

2021-08-17, Exam in

Turbulence modeling, MTF271

- **Time:** 8.30-12.30 **Location:** Zoom
 - **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
 - **The teacher is available on telephone**
 - **Checking the evaluation and results of your written exam at Canvas. If you have questions on the correction of the exam, add a comment at Canvas and send me an Email.**
 - **Grading:** 0-9p: 3, 10-19: 4, 20-30: 5.
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T1. Consider [Assignment 1](#) in the course. Choose the case **Large Wave**. Pick a vertical gridline close to the bottom of the wave. (10p)

1. Plot the exact production and the dissipation term of the $\overline{v_1'^2}$ equation along this line. Plot also the modeled production term.
2. Plot the modeled production and destruction term of the ε equation (use the AKN turbulence model).

Note: your Python/Matlab/Octave code must be uploaded to Canvas.

T2. Consider the same flow as above. (10p)

1. Pick the same gridline as above. Plot the $\overline{v_1'v_2'}$ stress using the ASM (Algebraic Stress Model). You may neglect the two wall pressure-strain terms. Compare it with the exact shear stress.
2. Pick the vertical gridline $i = 51$. Does the exact production term, P^k go negative anywhere along that gridline? If so, where? What is the physical meaning of negative production?

Note: your Python/Matlab/Octave code must be uploaded to Canvas.

T3. Consider the same flow as above.

(10p)

1. Pick the vertical gridline $i = 51$. In the eBook (Chapter 13), I present the limit on the turbulent viscosity to make the turbulence model realizable. Is this condition violated anywhere? If so, where? What is the consequence?

Note: your Python/Matlab/Octave code must be uploaded to Canvas.

MTF271 Turbulence modeling: Formula sheet

August 21, 2021

The continuity and Navier-Stokes equations for compressible flow read

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial v_k}{\partial x_k} \delta_{ij} \right\} + \rho g_i$$

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta (\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

► The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\frac{\partial \bar{v}_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v'_i v'_j} \right) - \rho_0 \beta (\bar{\theta} - \theta_0) g_i$$

$$\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} = \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v'_i \theta'}}{\partial x_i}$$

The Boussinesq assumption reads

$$\overline{v'_i v'_j} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) + \frac{2}{3} \delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3} \delta_{ij} k$$

The modeled $\overline{v'_i v'_j}$ equation with IP model reads

$$\begin{aligned}
& \bar{v}_k \frac{\partial \overline{v'_i v'_j}}{\partial x_k} = \text{(convection)} \\
& - \overline{v'_i v'_k} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v'_j v'_k} \frac{\partial \bar{v}_i}{\partial x_k} \text{ (production)} \\
& - c_1 \frac{\varepsilon}{k} \left(\overline{v'_i v'_j} - \frac{2}{3} \delta_{ij} k \right) \text{ (slow part)} \\
& - c_2 \left(P_{ij} - \frac{2}{3} \delta_{ij} P^k \right) \text{ (rapid part)} \\
& + c_{1w} \rho_0 \frac{\varepsilon}{k} \left[\overline{v'_k v'_m} n_k n_m \delta_{ij} - \frac{3}{2} \overline{v'_i v'_k} n_k n_j \right. \\
& \quad \left. - \frac{3}{2} \overline{v'_j v'_k} n_k n_i \right] f \left[\frac{\ell_t}{x_n} \right] \text{ (wall, slow part)} \\
& + c_{2w} \left[\Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2} \Phi_{ik,2} n_k n_j \right. \\
& \quad \left. - \frac{3}{2} \Phi_{jk,2} n_k n_i \right] f \left[\frac{\ell_t}{x_n} \right] \text{ (wall, rapid part)} \\
& + \nu \frac{\partial^2 \overline{v'_i v'_j}}{\partial x_k \partial x_k} \text{ (viscous diffusion)} \\
& + \frac{\partial}{\partial x_k} \left[c_k \overline{v'_k v'_m} \frac{k}{\varepsilon} \frac{\partial \overline{v'_i v'_j}}{\partial x_m} \right] \text{ (turbulent diffusion)} \\
& - g_i \beta \overline{v'_j \theta'} - g_j \beta \overline{v'_i \theta'} \text{ (buoyancy production)} \\
& - \frac{2}{3} \varepsilon \delta_{ij} \text{ (dissipation)}
\end{aligned}$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2} \frac{\partial A_i A_i}{\partial x_j}$$

