and $\Delta x_1 = \delta/20$



— : $L_t = k^{3/2}/\varepsilon$; - - - : $C_{DES}\Delta$, $\Delta x_1 = \delta/20$; - · - : $C_{DES}\Delta$, $\Delta x_1 = \delta$; thick dashed line: $\langle v_1 \rangle$

►It may occur that the F_{DES} term switches to DES in the boundary layer because Δx is too small (Δz is usually smaller than Δx)

Hence boundary layer is treated in LES mode with too a coarse mesh $\Rightarrow$ poorly resolved LES $\Rightarrow$ inaccurate predictions.

►The solution is DDES (Delayed DES)

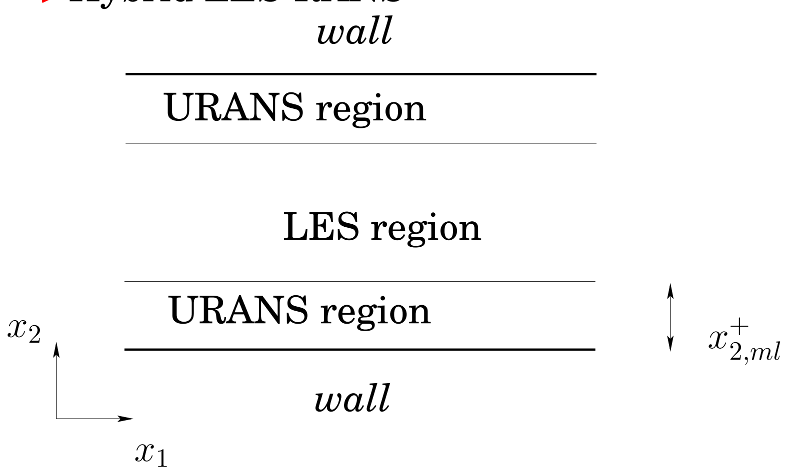
$$F_{DES} = \max\left\{\frac{L_t}{C_{DES}\Delta}(1 - F_S), 1\right\}$$

where F_S is taken as F_1 or F_2 of the SST model.

DES: The entire boundary layer is modelled with URANS

Hybrid LES-RANS: Only the inner part (inner part of the log region) is modelled with URANS. This is also call **WM-LES** (WM=Wall-modelled)

►Hybrid LES-RANS



One-equation model in both URANS and LES region

$$\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j}(\bar{v}_j k_T) = \frac{\partial}{\partial x_j}\left[(\nu + \nu_T)\frac{\partial k_T}{\partial x_j}\right] + P_{k_T} - C_\varepsilon \frac{k_T^{3/2}}{\ell}$$

$$P_{k_T} = -\tau_{ij}\bar{s}_{ij}, \quad \tau_{ij} = -2\nu_T \bar{s}_{ij}, \quad \nu_T \propto k^{1/2}\ell$$

Inner region ($x_2 \leq x_{2,ml}$): $\ell \propto \kappa x_2$; outer region: $\ell = \Delta$

▶The SAS model: This is a method to improve URANS. The objective is to reduce ν_t when the equations want to go into unsteady, resolving turbulence mode (LES mode).

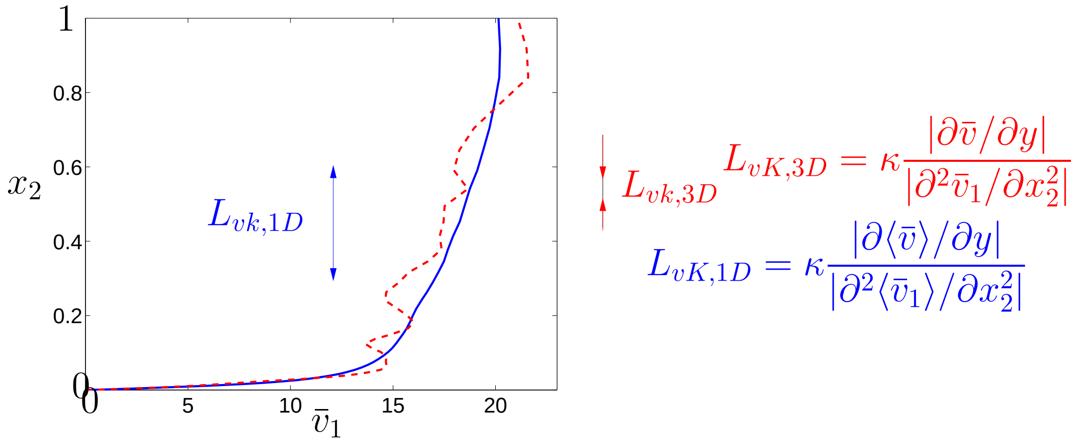
▶An additional source term, P_{SAS}, is introduced in the ω equation. $P_{SAS} \propto \frac{L_t}{L_{vK,3D}}$.

$$L_t \propto \frac{k^{1/2}}{\omega}, \quad L_{vK,3D} = \kappa \frac{|\bar{s}|}{|U''|}, \quad U'' = \left(\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} \frac{\partial^2 \bar{v}_i}{\partial x_k \partial x_k} \right)^{0.5}$$

▶The von Kármán length scale is used to detect unsteadiness; when it happens:

- the P_{SAS} term increases
- $\to \omega$ increases
- $\to \nu_t$ and k decrease
- $\to$ mom.eqns go into (or stay in) unsteady mode

▶In URANS, resolved fluctuations may be damped.



