2013-05-28, Exam in

Turbulence modeling, MTF270

- **Time:** 14.00–18.00 **Location:** V
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 2 pages, appended to the exam
- **Checking the evaluation and results of your written exam**: At 12-13: June 18 in Room Navier and June 19 in Room Alpha
- **Grading**: 20-29p: 3, 30-39: 4, 40-50: 5.

T1. a) Derive the exact transport equation for $\overline{v_i'\theta'}$. Which terms need to be modeled? (5p)

b) How is the production term modeled in the $k-\varepsilon$ model? Show how it can be expressed in $\bar{s}_{ij}$ (5p)

T2. a) Derive the ASM equation (given below). (5p)

$$\overline{v_i'v_j'} = \frac{2}{3}\delta_{ij}k + \frac{k}{\varepsilon}\frac{(1-c_2)\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) + \Phi_{ij,1w} + \Phi_{ij,2w}}{c_1 + P^k/\varepsilon - 1}$$

b) Discuss and show how the dissipation term, ε_{ij}, in the $\overline{v_i'v_j'}$ equation is modeled. (5p)

T3. a) Consider buoyancy-dominated flow with x_3 vertically upwards. The production term for the $\overline{v_i'v_j'}$ and the $\overline{v_i'\theta'}$ equations read (5p)

$$G_{ij} = -g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}, \quad P_{i\theta} = -\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k}$$

respectively. Show that the Reynolds stress model dampens and increases the vertical fluctuation in stable and unstable stratification, respectively, as it should.

b) Show how a sinus wave $\sin(\kappa_c x)$ corresponding to cut-off is represented on a grid with two and four nodes, respectively. How is κ_c related to the grid size Δx for these cases? (5p)

T4. a) Consider the energy spectrum. Show the three different regions (the large energy-containing scales, the $-5/3$ range and the dissipating scales). In which region should the cut-off be located? How are k, k_{res} and k_{sgs} computed from the energy spectrum? (5p)

b) Show that when a first-order upwind schemes is used for the convection term, an additional diffusion term appear in the momentum equations because of a numerical SGS viscosity. (5p)

T5. a) Explain what is DES. The length scale in the RANS S-A model reads $\left(\frac{\tilde{\nu}_t}{d}\right)^2$; how is it computed in the corresponding DES model? (5p)

b) Consider the SAS model. How is the von Kármán length scale defined? An additional source term is introduced in the ω equation: what is the form of this term? What is the object of this term? When is it large and small, respectively? (5p)

MTF270 Turbulence modelling: Formula sheet

May 25, 2013

The continuity, Navier-Stokes and temperature equations for incompressible flow with constant viscosity read (*conservative* form)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho_0 \beta(\theta - \theta_0) g_i$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

►The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The *time averaged* continuity equation, Navier-Stokes equation temperature equations read

$$\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\rho_0 \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho_0 \overline{v_i' v_j'}\right) - \rho_0 \beta(\bar{\theta} - \theta_0) g_i \\
\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} &= \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i}
\end{aligned}$$

The Boussinesq assumption reads

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

The modeled $\overline{v_i' v_j'}$ equation with IP model reads

$$\begin{aligned}
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} &= \quad \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k}\left(\overline{v_i' v_j'} - \frac{2}{3}\delta_{ij} k\right) & \quad \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right) & \quad \text{(rapid part)} \\
+c_{1w} \rho_0 \frac{\varepsilon}{k} \left[\, \overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2} \overline{v_i' v_k'} n_k n_j \right. & \\
\left. -\frac{3}{2} \overline{v_j' v_k'} n_k n_i \,\right] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, slow part)} \\
+c_{2w} \left[\, \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2} \Phi_{ik,2} n_k n_j \right. & \\
\left. -\frac{3}{2} \Phi_{jk,2} n_k n_i \,\right] f\left[\frac{\ell_t}{x_n}\right] & \quad \text{(wall, rapid part)} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} & \quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k}\left[c_k \, \overline{v_k' v_m'} \, \frac{k}{\varepsilon} \frac{\partial \overline{v_i' v_j'}}{\partial x_m}\right] & \quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} & \quad \text{(buoyancy production)} \\
-\frac{2}{3}\varepsilon \delta_{ij} & \quad \text{(dissipation)}
\end{aligned}$$

►The exact transport equation for turbulent heat heat flux vector $\overline{v_i'\theta'}$ reads

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$

$$\underbrace{+\alpha\overline{\frac{\partial}{\partial x_k}\left(v_i'\frac{\partial \theta'}{\partial x_k}\right)} + \nu\overline{\frac{\partial}{\partial x_k}\left(\theta'\frac{\partial v_i'}{\partial x_k}\right)}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}$$

►The exact k equation reads

$$\frac{\partial k}{\partial t} + \frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

►The exact $\overline{v_i'v_j'}$ equation reads

$$\frac{\partial \overline{v_i'v_j'}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) = -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right)$$
$$+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}$$

►The modeled k and ε equations

$$\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} = \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i}$$
$$-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j}$$
$$+ c_{\varepsilon 1}g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$