2019-08-20, Exam in

Turbulence modeling, MTF270

- **Time:** 14.00–18.00 **Location:** M
- **Teacher:** Lars Davidson, phone 1404
- **Aids**
  - None.
- **The teacher will come at 14.30 and 16.00**
- **Checking the evaluation and results of your written exam**: at 12-13 on 3 and 10 September in my office
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Show how the turbulent diffusion (i.e. the term which includes the triple correlation) in the k equation is modeled. (5p)

b) Derive the Boussinesq assumption (5p)

T2. a) The slow pressure-strain model reads $\Phi_{ij,1} = -c_1\rho\frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij}k\right)$. The anisotropy tensor is defined as $a_{ij} = \frac{\overline{v_i'v_j'}}{k} - \frac{2}{3}\delta_{ij}$. Show that for decaying grid turbulence, the model for the slow pressure-strain model indeed acts as to make the turbulence more isotropic if $c_1 > 1$. (5p)

b) Streamline curvature: consider a boundary layer where the streamlines are curved away from the wall (concave curvature). Show that the Reynolds stress model gives an enhanced turbulence production (as it should) because of positive feedback between the production terms. Why is the effect of streamline curvature in the $k-\varepsilon$ model much smaller? (5p)

T3. a) What is a non-linear eddy-viscosity model? When formulating a non-linear model, the anisotropy tensor $a_{ij} = -2\nu_t\bar{s}_{ij}/k$ is often used. The three terms read (5p)

$$
\begin{aligned}
c_1\tau^2\left(\bar{s}_{ik}\bar{s}_{kj} - \frac{1}{3}\bar{s}_{\ell k}\bar{s}_{\ell k}\delta_{ij}\right)& \\
+c_2\tau^2\left(\bar{\Omega}_{ik}\bar{s}_{kj} - \bar{s}_{ik}\bar{\Omega}_{kj}\right)& \\
+c_3\tau^2\left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{\ell k}\bar{\Omega}_{\ell k}\delta_{ij}\right)&
\end{aligned}
$$

Show that each term has the same properties as a_{ij}, i.e. non-dimensional, traceless and symmetric.

b) Consider the energy spectrum. Show the three different regions (the large energy-containing scales, the $-5/3$ range and the dissipating scales). Where should the cut-off be located? Show where the SGS scales, grid (i.e resolved) scales and the cut-off, κ_c are located in the spectrum. (5p)

T4. a) When doing LES, how fine does the mesh need to be in the near-wall region? Why does it need to be that fine? (5p)

b) What is the physical meaning of f_k in PANS? (5p)

T5. a) Describe the SAS model. How is the von Kármán length scale defined? (5p)

b) What is embedded LES? Give an example when we may use it. (5p)

MTF270 Turbulence modeling: Formula sheet

The continuity and Navier-Stokes equations for compressible flow read

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_i}{\partial x_i}=0 \\
\frac{\partial \rho v_i}{\partial t}+\frac{\partial \rho v_i v_j}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\}+\rho g_i
\end{aligned}
$$

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$
\begin{aligned}
\frac{\partial v_i}{\partial x_i}=0 \\
\rho_0\frac{\partial v_i}{\partial t}+\rho_0\frac{\partial v_i v_j}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}-\rho_0\beta(\theta-\theta_0)g_i \\
\frac{\partial \theta}{\partial t}+\frac{\partial v_i\theta}{\partial x_i}=\alpha\frac{\partial^2\theta}{\partial x_i\partial x_i}
\end{aligned}
$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p=0$ if $v_i=0$)

$$
\rho_0\frac{\partial v_i}{\partial t}+\rho_0 v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}
$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$
\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j}-\rho_0\overline{v_i' v_j'}\right)-\rho_0\beta(\bar{\theta}-\theta_0)g_i \\
\frac{\partial \bar{v}_i\bar{\theta}}{\partial x_i} &= \alpha\frac{\partial^2\bar{\theta}}{\partial x_i\partial x_i}-\frac{\partial \overline{v_i'\theta'}}{\partial x_i}
\end{aligned}
$$

The Boussinesq assumption reads

$$
\overline{v_i' v_j'}=-\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)+\frac{2}{3}\delta_{ij}k=-2\nu_t\bar{s}_{ij}+\frac{2}{3}\delta_{ij}k
$$

The modeled $\overline{v_i'v_j'}$ equation with IP model reads

$$
\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k} = & \quad \text{(convection)} \\
-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij} k\right) & \quad \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & \quad \text{(rapid part)} \\
+c_{1w}\rho_0 \frac{\varepsilon}{k}\ [\ \overline{v_k'v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i'v_k'} n_k n_j & \\
-\frac{3}{2}\overline{v_j'v_k'} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, slow part)} \\
+c_{2w}\ [\ \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j & \\
-\frac{3}{2}\Phi_{jk,2} n_k n_i\ ] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k} & \quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\ \overline{v_k'v_m'}\ \frac{k}{\varepsilon}\frac{\partial \overline{v_i'v_j'}}{\partial x_m}\right] & \quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} & \quad \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & \quad \text{(dissipation)}
\end{aligned}
$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2} \frac{\partial A_i A_i}{\partial x_j}$$

▶The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k} \bar{v}_k \overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{- \overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{- \frac{\partial}{\partial x_k}\overline{v_k' v_i' \theta'}}_{D_{i\theta,t}}$$
$$\underbrace{+ \alpha \overline{\frac{\partial}{\partial x_k}\left(v_i' \frac{\partial \theta'}{\partial x_k}\right)} + \nu \overline{\frac{\partial}{\partial x_k}\left(\theta' \frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{- (\nu + \alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i \beta \overline{\theta'^2}}_{G_{i\theta}}$$

▶The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu \frac{\partial k}{\partial x_j}\right] - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} - g_i \beta \overline{v_i'\theta'}$$

▶The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}
\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k \overline{v_i'v_j'}) &= -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} \\
-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu \frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right) \\
+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - g_i \beta \overline{v_j'\theta'} - g_j \beta \overline{v_i'\theta'} - 2\nu \overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}
\end{aligned}$$

▶The modelled k and ε equations

$$\begin{aligned}
\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} &= \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} \\
&\quad -\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k} c_{\varepsilon 1} \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} \\
&\quad + c_{\varepsilon 1} g_i \frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]
\end{aligned}$$

▶DES

$$L_t = \frac{k^{3/2}}{\varepsilon} = \frac{k^{1/2}}{\beta^* \omega} : \text{RANS lengthscale}$$
$$C_{DES}\Delta, \quad \Delta = \max(\Delta x, \Delta y, \Delta z) : \text{LES lengthscale}$$