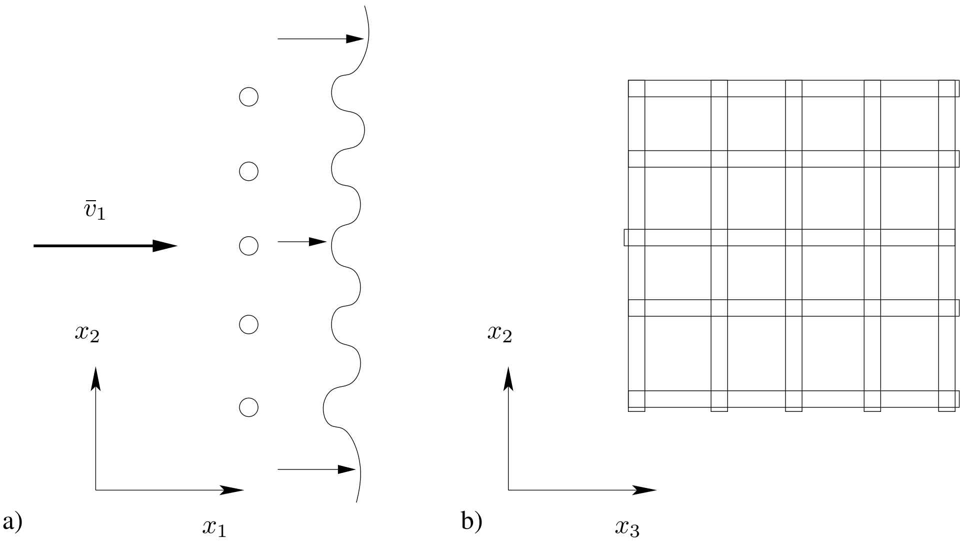

Figure 11.2: Decaying grid turbulence. The circles (a) and the thin rectangles (b) illustrates part of the grid which consists of a mesh of circular cylinders.

Let us investigate how Eq. 11.57 behaves for decaying grid turbulence, see Fig. 11.2. Flow from left with velocity $\bar{v}_1$ passes through a grid. The grid creates velocity gradients behind the grid which generates turbulence. Further downstream the velocity gradients are smoothed out and the mean flow becomes constant. From this point and further downstream the flow represents anisotropic turbulence (homogeneous in the x_2 and x_3 directions) which is slowly approaching isotropic turbulence; furthermore the turbulence is slowly dying (i.e. decaying) due to dissipation. The exact $\overline{v'_i v'_j}$ equation for this flow reads (no production or diffusion because of homogeneity)

$$\bar{v}_1 \frac{d\overline{v'_i v'_j}}{dx_1} = \overline{\frac{p'}{\rho}\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)} - \varepsilon_{ij} \tag{11.58}$$

Rotta's pressure-strain model is supposed to reduce anisotropy. Thus it should be interesting to re-write Eq. 11.58 expressed in the normalized anisotropy Reynolds stress tensor which is defined as

$$a_{ij} = \frac{\overline{v'_i v'_j}}{k} - \frac{2}{3}\delta_{ij} \tag{11.59}$$

Note that when the turbulence is isotropic, then $a_{ij} = 0$. We introduce a_{ij} (Eq. 11.59), Rotta's model (Eq. 11.57) and the model for the dissipation tensor (11.49) into Eq. 11.58 so that

$$\bar{v}_1 \left(\frac{d(k a_{ij})}{dx_1} + \delta_{ij}\frac{2}{3}\frac{dk}{dx_1}\right) = -c_1 \varepsilon a_{ij} - \frac{2}{3}\delta_{ij}\varepsilon \tag{11.60}$$

