Next, we use **Trick 2**

$$\bar{v}_j \left(\overline{v'_i \frac{\partial v'_i}{\partial x_j}} \right) = \bar{v}_j \frac{\partial}{\partial x_j} \overline{\left(\frac{1}{2} v'_i v'_i \right)} = \bar{v}_j \frac{\partial}{\partial x_j} (k) = \frac{\partial}{\partial x_j} (\bar{v}_j k) \tag{8.9}$$

The third term in Eq. 8.6 can be written as (replace $\bar{v}_j$ by v'_j and use the same technique as in Eq. 8.9)

$$\frac{1}{2} \frac{\partial}{\partial x_j} \overline{(v'_j v'_i v'_i)}. \tag{8.10}$$

The first term on the right side of Eq. 8.5 is re-written using the continuity equation as

$$-\frac{1}{\rho} \overline{v'_i \frac{\partial p'}{\partial x_i}} = -\frac{1}{\rho} \frac{\partial \overline{p' v'_i}}{\partial x_i} \tag{8.11}$$

The second term on the right side of Eq. 8.5 can be written

$$\nu \overline{v'_i \frac{\partial^2 v'_i}{\partial x_j \partial x_j}} = \nu \overline{\frac{\partial}{\partial x_j} \left(\frac{\partial v'_i}{\partial x_j} v'_i \right)} - \nu \overline{\frac{\partial v'_i}{\partial x_j} \frac{\partial v'_i}{\partial x_j}} \tag{8.12}$$

applying **Trick 1** (if we apply the product rule on the first term on the right side of Eq. 8.12 we get the left side and the second term on the right side). For the first term in Eq. 8.12 we use the same trick as in Eq. 8.9 so that

$$\begin{aligned} \nu \overline{\frac{\partial}{\partial x_j} \left(\frac{\partial v'_i}{\partial x_j} v'_i \right)} &= \nu \overline{\frac{\partial}{\partial x_j} \left(\frac{1}{2} \left(\frac{\partial v'_i}{\partial x_j} v'_i + \frac{\partial v'_i}{\partial x_j} v'_i \right) \right)} = \\ \nu \overline{\frac{\partial}{\partial x_j} \left(\frac{1}{2} \left(\frac{\partial v'_i v'_i}{\partial x_j} \right) \right)} &= \nu \frac{1}{2} \frac{\partial^2 \overline{v'_i v'_i}}{\partial x_j \partial x_j} = \nu \frac{\partial^2 k}{\partial x_j \partial x_j} \end{aligned} \tag{8.13}$$

The last term on the right side of Eq. 8.5 is zero because it is time averaging of a fluctuation, i.e. $\overline{\bar{a} b'} = \bar{a} \overline{b'} = 0$. Now we can assemble the transport equation for the turbulent kinetic energy. Equations 8.7, 8.9, 8.11, 8.12 and 8.13 give

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{I} = \underbrace{-\overline{v'_i v'_j} \frac{\partial \bar{v}_i}{\partial x_j}}_{II} \underbrace{-\frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \overline{v'_j p'} + \frac{1}{2} \overline{v'_j v'_i v'_i} - \nu \frac{\partial k}{\partial x_j} \right]}_{III} \underbrace{-\nu \overline{\frac{\partial v'_i}{\partial x_j} \frac{\partial v'_i}{\partial x_j}}}_{IV} \tag{8.14}$$

The terms in Eq. 8.14 have the following meaning.

I. Convection.

II. Production, P^k. The large turbulent scales extract energy from the mean flow. This term (including the minus sign) is almost always positive. It is largest for the energy-containing eddies, i.e. for small wavenumbers, see Fig. 5.2. This term originates from the convection term (the first term on the right side of Eq. 8.6). It can be noted that the production term is an acceleration term, $v'_j \partial \bar{v}_i / \partial x_j$, multiplied by a fluctuating velocity, v'_i, i.e. the product of an inertial force per unit mass (acceleration) and a fluctuating velocity. A force multiplied with a velocity corresponds to energy per unit time (energy transfer per unit time). When the acceleration term and the fluctuating velocity are in opposite directions (i.e. when $P^k > 0$), the fluctuating velocity field extracts energy from the mean field.