2017-05-30, Exam in

Turbulence modeling, MTF270

- **Time:** 08.30-12.30 **Location:** M
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 2 pages, appended to the exam
- **Checking the evaluation and results of your written exam**: at 12-13 on 19 and 20 June in room Navier
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) What is the expression for the exact total heat flux that appears in the $\bar{\theta}$ equation? (5p)

b) Consider the equation below. (5p)

$$\frac{\partial \overline{v_i'v_j'}}{\partial t} + \underbrace{\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)}}_{\Pi_{ij}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_i'v_j'v_k'} + \frac{\overline{p'v_j'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu\frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}}_{G_{ij}} - \underbrace{2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

Explain the physical meaning of the different terms. Which terms need to be modeled?

T2. a) How is the production term modeled in the $k-\varepsilon$ model? Show how it can be expressed in $\bar{s}_{ij}$ (5p)

b) Consider the exact equation for the pressure fluctuation. (3p)

$$p'(\mathbf{x}) = \frac{\rho}{4\pi}$$
$$\int_V \left[2\frac{\partial \bar{v}_i(\mathbf{y})}{\partial y_j}\frac{\partial v_j'(\mathbf{y})}{\partial y_i} + \frac{\partial^2}{\partial y_i \partial y_j}\left(v_i'(\mathbf{y})v_j'(\mathbf{y}) - \overline{v_i'(\mathbf{y})v_j'(\mathbf{y})}\right)\right]\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|}$$

Which term is the “slow” and “rapid” terms? Why are they called “slow” and “rapid”?

T3. a) When formulating a non-linear model, the anisotropy tensor $a_{ij} = -2\nu_t \bar{s}_{ij}/k$ is often used. The first four terms read (7p)

$$
\begin{aligned}
a_{ij} = -2c_\mu \tau \bar{s}_{ij} + c_1\tau^2 \left(\bar{s}_{ik}\bar{s}_{kj} - \frac{1}{3}\bar{s}_{\ell k}\bar{s}_{\ell k}\delta_{ij}\right) \\
+c_2\tau^2 \left(\bar{\Omega}_{ik}\bar{s}_{kj} - \bar{s}_{ik}\bar{\Omega}_{kj}\right) \\
+c_3\tau^2 \left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{\ell k}\bar{\Omega}_{\ell k}\delta_{ij}\right)
\end{aligned}
$$

Show that each term has the same properties as a_{ij}, i.e. non-dimensional, traceless and symmetric.

b) Derive a transport equation for ω from the k and ε transport equations; you only need to do the production, the destruction and the viscous diffusion terms. (5p)

T4. a) Consider Fig. 1. A-F denotes energy transfer. Identify A-F with one of the following expression (5p)

$$
-\langle \bar{v}_i'\bar{v}_j'\rangle \frac{\partial \langle \bar{v}_i\rangle}{\partial x_j}, \quad 2\langle \nu_{sgs}\rangle\langle \bar{s}_{ij}\rangle\langle \bar{s}_{ij}\rangle, \quad \nu\frac{\partial \langle \bar{v}_i\rangle}{\partial x_j}\frac{\partial \langle \bar{v}_i\rangle}{\partial x_j}, \quad \nu\left\langle \frac{\partial \bar{v}_i'}{\partial x_j}\frac{\partial \bar{v}_i'}{\partial x_j}\right\rangle, \quad \varepsilon_{sgs}', \quad \varepsilon
$$

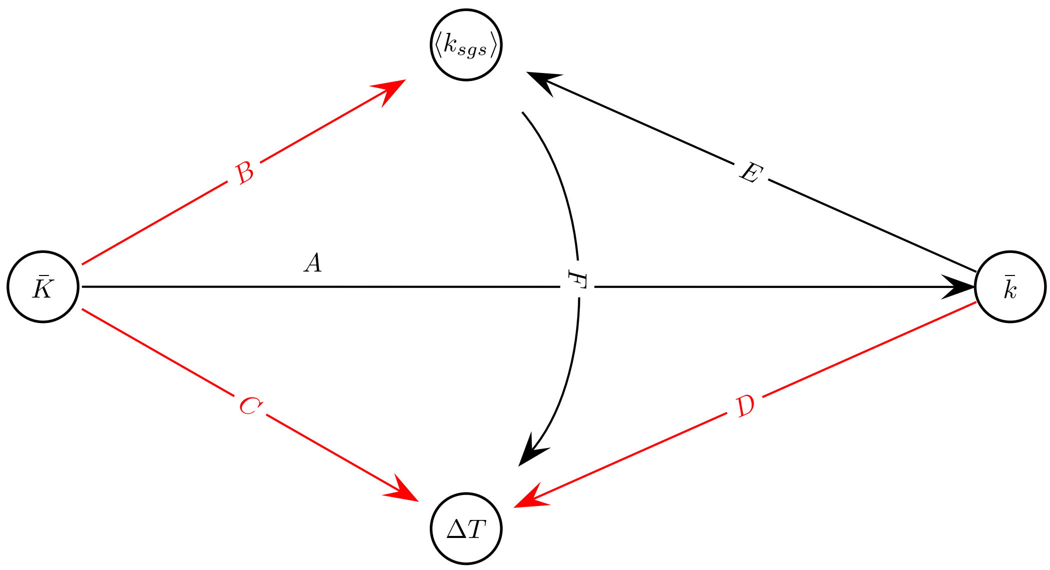

Figure 1: Transfer of kinetic turbulent energy. $\bar{K} = \frac{1}{2}\langle \bar{v}_i\rangle\langle \bar{v}_i\rangle$ and $\bar{k} = \frac{1}{2}\langle \bar{v}_i'\bar{v}_i'\rangle$ denote time-averaged kinetic and resolved turbulent kinetic energy, respectively. ΔT denotes increase in internal energy, i.e. dissipation. The cascade process assumes that the terms in red are negligible.

b) The filtered non-linear term on the left side has the form (5p)

$$\overline{\frac{\partial v_i v_j}{\partial x_j}}$$

Show that it can be re-written as

$$\frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j}$$

giving an additional term

$$-\frac{\partial \tau_{ij}}{\partial x_j}$$

on the right side.