Analogously to Eq, 11.58, the k equation in decaying grid turbulence reads

$$\bar{v}_1 \frac{dk}{dx_1} = -\varepsilon \tag{11.61}$$

Inserting Eq. 11.61 in Eq. 11.60, the left side reads

$$\begin{aligned}\bar{v}_1 a_{ij}\frac{dk}{dx_1} + \bar{v}_1 k\frac{da_{ij}}{dx_1} + \frac{2}{3}\delta_{ij}\bar{v}_1\frac{dk}{dx_1} &= \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)\bar{v}_1\frac{dk}{dx_1} + k\bar{v}_1\frac{da_{ij}}{dx_1}\\ &= -\left(a_{ij} + \frac{2}{3}\delta_{ij}\right)\varepsilon + k\bar{v}_1\frac{da_{ij}}{dx_1}\end{aligned}$$

Dividing by k and inserting into Eq.Eq. 11.60 we get

$$\bar{v}_1\frac{da_{ij}}{dx_1} = -c_1\frac{\varepsilon}{k}a_{ij} - \frac{2}{3}\delta_{ij}\frac{\varepsilon}{k} + \frac{\varepsilon}{k}a_{ij} + \frac{2}{3}\delta_{ij}\frac{\varepsilon}{k} = \frac{\varepsilon}{k}a_{ij}(1 - c_1) \tag{11.62}$$

Provided that $c_1 > 1$ Rotta's model does indeed reduce non-isotropy as it should.

The model of the slow pressure-strain term in Eq. 11.57 can be extended by including terms which are non-linear in $\overline{v_i'v_j'}$. To make it general it is enough to include terms which are quadratic in $\overline{v_i'v_j'}$, since according to the Cayley-Hamilton theorem, a second-order tensor satisfies its own characteristic equation (see Section 1.20 in [28]); this means that terms that are cubic in $\overline{v_i'v_j'}$ (i.e. $\overline{v_i'v_j'}^3 = \overline{v_i'v_k'}\;\overline{v_k'v_m'}\;\overline{v_m'v_j'}$) can be expressed in terms that are linear and quadratic in $\overline{v_i'v_j'}$. The most general form of $\Phi_{ij,1}$ can be formulated as [29]

$$\begin{aligned}\Phi_{ij,1} &= -c_1\rho\left[\varepsilon a_{ij} + c_1'\left(a_{ik}a_{kj} - \frac{1}{3}\delta_{ij}a_{k\ell}a_{\ell k}\right)\right]\\ a_{ij} &= \frac{\overline{v_i'v_j'}}{k} - \frac{2}{3}\delta_{ij}\end{aligned} \tag{11.63}$$

a_{ij} is an anisotropy tensor whose trace is zero. In isotropic flow all its components are zero. Note that the right side is trace-less (i.e. the trace is zero). This should be so since the exact form of Φ_{ij} is trace-less, i.e. $\Phi_{ii} = \overline{2p'\partial v_i'/\partial x_i} = 0$.

11.7.5 Rapid pressure-strain term

Above a model for the slow part of the pressure-strain term was developed using physical arguments. Here we will carry out a mathematical derivation of a model for the rapid part of the pressure-strain term.

The notation "rapid" comes from a classical problem in turbulence called the rapid distortion problem, where a very strong velocity gradient $\partial\bar{v}_i/\partial x_j$ is imposed so that initially the second term (the slow term) can be neglected, see Eq. 11.66. It is assumed that the effect of the mean gradients is much larger than the effect of the turbulence, i.e.

$$\left|\frac{\partial\bar{v}_i}{\partial x_j}\right| \Big/ (\varepsilon/k) \to \infty \tag{11.64}$$

Thus in this case it is the first term in Eq. 11.66 which gives the most "rapid" response in p'. The second "slow" term becomes important first at a later stage when turbulence has been generated.

Now we want to derive an exact equation for the pressure-strain term, Π_{ij}. Since it includes the fluctuating pressure, p', we start by deriving an exact equation for p' starting from Navier-Stokes equations.

1. Take the divergence of the incompressible Navier-Stokes equation assuming constant viscosity (see Eq. 6.6) i.e. $\dfrac{\partial}{\partial x_i}\left(v_j\dfrac{\partial v_i}{\partial x_j}\right) = \ldots \Rightarrow$ Equation **A**.