2018-10-12, Exam in

Turbulence modeling, MTF270

- **Time:** 14.00–18.00 **Location:** M
- **Teacher:** Hamidreza Abedi, Lars Davidson, phone 772 1390
- **Aids**
  - None.
- **The teacher will come at 15.00 and 16.00**
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Show the principles how to derive the transport equation for $\overline{v_i'v_j'}$. Explain the physical meaning of the different terms (see Formula sheet). (5p)

b) The exact Poisson equation for the pressure fluctuation reads (5p)

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_j \partial x_j} = \underbrace{-2\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial v_j'}{\partial x_i}}_{\text{rapid term}} \underbrace{- \frac{\partial^2}{\partial x_i \partial x_j}\left(v_i'v_j' - \overline{v_i'v_j'}\right)}_{\text{slow term}} \tag{1}$$

Derive this equation.

T2. a) The Boussinesq assumption reads (5p)

$$\overline{v_i'v_j'} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}k = -2\nu_t\bar{s}_{ij} + \frac{2}{3}\delta_{ij}k$$

Derive this expression.

b) The slow pressure-strain model reads $\Phi_{ij,1} = -c_1\rho\frac{\varepsilon}{k}\left(\overline{v_i'v_j'} - \frac{2}{3}\delta_{ij}k\right)$. The anisotropy tensor is defined as $a_{ij} = \frac{\overline{v_i'v_j'}}{k} - \frac{2}{3}\delta_{ij}$. Show that for decaying grid turbulence, the model for the slow pressure-strain model indeed acts as to make the turbulence more isotropic if $c_1 > 1$. (5p)
Hint: the k equation in decaying grid turbulence read $\bar{v}_1\frac{dk}{dx_1} = \Phi_{11} - \frac{2}{3}\varepsilon_{11}$

T3. a) The f equation in the V2F model reads (7p)

$$L^2 \frac{\partial^2 f}{\partial x_2^2} - f = -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right), \quad T \propto \frac{k}{\varepsilon}, \quad L \propto \frac{k^{3/2}}{\varepsilon}$$

In the V2F model, the v^2 equation is solved: what is the difference between $\overline{v_2'^2}$ and v^2? Explain which term it is that makes $\overline{v_2'^2}$ and v^2 different.

b) When formulating a non-linear model, the anisotropy tensor $a_{ij} = -2\nu_t \bar{s}_{ij}/k$ is often used. The three terms read (3p)

$$\begin{aligned} a_{ij} = {} & c_1 \tau^2 \left(\bar{s}_{ik}\bar{s}_{kj} - \frac{1}{3}\bar{s}_{\ell k}\bar{s}_{\ell k}\delta_{ij}\right) \\ & + c_2 \tau^2 \left(\bar{\Omega}_{ik}\bar{s}_{kj} - \bar{s}_{ik}\bar{\Omega}_{kj}\right) \\ & + c_3 \tau^2 \left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{\ell k}\bar{\Omega}_{\ell k}\delta_{ij}\right) + \ldots \end{aligned}$$

Show that each term has the same properties as a_{ij}, i.e. non-dimensional, traceless and symmetric.

T4. a) Consider the SST $k - \omega$ model. Show that the eddy-viscosity assumption gives too high a shear stress in adverse-pressure gradient (APG) flow since $P^k/\varepsilon \gg 1$. Assume boundary-layer flow. (5p)
Hint: experiments show that $-\overline{v_1' v_2'} \simeq c_\mu^{1/2} k$.