T5. a) What is a test filter? Grid and test filter Navier-Stokes equation and derive the relation (5p)

$$\widehat{\bar{v}_i \bar{v}_j} - \widehat{\bar{v}}_i \widehat{\bar{v}}_j + \widehat{\tau}_{ij} = \mathcal{L}_{ij} + \widehat{\tau}_{ij} = T_{ij} \quad (1)$$

b) The ε_u equation in the PANS model reads (5p)

$$\frac{\partial \varepsilon_u}{\partial t} + \frac{\partial (\varepsilon_u \bar{v}_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_u}{\sigma_{\varepsilon u}}\right)\frac{\partial \varepsilon_u}{\partial x_j}\right] + C_{\varepsilon 1} P_u \frac{\varepsilon_u}{k_u} - C^*_{\varepsilon 2}\frac{\varepsilon_u^2}{k_u}$$

where

$$C^*_{\varepsilon 2} = C_{\varepsilon 1} + \frac{f_k}{f_\varepsilon}(C_{\varepsilon 2} - C_{\varepsilon 1}), \quad C_{\varepsilon 1} = 1.5, \quad C_{\varepsilon 2} = 1.9$$

Assume that $f_\varepsilon = 1$. Consider the destruction term in the ε equation and the coefficient $C^*_{\varepsilon 2}$. Explain what happens if f_k is reduced from 1 (RANS mode) to $f_k = 0.4$ (LES mode).

MTF270 Turbulence modeling: Formula sheet

March 25, 2019

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v'_i v'_j}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v'_i \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v'_i v'_j} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v'_i v'_j}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v'_i v'_j}}{\partial x_k} &= && \text{(convection)} \\
-\overline{v'_i v'_k} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v'_j v'_k} \frac{\partial \bar{v}_i}{\partial x_k} & && \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v'_i v'_j} - \frac{2}{3}\delta_{ij} k\right) & && \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & && \text{(rapid part)} \\
+c_{1w} \rho_0 \frac{\varepsilon}{k} \left[\ \overline{v'_k v'_m} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v'_i v'_k} n_k n_j\right. & && \\
\left. -\frac{3}{2}\overline{v'_j v'_k} n_k n_i\ \right] f\left[\frac{\ell_t}{x_n}\right] & && \text{(wall, slow part)} \\
+c_{2w} \left[\ \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j\right. & && \\
\left. -\frac{3}{2}\Phi_{jk,2} n_k n_i\ \right] f\left[\frac{\ell_t}{x_n}\right] & && \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v'_i v'_j}}{\partial x_k \partial x_k} & && \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\ \overline{v'_k v'_m}\ \frac{k}{\varepsilon} \frac{\partial \overline{v'_i v'_j}}{\partial x_m}\right] & && \text{(turbulent diffusion)} \\
-g_i \beta \overline{v'_j \theta'} - g_j \beta \overline{v'_i \theta'} & && \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & && \text{(dissipation)}
\end{aligned}$$

▶The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial\overline{v_i'\theta'}}{\partial t}+\frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'}=\underbrace{-\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k}-\overline{v_k'\theta'}\frac{\partial\bar{v}_i}{\partial x_k}}_{P_{i\theta}}\underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}}\underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$
$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial\theta'}{\partial x_k}\right)}+\nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}}\underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial\theta'}{\partial x_k}}}_{\varepsilon_{i\theta}}\underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

▶The exact k equation reads

$$\frac{\partial k}{\partial t}+\frac{\partial\bar{v}_jk}{\partial x_j}=-\overline{v_i'v_j'}\frac{\partial\bar{v}_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'}+\frac{1}{2}\overline{v_j'v_i'v_i'}-\nu\frac{\partial k}{\partial x_j}\right]-\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}\frac{\partial\overline{v_i'v_j'}}{\partial t}+\frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'})&=-\overline{v_j'v_k'}\frac{\partial\bar{v}_i}{\partial x_k}-\overline{v_i'v_k'}\frac{\partial\bar{v}_j}{\partial x_k}\\ -\frac{\partial}{\partial x_k}&\left(\overline{v_i'v_j'v_k'}+\frac{1}{\rho}\delta_{jk}\overline{v_i'p'}+\frac{1}{\rho}\delta_{ik}\overline{v_j'p'}-\nu\frac{\partial\overline{v_i'v_j'}}{\partial x_k}\right)\\ +\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j}+\frac{\partial v_j'}{\partial x_i}\right)}&-g_i\beta\overline{v_j'\theta'}-g_j\beta\overline{v_i'\theta'}-2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}\end{aligned}$$

▶The modelled k and ε equations

$$\begin{aligned}\frac{\partial k}{\partial t}+\bar{v}_j\frac{\partial k}{\partial x_j}&=\nu_t\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)\frac{\partial\bar{v}_i}{\partial x_j}+g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i}\\ &-\varepsilon+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial\varepsilon}{\partial t}+\bar{v}_j\frac{\partial\varepsilon}{\partial x_j}&=\frac{\varepsilon}{k}c_{\varepsilon1}\nu_t\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)\frac{\partial\bar{v}_i}{\partial x_j}\\ &+c_{\varepsilon1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i}-c_{\varepsilon2}\frac{\varepsilon^2}{k}+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial x_j}\right]\end{aligned}$$