2013-08-26, Exam in

Turbulence modeling, MTF270

- **Time:** 8.30–12.30 **Location:** M
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 2 pages, appended to the exam
- **Checking the evaluation and results of your written exam**: contact the assistant, Hamid Abedi
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Derive the Boussinesq assumption. (5p)

b) Discuss and show how the dissipation term, ε_{ij}, is modeled. (5p)

T2. a) Describe the physical effect of stable stratification and unstable stratification on turbulence. (5p)

b) Consider streamline curvature for a streamline formed as a circular arc (convex curvature). Show that the turbulence is dampened if $\partial v_\theta/\partial r > 0$ and that it is enhanced if the sign of $\partial v_\theta/\partial r$ is negative. (5p)

T3. a) Derive a transport equation for ω from the k and ε transport equations. (5p)
Hint:

$$D_\omega^T \simeq \frac{2\nu_t}{\sigma_\varepsilon k}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \omega}{\partial x_j}\right)$$

b) Consider the energy spectrum. Show the three different regions (the large energy-containing scales, the $-5/3$ range and the dissipating scales). Where should the cut-off be located? Show where the SGS scales, grid (i.e resolved) scales and the cut-off, κ_c are located in the spectrum. (5p)

T4. a) Show that when a first-order upwind schemes is used for the convection term, an additional diffusion term and dissipation terms appear because of a numerical SGS viscosity. (5p)

b) The modified (reduced) length scale in two-equation DES models can be introduced in different equations. Which equations and which term? What is the effect on the modeled, turbulent quantities? (5p)

T5. a) Describe URANS. How is the instantaneous velocity decomposed? What turbulence models are used? What is scale separation? (5p)

b) Describe the PANS model. What is the main modification compared to the standard $k-\varepsilon$ model? What is the physical meaning of f_k? Describe what happens to the equation system when f_k is reduced. (5p)

MTF270 Turbulence modelling: Formula sheet

August 21, 2013

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v_i' v_j'}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v_i' v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} &= && \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} & && \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i' v_j'} - \frac{2}{3}\delta_{ij} k\right) & && \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & && \text{(rapid part)} \\
+c_{1w} \rho_0 \frac{\varepsilon}{k} \left[\, \overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2} \overline{v_i' v_k'} n_k n_j \right. & && \\
\left. - \frac{3}{2} \overline{v_j' v_k'} n_k n_i \,\right] f \left[\frac{\ell_t}{x_n}\right] & && \text{(wall, slow part)} \\
+c_{2w} \left[\, \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2} \Phi_{ik,2} n_k n_j \right. & && \\
\left. - \frac{3}{2} \Phi_{jk,2} n_k n_i \,\right] f \left[\frac{\ell_t}{x_n}\right] & && \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} & && \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k \, \overline{v_k' v_m'} \, \frac{k}{\varepsilon} \frac{\partial \overline{v_i' v_j'}}{\partial x_m}\right] & && \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} & && \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & && \text{(dissipation)}
\end{aligned}$$

▶The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$

$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial \theta'}{\partial x_k}\right)} + \nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

▶The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶The exact $\overline{v_i'v_j'}$ equation reads

$$\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) = -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right)$$
$$+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}$$

▶The modeled k and ε equations

$$\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} = \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i}$$
$$-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j}$$
$$+ c_{\varepsilon 1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$