front of the integral in Eq. 18.11 (the Fourier par is *symmetric*); as a consequence it is also absent in Eq. 18.12.

Note that it is physically meaningful to use Fourier transforms only in a homogeneous coordinate direction; in non-homogeneous directions the Fourier coefficients – which are not a function of space – have no meaning. Using the convolution theorem (saying that the integrated product of two functions is equal to the product of their Fourier transforms), we can filter $\hat{v}$ using Eqs. 18.10 and 18.8

$$\begin{aligned}\bar{\hat{v}}(\kappa) = \widehat{\bar{v}}(\kappa) = \int_0^\infty \bar{v}(\eta)\exp(-2\pi\imath\kappa\eta)d\eta \\ = \int_0^\infty\int_0^\infty \exp(-2\pi\imath\kappa\eta)G(\alpha)v(\eta-\alpha)d\alpha d\eta \\ = \int_0^\infty\int_0^\infty \exp(-2\pi\imath\kappa\alpha)\exp(-2\pi\imath\kappa(\eta-\alpha))G(\alpha)v(\eta-\alpha)d\alpha d\eta \\ = \int_0^\infty\int_0^\infty \exp(-2\pi\imath\kappa\alpha)\exp(-2\pi\imath\kappa\xi)G(\alpha)v(\xi)d\xi d\alpha = \hat{G}(\kappa)\hat{v}(\kappa)\end{aligned} \tag{18.12}$$

(in the last line we used $\xi = \eta - \alpha$). If we use the cut-off filter and filter twice we get

$$\bar{\bar{\hat{v}}} = \hat{G}_C\hat{G}_C\hat{v} = \hat{G}_C\hat{v} \tag{18.13}$$

since $\hat{G}_C^2 = \hat{G}_C$, see Eq 18.9. Since $\bar{\bar{\hat{v}}} = \bar{\hat{v}}$ for the Fourier transform $\hat{v}$, we know that – when using the cuf-off fiter – $\bar{\bar{v}} = \bar{v}$. Thus, contrary to the box-filter (see Eq. 18.7), nothing happens when we filter twice in spectral space. The box filter is sharp in physical space but not in wavenumber space; for the cut-off filter it is vice versa.

In finite volume methods box filtering is always used. Furthermore *implicit* filtering is employed. This means that the filtering is the same as the discretization (=integration over the control volume is equal to the filter volume, see Eq. 18.17).

18.5 Highest resolved wavenumbers

Any function can be expressed as a Fourier series such as Eq. 18.11 (see Section 5.3, Eq. D.28 and Section E) provided that the coordinate direction is homogeneous. Let's choose the fluctuating velocity in the x_1 direction, i.e. v_1', and let it be a function of x_1. We require it to be homogeneous, i.e. its RMS, $v_{1,rms}$, does not vary with x_1. Now we ask the question: on a given grid, what is the highest wavenumber that is resolved? Or, in other words, what is the cut-off wavenumber?

Consider Fig. 18.4 (cf. Section E). We assume that v_2' is periodic which makes it convenient to use Fourier transform. We construct v_2' as a sum of four Fourier components

$$v_2'(x_2) = b_1\cos\left(\frac{2\pi}{L/1}x_2\right)+b_2\cos\left(\frac{2\pi}{L/2}x_2\right)+b_3\cos\left(\frac{2\pi}{L/3}x_2\right)+b_4\cos\left(\frac{2\pi}{L/4}x_2\right)$$

The thick line in Fig. 18.4 shows how v_2' varies with x_2. The blue circles show the Fourier component with the highest wave number, $m = 4$. How many grid point does it take to resolve this Fourier component?

Figure 18.5 shows an example how to resolve the highest Fourier component on two different grids. The wave shown in Fig. 18.5a reads

$$v_1' = 0.25\left[1 + 0.8\sin(\kappa_1 x_1)\right] \tag{18.14}$$

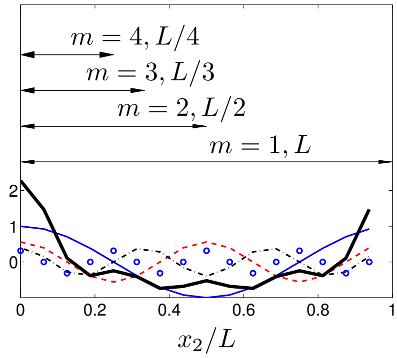
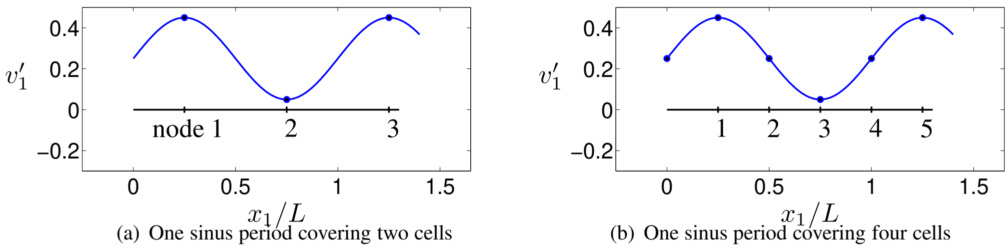


Figure 18.4: v_2' vs. x_2/L. ▬▬ : term 1 ($m = 1$); ▬ ▬ : term 2 ($m = 2$); ▬ · ▬ : term 3 ($m = 3$); ∘: term 4 ($m = 4$); thick line: v_2'. Matlab code is given in Section E.3.



(a) One sinus period covering two cells (b) One sinus period covering four cells

Figure 18.5: Physical and wavenumber space. Sinus curves with different wavenumbers illustrated in physical space.

and it covers two cells ($\Delta x_1/L = 0.5$). If we define this as the cut-off wavenumber we get $\kappa_{1,c}L = \kappa_{1,c}2\Delta x_1 = 2\pi$ (i.e. $\sin(\kappa_{1,c}2\Delta x_1) = \sin(2\pi)$; recall that 2π is one period) so that

$$\kappa_{1,c} = 2\pi/(2\Delta x_1) = \pi/\Delta x_1 \tag{18.15}$$

It is of course questionable if v_1' in Fig. 18.5a really is resolved since the sinus wave covers only two cells. However this is the usual definition of the cut-off wavenumber.

If we require that the highest resolved wavenumber should be covered by four cells ($\Delta x_1/L = 0.25$), as in Fig. 18.5b, then the cut-off wavenumber is given by $\kappa_{1,c} = 2\pi/(4\Delta x_1) = \pi/(2\Delta x_1)$.

18.6 Subgrid model

We need a subgrid model to model the turbulent scales which cannot be resolved by the grid and the discretization scheme.

The simplest model is the Smagorinsky model [70]:

$$\begin{aligned}\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} &= -\nu_{sgs}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) = -2\nu_{sgs}\bar{s}_{ij} \\ \nu_{sgs} &= (C_S\Delta)^2\sqrt{2\bar{s}_{ij}\bar{s}_{ij}} \equiv (C_S\Delta)^2\,|\bar{s}|\end{aligned} \tag{18.16}$$

and the filter-width is taken as the local grid size

$$\Delta = (\Delta V_{IJK})^{1/3} \tag{18.17}$$