23.3 A new formulation of f_k for the PANS model

It is natural to link f_k to the mesh resolution and in that aim several proposals have been made on how to compute f_k. Girimaji and Abdol-Hamid [162] proposed one way to compute f_k:

$$f_k = C_\mu^{-1/2}\left(\frac{\Delta}{L_t}\right)^{2/3}, L_t = \frac{k_{tot}^{3/2}}{\varepsilon} \tag{23.38}$$

using $\Delta = \Delta_{min}$, the smallest grid cell size. Basara *et al.* also used f_k prescribed from Eq. 23.38, however taking the geometric average $\Delta = (\Delta V)^{1/3}$. [167] they have made a slightly different proposal which reads

$$f_k = \frac{\Delta}{L_t} \tag{23.39}$$

More recently, in [168] they derived an expression from the Kolmogorov energy spectrum which reads

$$f_k = 1 - \left[\frac{(\Lambda/\Delta)^{2/3}}{0.23 + (\Lambda/\Delta)^{2/3}}\right]^{9/2} \tag{23.40}$$

Rccently, Davidson and Friess [169][5] proposed a new formulation for f_k. It is based on the H-equivalence introduced by Friess *et al.* (2015). In this formulation the expression of f_k is derived to mimic DES. This new formulation behaves very much like "classic DES", even though tthe two formulations use different mechanisms to separate modeled and resolved scales. They show very similar performance in separated flows as well as in attached boundary layers. Moreover, the new formulation exhibits similar robustness features as DES.

23.3.1 f_k derived from the equivalence criterion

In [170] a relation between f_k and the grid step is derived, through the establishment of a statistical equivalence between DES and PITM. To that aim, they performed perturbation analyses about the equilibrium states, representing small variation of the energy partition. They did the analysis with and without considering inhomogeneity. That derivation is summarized here in a homogeneous framework, as a first step. Let us first consider the PANS/PITM equations. For equilibrium turbulence $d\tau/dt = 0$ where $\tau = k/\varepsilon$, Eq. 23.19 gives

$$\begin{aligned} \frac{d\tau}{dt} &= \frac{1}{\varepsilon}\frac{dk}{dt} - \frac{k}{\varepsilon^2}\frac{d\varepsilon}{dt} = \frac{1}{\varepsilon}\left(P^k + D^k - \varepsilon\right) \\ &- \frac{k}{\varepsilon^2}\left(C_{\varepsilon 1}\frac{\varepsilon}{k}P^k + D^\varepsilon - C^*_{\varepsilon 2}\frac{\varepsilon^2}{k}\right) = 0 \end{aligned} \tag{23.41}$$

where D^k and D^ε denote the diffusion term for k and ε, respectively. For local homogeneous turbulence (i.e. $D^k = D^\varepsilon = 0$), it can be written

$$\begin{aligned} &\gamma(C_{\varepsilon 1} - 1)Sk = (C^*_{\varepsilon 2} - 1)\varepsilon \\ &\gamma = \frac{P^k}{Sk}, \quad S = (2\bar{s}_{ij}\bar{s}_{ij})^{1/2} \end{aligned} \tag{23.42}$$

[5] L. Davidson and C. Friess "A new formulation of fk for the PANS model", Journal of Turbulence, doi = 10.1080/14685248.2019.1641605, 2019.

The quantities that are affected by the partition between modeled and resolved turbulence (i.e. f_k) in Eq. 23.42 are γ, S, k and $C^*_{\varepsilon 2}$.[6] Differentiation of Eq. 23.42, by considering infinitesimal perturbations $\delta\gamma$, δS, δk and $\delta C^*_{\varepsilon 2}$ of the variables, yields:

$$\delta\gamma S k + \delta S \gamma k + \delta k \gamma S = \frac{\delta C^*_{\varepsilon 2}\varepsilon}{C_{\varepsilon 1} - 1} \tag{23.43}$$

so that

$$\frac{\delta\gamma}{\gamma} + \frac{\delta S}{S} + \frac{\delta k}{k} = \frac{\delta C^*_{\varepsilon 2}\varepsilon}{(C_{\varepsilon 1} - 1)\gamma S k} = \frac{\delta C^*_{\varepsilon 2}}{C^*_{\varepsilon 2} - 1} \tag{23.44}$$

Equation 23.44 was derived for the PANS/PITM equations. Now we repeat the derivation for the DES equations. The differences between DES and PITM/PANS are that in DES (i) $C^*_{\varepsilon 2} = C_{\varepsilon 2}$ is constant and (ii) the dissipation term in the equation for modeled energy k is replaced with $\psi\varepsilon$, i.e.

$$\frac{dk}{dt} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \psi\varepsilon \tag{23.45}$$

$$\psi = \max\left(1, \frac{k^{3/2}/\varepsilon}{C_{DES}\Delta_{max}}\right), \quad \Delta_{max} = \max(\Delta x_1, \Delta x_2, \Delta x_3) \tag{23.46}$$

$$\frac{d\varepsilon}{dt} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1}P_k\frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

Assuming $d\tau/dt = 0$ and local homogeneous turbulence gives

$$\gamma(C_{\varepsilon 1} - 1)Sk = (C_{\varepsilon 2} - \psi)\varepsilon \tag{23.47}$$

We differentiate so that

$$\frac{\delta\gamma}{\gamma} + \frac{\delta S}{S} + \frac{\delta k}{k} = -\frac{d\psi\varepsilon}{(C_{\varepsilon 1} - 1)Sk\gamma} = -\frac{d\psi}{C_{\varepsilon 2} - \psi} \tag{23.48}$$

Equations 23.43 and 23.48 describe how $C^*_{\varepsilon 2}$ and ψ depend on variations in γ, S and k. The parameters $C^*_{\varepsilon 2}$ and ψ vary from $C_{\varepsilon 2}$ and 1 (RANS values), respectively, to $C^*_{\varepsilon 2}$ and $\psi(\Delta)$ (LES values). Combining Eqs. 23.43 and 23.48 and integrating from RANS to LES conditions ($C^*_{\varepsilon 2}$ and ψ)

$$\begin{aligned}\int_{C_{\varepsilon 2}}^{C^*_{\varepsilon 2}} \frac{dC^*_{\varepsilon 2}}{C^*_{\varepsilon 2} - 1} &= \int_1^{\psi} -\frac{d\psi}{C_{\varepsilon 2} - \psi} \Rightarrow \\ \ln\left(\frac{C^*_{\varepsilon 2} - 1}{C_{\varepsilon 2} - 1}\right) &= \ln\left(\frac{C_{\varepsilon 2} - \psi}{C_{\varepsilon 2} - 1}\right)\end{aligned} \tag{23.49}$$

By using the expression for $C^*_{\varepsilon 2}$ in Eq. 23.19 (with $f_2 = f_\varepsilon = 1$), and ensuring that $0 < f_k \le 1$ we finally get

$$f_k = \max\left[0, \min\left(1, 1 - \frac{\psi - 1}{C_{\varepsilon 2} - C_{\varepsilon 1}}\right)\right] \tag{23.50}$$

where ψ is given by Eq. 23.46. This model is evaluated in [169]. It gives much better results than the old PANS model and very similar results to the DES model. What is the advantage of the new PANS model vs. the DES model? The PANS model is based on a rigorous derivation whereas DES is based on an ad-hoc modification of RANS models.

[6] ε is independent of f_k provided that no dissipation is resolved, which corresponds to $f_\varepsilon = 1$