Solid line: $L_{vk,1D}$; dashed line: $L_{vk,3D}$

Lecture 11

▶PANS: Partial-Averaging Navier-Stokes. The k and ε eqns read (subscript u=$u$nresolved)

$$\frac{\partial k_u}{\partial t}+\frac{\partial(k_u\bar{v}_j)}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_u}{\sigma_{ku}}\right)\frac{\partial k_u}{\partial x_j}\right]+P_u-\varepsilon_u$$
$$\frac{\partial \varepsilon_u}{\partial t}+\frac{\partial(\varepsilon_u\bar{v}_j)}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_u}{\sigma_{\varepsilon u}}\right)\frac{\partial \varepsilon_u}{\partial x_j}\right]+C_{\varepsilon 1}P_u\frac{\varepsilon_u}{k_u}-C^*_{\varepsilon 2}\frac{\varepsilon_u^2}{k_u}$$

where

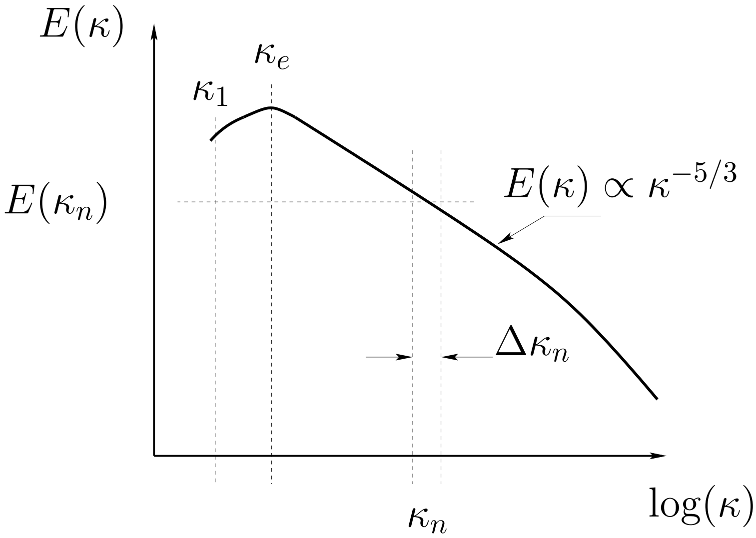
$$C^*_{\varepsilon 2}=C_{\varepsilon 1}+\frac{f_k}{f_\varepsilon}(C_{\varepsilon 2}-C_{\varepsilon 1}),\quad C_{\varepsilon 1}=1.44,\quad C_{\varepsilon 2}=1.92$$
$$\Rightarrow C^*_{\varepsilon 2}=1.44+\frac{f_k}{f_\varepsilon}0.48$$

- $f_k=k/(k+k_{res})$ and $f_\varepsilon=\varepsilon/(\varepsilon+\varepsilon_{res})$ denote the ratio of modelled to total k and ε, respectively.
- Usually $f_\varepsilon=1$; $f_\varepsilon<1$ implies that dissipative scales are resolved.
- If f_k is smaller than one (say, $f_k=0.4$) then:
  - $C^*_{\varepsilon 2}\frac{\varepsilon_u^2}{k_u}$ is reduced
  - $\Rightarrow \varepsilon_u$ is increased
  - $\Rightarrow k_u$ and ν_u are decreased
  - $\Rightarrow$ the momentum eqns go into LES mode.

▶In LES and DES the large-scale turbulence is resolved: hence, turbulent fluctuations should be provided as inlet boundary conditions

▶Synthetic fluctuations is one method. The inlet velocity can be written as a Fourier series

$$\mathbf{v}'(\mathbf{x}) = 2\sum_{n=1}^{N} \hat{u}^n \cos(\boldsymbol{\kappa}^n \cdot \mathbf{x} + \psi^n)\boldsymbol{\sigma}^n$$



▶Amplitude $\hat{u}^n$ related to energy spectrum: $\hat{u}^n = (E(\kappa)\Delta\kappa)^{1/2}$

▶Usually we generate energy spectra from turbulent fluctuations. Here we do the opposite: we assume a spectrum and generate turbulent fluctuations. A $-5/3$ spectrum is assumed: this gives the amplitude $\hat{u}^n$ for wavenumber κ_n

▶κ_{max} from the cells size

▶$\kappa_e \propto 1/L_t$ from the integral turbulent length scale

▶$\kappa_{min} = \kappa_e/2$ $\Delta\kappa = (\kappa_{max} - \kappa_{min})/N \Rightarrow \kappa_1 = \kappa_{min}$, $\kappa_2 = \kappa_1 + \Delta\kappa, \ldots \kappa_N = \kappa_{max}$

►With this method synthetic turbulent fluctuations are generated at the inlet plane for M time steps.

►However, they are not correlated in time. An asymmetric time filter is used $(\mathcal{V}_1')^m = a(\mathcal{V}_1')^{m-1} + b(v_1')^m$

The coefficient a is related to the turbulent integral timescale, $\mathcal{T}$, as $a = \exp(-\Delta t/\mathcal{T})$

►The coefficient b is computed as $b = (1 - a^2)^{1/2}$ which ensures that $\mathcal{V}_{1,rms}' = v_{1,rms}'$

►Finally, the turbulent synthetic fluctuations are superimposed to the inlet mean velocity.