2014-05-27, Exam in

Turbulence modeling, MTF270

- **Time:** 14.00–18.00 **Location:** M
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 2 pages, appended to the exam
- **Checking the evaluation and results of your written exam**: contact the assistant, Hamid Abedi
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Show the principles how to derive the transport equation for $\overline{v_i'v_j'}$ (no derivation). (5p)

b) Derive a transport equation for ω from the k and ε transport equations; you only need to do the production and the destruction terms. (5p)

T2. a) How are the Reynolds stress, $\overline{v_i'v_j'}$, and the turbulent heat flux, $\overline{v_i'\theta'}$, modeled in the Boussinesq approach? (5p)

b) Describe the physical effect of stable stratification and unstable stratification on turbulence. (5p)

T3. a) What is a realizability constraint? Give the main realizability constraint for the normal stress and the shear stress. The Boussinesq assumption may violate one of those two contraints: show which one. (5p)

b) Consider the V2F model. It includes the f equation which reads (5p)

$$
\begin{aligned}
L^2\frac{\partial^2 f}{\partial x_2^2} - f &= -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right)\\
T &= \max\left\{\frac{k}{\varepsilon}, C_T\left(\frac{\nu}{\varepsilon}\right)^{1/2}\right\}\\
\frac{\Phi_{22}}{k} &= \frac{C_1}{T}\left(\frac{2}{3} - \frac{\overline{v_2'^2}}{k}\right) + C_2\frac{\nu_t}{k}\left(\frac{\partial\bar{v}_1}{\partial x_2}\right)^2\\
L &= C_L\max\left\{\frac{k^{3/2}}{\varepsilon}, C_\eta\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}\right\}
\end{aligned}
$$

Explain how the magnitude of the right side and L affect f.

T4. a) The filtered non-linear term has the form (5p)

$$\overline{\frac{\partial v_i v_j}{\partial x_j}}$$

Show that it can be re-written as

$$\frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j}$$

What is the resulting additional term on the right side?

b) Derive the Smagorinsky model from the k_{sgs} equation. (5p)

T5. a) We usually define the SGS stress tensor as $\tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$. In scale-similarity models τ_{ij} is written as three different terms. Derive these three terms. What does the word "scale-similar" mean? (5p)

b) The modified (reduced) length scale in two-equation DES models can be introduced in different equations. Which equations and which term? How is the turbulent viscosity reduced in the LES region? (5p)

MTF270 Turbulence modelling: Formula sheet

May 21, 2014

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v_i' v_j'}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v_i' v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} = \quad & \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} \quad & \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i' v_j'} - \frac{2}{3}\delta_{ij} k\right) \quad & \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) \quad & \text{(rapid part)} \\
+c_{1w} \rho_0 \frac{\varepsilon}{k} \left[\ \overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{v_i' v_k'} n_k n_j \right. \quad & \\
\left. -\frac{3}{2}\overline{v_j' v_k'} n_k n_i\ \right] f\left[\frac{\ell_t}{x_n}\right] \quad & \text{(wall, slow part)} \\
+c_{2w} \left[\ \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j \right. \quad & \\
\left. -\frac{3}{2}\Phi_{jk,2} n_k n_i\ \right] f\left[\frac{\ell_t}{x_n}\right] \quad & \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} \quad & \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k\ \overline{v_k' v_m'}\ \frac{k}{\varepsilon} \frac{\partial \overline{v_i' v_j'}}{\partial x_m}\right] \quad & \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} \quad & \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} \quad & \text{(dissipation)}
\end{aligned}$$

►The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$

$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial \theta'}{\partial x_k}\right)} + \nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

►The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

►The exact $\overline{v_i'v_j'}$ equation reads

$$\begin{aligned}\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) &= -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} \\ -\frac{\partial}{\partial x_k}&\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right) \\ &+\overline{\frac{1}{\rho}p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}\end{aligned}$$

►The modeled k and ε equations

$$\begin{aligned}\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} &= \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} \\ &-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k}c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} \\ &+ c_{\varepsilon 1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]\end{aligned}$$