2019-06-04, Exam in

Turbulence modeling, MTF270

- **Time:** 8.30–12.30 **Location:** M
- **Teacher:** Lars Davidson, phone 1404
- **Aids**
  - None.
- **The teacher will come at 9.30 and 11.00**
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Given the transport equation of $\overline{v_i'v_j'}$, derive the exact k equation. (5p)

b) Which terms in the $\overline{v_i'v_j'}$ equation need to be modeled? Explain the physical meaning of the different terms in the $\overline{v_i'v_j'}$ equation. (5p)

T2. a) For a Poisson equation (5p)

$$\frac{\partial^2 \varphi}{\partial x_j \partial x_j} = f$$

there exists an exact analytical solution

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{f(\mathbf{y})dy_1 dy_2 dy_3}{|\mathbf{y}-\mathbf{x}|} \tag{1}$$

Use the equation for p', i.e.

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_j \partial x_j} = -2\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial v_j'}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}\left(v_i'v_j' - \overline{v_i'v_j'}\right)$$

and Eq. 1 to derive the exact analytical solution for the fluctuating pressure. Which are the “slow” and “rapid” terms? Why are they called “slow” and “rapid”?

b) Consider buoyancy-dominated flow with x_3 vertically upwards. The production term for the $\overline{v_i'v_j'}$ and the $\overline{v_i'\theta'}$ equations read (5p)

$$G_{ij} = -g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}, \quad P_{i\theta} = -\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k}$$

respectively (we assume that the velocity gradient is negligible). Show that the Reynolds stress model dampens and increases the vertical fluctuation in stable and unstable stratification, respectively, as it should.

T3. a) Show that the Boussinesq assumption may give negative normal stresses. In which coordinate system is the risk largest for negative normal stresses? Derive an expression (2D) how to avoid negative normal stresses by reducing the turbulent viscosity. (7p)
<u>Hint:</u> the eigenvalues, λ_1, λ_2, are obtained from $|\bar{s}_{ij} - \delta_{ij}\lambda| = 0$, $I_2^{2D} = \frac{1}{2}(C_{ii}C_{jj} - C_{ij}C_{ij})$

b) Show the difference between volume averaging (filtering) in LES and time-averaging in RANS. (3p)

T4. a) Mention four different ways to estimate the resolution of an LES that you have made. Which method is good/bad? Which is best? (5p)

b) Consider the energy spectrum and discuss the physical meaning of $P_{k_{sgs}}$ and ε_{sgs}. (5p)

T5. a) What is DES? The destruction term in the RANS S-A model reads $\left(\frac{\tilde{\nu}_t}{d}\right)^2$; how is it computed in the S-A DES model? (5p)

b) How is the matching line between RANS and LES defined in S-A DES and $k-\omega$ SST-DES? (5p)
<u>Hint:</u> $L_t = k^{1/2}/(\beta^*\omega)$

MTF270 Turbulence modeling: Formula sheet

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v_i' v_j'}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v_i' v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} &= && \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} & && \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i' v_j'} - \frac{2}{3}\delta_{ij} k\right) & && \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & && \text{(rapid part)} \\
+c_{1w} \rho_0 \frac{\varepsilon}{k}\left[\, \overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i' v_k'} n_k n_j \right. & && \\
\left. -\frac{3}{2}\overline{v_j' v_k'} n_k n_i \,\right] f\left[\frac{\ell_t}{x_n}\right] & && \text{(wall, slow part)} \\
+c_{2w}\left[\, \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j \right. & && \\
\left. -\frac{3}{2}\Phi_{jk,2} n_k n_i \,\right] f\left[\frac{\ell_t}{x_n}\right] & && \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} & && \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k \, \overline{v_k' v_m'} \, \frac{k}{\varepsilon} \frac{\partial \overline{v_i' v_j'}}{\partial x_m}\right] & && \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} & && \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & && \text{(dissipation)}
\end{aligned}$$

▶The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t}+\frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'}=\underbrace{-\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k}-\overline{v_k'\theta'}\frac{\partial\bar{v}_i}{\partial x_k}}_{P_{i\theta}}\underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}}\underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$

$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial\theta'}{\partial x_k}\right)}+\nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}}\underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial\theta'}{\partial x_k}}}_{\varepsilon_{i\theta}}\underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

▶The exact k equation reads

$$\frac{\partial k}{\partial t}+\frac{\partial\bar{v}_jk}{\partial x_j}=-\overline{v_i'v_j'}\frac{\partial\bar{v}_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'}+\frac{1}{2}\overline{v_j'v_i'v_i'}-\nu\frac{\partial k}{\partial x_j}\right]-\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}-g_i\beta\overline{v_i'\theta'}$$

▶The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}\frac{\partial\overline{v_i'v_j'}}{\partial t}+\frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'})&=-\overline{v_j'v_k'}\frac{\partial\bar{v}_i}{\partial x_k}-\overline{v_i'v_k'}\frac{\partial\bar{v}_j}{\partial x_k}\\-\frac{\partial}{\partial x_k}&\left(\overline{v_i'v_j'v_k'}+\frac{1}{\rho}\delta_{jk}\overline{v_i'p'}+\frac{1}{\rho}\delta_{ik}\overline{v_j'p'}-\nu\frac{\partial\overline{v_i'v_j'}}{\partial x_k}\right)\\+\overline{\frac{1}{\rho}p'\left(\frac{\partial v_i'}{\partial x_j}+\frac{\partial v_j'}{\partial x_i}\right)}&-g_i\beta\overline{v_j'\theta'}-g_j\beta\overline{v_i'\theta'}-2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}\end{aligned}$$

▶The modelled k and ε equations

$$\begin{aligned}\frac{\partial k}{\partial t}+\bar{v}_j\frac{\partial k}{\partial x_j}&=\nu_t\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)\frac{\partial\bar{v}_i}{\partial x_j}+g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i}\\&-\varepsilon+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial\varepsilon}{\partial t}+\bar{v}_j\frac{\partial\varepsilon}{\partial x_j}&=\frac{\varepsilon}{k}c_{\varepsilon1}\nu_t\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)\frac{\partial\bar{v}_i}{\partial x_j}\\&+c_{\varepsilon1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i}-c_{\varepsilon2}\frac{\varepsilon^2}{k}+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial x_j}\right]\end{aligned}$$

▶DES

$$L_t=\frac{k^{3/2}}{\varepsilon}=\frac{k^{1/2}}{\beta^*\omega}:\text{RANS lengthscale}$$

$$C_{DES}\Delta,\quad \Delta=\max(\Delta x,\Delta y,\Delta z):\text{LES lengthscale}$$