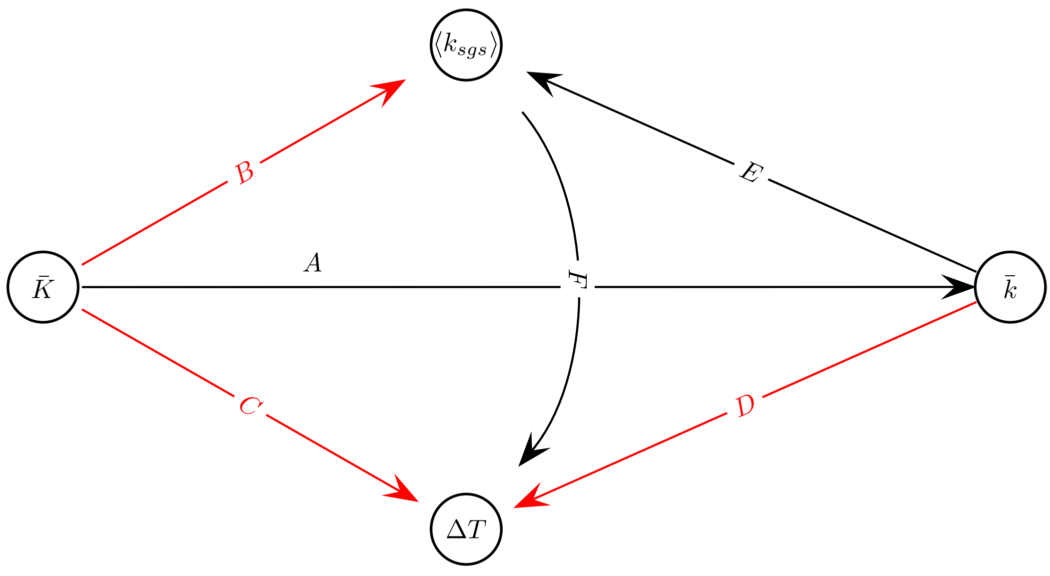
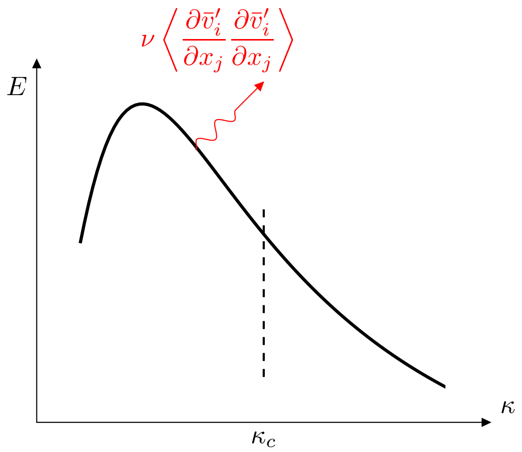


Figure 1: Transfer of kinetic turbulent energy. $\bar{K} = \frac{1}{2}\langle \bar{v}_i \rangle \langle \bar{v}_i \rangle$ and $\bar{k} = \frac{1}{2}\langle \bar{v}_i' \bar{v}_i' \rangle$ denote time-averaged kinetic and resolved turbulent kinetic energy, respectively. ΔT denotes increase in internal energy, i.e. dissipation. The cascade process assumes that the terms in red are negligible.

b) Consider the flow of energy in Fig. 1. The arrow corresponding to viscous dissipation of resolved turbulent kinetic energy (from $\bar{k}$ to ΔT) is shown in the energy spectrum below. Draw the other five arrows in Fig. 1 in the energy spectrum. (5p)



T5. a) Give a short description of the method to generate synthetic turbulent inlet fluctuations. What form on the spectrum is assumed? How are the maximum and minimum wavelengths, κ_{max}, κ_{min}, determined? (5p)

b) The one-equation RANS model of Spalart & Allmaras reads (5p)

$$\frac{\partial \rho \tilde{\nu}_t}{\partial t} + \frac{\partial \rho \bar{v}_j \tilde{\nu}_t}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\mu + \mu_t}{\sigma_{\tilde{\nu}_t}} \frac{\partial \tilde{\nu}_t}{\partial x_j}\right) + \frac{C_{b2}\rho}{\sigma_{\tilde{\nu}_t}} \frac{\partial \tilde{\nu}_t}{\partial x_j} \frac{\partial \tilde{\nu}_t}{\partial x_j} + P - C_{w1}\rho f_w \left(\frac{\tilde{\nu}_t}{d}\right)^2$$

$$\nu_t = \tilde{\nu}_t f_1$$

where $\tilde{\nu}_t$, P and d denote turbulent viscosity, production and wall distance, respectively. This model was in 1997 converted into a DES model. How did they do that? What is the physical idea of DES?

MTF270 Turbulence modeling: Formula sheet

March 25, 2019

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v_i' v_j'}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v_i' v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} &= && \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} & && \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i' v_j'} - \frac{2}{3}\delta_{ij} k\right) & && \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & && \text{(rapid part)} \\
+c_{1w} \rho_0 \frac{\varepsilon}{k} \left[\, \overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i' v_k'} n_k n_j \right. & && \\
\left. -\frac{3}{2}\overline{v_j' v_k'} n_k n_i \,\right] f\left[\frac{\ell_t}{x_n}\right] & && \text{(wall, slow part)} \\
+c_{2w} \left[\, \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j \right. & && \\
\left. -\frac{3}{2}\Phi_{jk,2} n_k n_i \,\right] f\left[\frac{\ell_t}{x_n}\right] & && \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} & && \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k \, \overline{v_k' v_m'} \, \frac{k}{\varepsilon} \frac{\partial \overline{v_i' v_j'}}{\partial x_m}\right] & && \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} & && \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & && \text{(dissipation)}
\end{aligned}$$

►The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$
$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial \theta'}{\partial x_k}\right)} + \nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

►The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} - g_i\beta\overline{v_i'\theta'}$$

►The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}
&\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) = -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k}\\
&-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right)\\
&+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}
\end{aligned}$$

►The modelled k and ε equations

$$\begin{aligned}
\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} &= \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i}\\
&-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k}c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j}\\
&+ c_{\varepsilon 1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]
\end{aligned}$$