(assuming that C varies slowly), substituting this equation and Eq. 18.37 into Eq. 18.30 gives

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C\left(\widehat{\Delta}^2 |\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2 \widehat{|\bar{s}|\bar{s}_{ij}}\right) \tag{18.38}$$

where we used the relatation

$$\frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = \frac{1}{3}\delta_{ij}T_{kk} - \frac{1}{3}\delta_{ij}\widehat{\tau}_{kk}$$

obtaioned from Eq. 18.29.

Note that the "constant" C in Eq. 18.38 really is a function of both space and time, i.e. $C = C(x_i, t)$.

Equation 18.38 is a tensor equation, and we have five ($\bar{s}_{ij}$ is symmetric and traceless) equations for C. Lilly [77] suggested to satisfy Eq. 18.38 in a least-square sense. Let us define the error as the difference between the left-hand side and the right-hand side of Eq. 18.38 raised to the power of two, i.e.

$$Q = \left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij}\right)^2 \tag{18.39a}$$

$$M_{ij} = \left(\widehat{\Delta}^2 |\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2 \widehat{|\bar{s}|\bar{s}_{ij}}\right) \tag{18.39b}$$

The error, Q, has a minimum (or maximum) when $\partial Q/\partial C = 0$. Carrying out the derivation of 18.39a gives

$$\frac{\partial Q}{\partial C} = 4M_{ij}\left(\mathcal{L}_{ij} + 2CM_{ij}\right) = 0 \tag{18.40}$$

Note that $\frac{1}{3}\delta_{ij}\mathcal{L}_{kk}M_{ij} = \frac{1}{3}\mathcal{L}_{kk}M_{ii} = 0$ since $\widehat{\bar{s}}_{ii} = \bar{s}_{ii} = 0$ thanks to contiuity. Since $\partial^2 Q/\partial C^2 = 8M_{ij}M_{ij} > 0$ we find that Eq. 18.40 represents indeed a minimum. Equation 18.40 finally gives

$$C = -\frac{\mathcal{L}_{ij}M_{ij}}{2M_{ij}M_{ij}} \tag{18.41}$$

It turns out that the dynamic coefficient C fluctuates wildly both in space and time. This causes numerical problems, and it has been found necessary to average C in homogeneous direction(s). Furthermore, C must be clipped to ensure that the total viscosity stays positive ($\nu + \nu_{sgs} \geq 0$).

In real 3D flows, there is no homogeneous direction. Usually local averaging and clipping (i.e. requiring that C stays within pre-defined limits) of the dynamic coefficient is used.

Use of one-equation models solve these numerical problems (see p. 190).

18.14 Numerical dissipation

The main function of an SGS model is to dissipate (i.e. to dampen) resolved turbulent fluctuations. The SGS model is – hopefully – designed to give a proper amount of dissipation. This is the reason why in LES we should use a central differencing scheme, because this class of schemes does not give any *numerical* dissipation. All upwind schemes give numerical dissipation in addition to the modeled SGS dissipation. Indeed,