► The exact transport equation for turbulent heat flux vector $\overline{v'_i \theta'}$ reads

$$\begin{aligned} \frac{\partial \overline{v'_i \theta'}}{\partial t} + \frac{\partial}{\partial x_k} \overline{v_k v'_i \theta'} &= \underbrace{-\overline{v'_i v'_k} \frac{\partial \bar{\theta}}{\partial x_k}}_{P_{i\theta}} - \underbrace{v'_k \theta' \frac{\partial \bar{v}_i}{\partial x_k}}_{\Pi_{i\theta}} - \underbrace{\frac{\theta'}{\rho} \frac{\partial \bar{p}'}{\partial x_i}}_{D_{i\theta,t}} - \underbrace{\frac{\partial}{\partial x_k} \overline{v'_k v'_i \theta'}}_{D_{i\theta,t}} \\ + \alpha \underbrace{\frac{\partial}{\partial x_k} \left(\overline{v'_i \frac{\partial \theta'}{\partial x_k}} \right)}_{D_{i\theta,\nu}} + \nu \underbrace{\frac{\partial}{\partial x_k} \left(\overline{\theta' \frac{\partial v'_i}{\partial x_k}} \right)}_{\varepsilon_{i\theta}} &- (\nu + \alpha) \underbrace{\frac{\partial v'_i}{\partial x_k} \frac{\partial \theta'}{\partial x_k}}_{\varepsilon_{i\theta}} - \underbrace{g_i \beta \overline{\theta'^2}}_{G_{i\theta}} \end{aligned}$$

► The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v'_i v'_j} \frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \overline{v'_j p'} + \frac{1}{2} \overline{v'_j v'_i v'_i} - \nu \frac{\partial k}{\partial x_j} \right] - \nu \frac{\partial v'_i}{\partial x_j} \frac{\partial v'_i}{\partial x_j} - g_i \beta \overline{v'_i \theta'}$$

► The exact $\overline{v'_i v'_j}$ equation reads

$$\begin{aligned} \frac{\partial \overline{v'_i v'_j}}{\partial t} + \frac{\partial}{\partial x_k} (\overline{v_k v'_i v'_j}) &= -\overline{v'_j v'_k} \frac{\partial \bar{v}_i}{\partial x_k} - \overline{v'_i v'_k} \frac{\partial \bar{v}_j}{\partial x_k} \\ - \frac{\partial}{\partial x_k} \left(\overline{v'_i v'_j v'_k} + \frac{1}{\rho} \delta_{jk} \overline{v'_i p'} + \frac{1}{\rho} \delta_{ik} \overline{v'_j p'} - \nu \frac{\partial \overline{v'_i v'_j}}{\partial x_k} \right) \\ + \frac{1}{\rho} p' \left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i} \right) &- g_i \beta \overline{v'_j \theta'} - g_j \beta \overline{v'_i \theta'} - 2\nu \frac{\partial v'_i}{\partial x_k} \frac{\partial v'_j}{\partial x_k} \end{aligned}$$

► The modelled k and ε equations

$$\begin{aligned} \frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} &= \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} \\ - \varepsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \\ \frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k} c_{\varepsilon 1} \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} \\ + c_{\varepsilon 1} g_i \frac{\varepsilon}{k} \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] \end{aligned}$$

► DES

$$L_t = \frac{k^{3/2}}{\varepsilon} = \frac{k^{1/2}}{\beta^* \omega} : \text{RANS lengthscale}$$

$$C_{DES} \Delta, \quad \Delta = \max(\Delta x, \Delta y, \Delta z) : \text{LES lengthscale}$$