2016-08-23, Exam in

Mechanics of fluids, TME225

- **Time:** 8.30–12.30 **Location:** M
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 4 pages, appended to the exam
- **Checking the evaluation and the results of your written exam**: 5 and 6 September at 12-13 in Room Navier. hamed.jamshidi@chalmers.se
- **Grading**: 24-35p: 3, 36-47: 4, 48-60: 5.

1

- Consider two-dimensional flow. The velocity components v_1 and v_2 at point (x_1, x_2) and its surrounding are given in the tables below.

	$v_1(x_1, x_2, t)$		
	$t - \Delta t$	$t = 0$	$t + \Delta t$
$(x_1 - \Delta, x_2)$	18	22	21
(x_1, x_2)	16	20	26
$(x_1 + \Delta, x_2)$	14	17	19

	$v_2(x_1, x_2, t)$		
	$t - \Delta t$	$t = 0$	$t + \Delta t$
$(x_1 - \Delta, x_2)$	5	-2	0
(x_1, x_2)	3	2	-1
$(x_1 + \Delta, x_2)$	-1	1	3

	$v_1(x_1, x_2, t)$		
	$t - \Delta t$	$t = 0$	$t + \Delta t$
$(x_1, x_2 - \Delta)$	21	20	18
$(x_1, x_2 + \Delta)$	14	15	19

	$v_2(x_1, x_2, t)$		
	$t - \Delta t$	$t = 0$	$t + \Delta t$
$(x_1, x_2 - \Delta)$	1	-3	-4
$(x_1, x_2 + \Delta)$	4	0	1

Compute $\partial v_1/\partial t$, dv_1/dt and dv_2/dt in (x_1, x_2) at $t = 0$. $\Delta x_1 = \Delta x_2 = \Delta = 0.25$, $\Delta t = 0.1$.

<u>Hint:</u> Use central differencing, e.g. $\partial\phi/\partial t$ at time t is computed as $\frac{\phi(t+\Delta t)-\phi(t-\Delta t)}{2\Delta t}$

(10p)

2

a) The Navier-Stokes equation reads

$$\rho\frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ij} - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) + \rho f_i$$

Simplify the Navier-Stokes equation for incompressible flow and constant viscosity.

(5p)

b) The internal energy equation reads

$$\rho\frac{du}{dt} = \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

The equation for kinetic energy reads

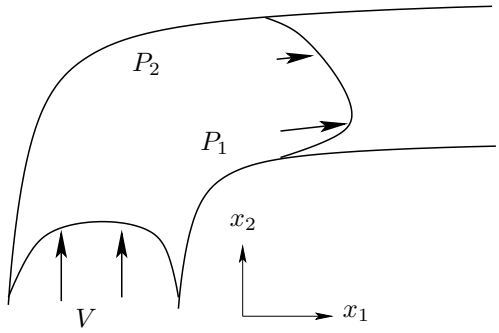
$$\rho \frac{dk}{dt} = \frac{\partial v_i \sigma_{ji}}{\partial x_j} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} + \rho v_i f_i$$

Explain the energy transfer between kinetic energy and internal energy. Which part is reversible and which one is irreversible?

(5p)

3

a) Consider fluid flow in a channel bend (see figure below). Explain why the velocity after the bend is largest near the inner (lower) wall.



(5p)

b) Explain the cascade process. In a boundary layer, how large are the largest scales? Which eddies extract energy from the mean flow?

(5p)

4

a) What does isotropic turbulence mean? Write the turbulent Reynolds stress tensor for isotropic turbulence.

(5p)

b) Consider a turbulent boundary layer. What are the relevant velocity and length scales in the viscous-dominated region?

(5p)

5

a) Consider channel flow and use physical reasoning to show that $\overline{v_1' v_2'}$ must be negative and positive in the lower and upper half of the channel, respectively.

(5p)

b) The exact $\overline{v_i' v_j'}$ equation reads

$$\frac{\partial}{\partial x_k}(\bar{v}_k \overline{v_i' v_j'}) = -\overline{v_j' v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i' v_k'}\frac{\partial \bar{v}_j}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}\left(\overline{v_i' v_j' v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i' p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j' p'} - \nu\frac{\partial \overline{v_i' v_j'}}{\partial x_k}\right)$$
$$+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}$$

Use this equation to derive the exact k equation.

(5p)

6

a) Consider the pressure-strain term in the $\overline{v_i' v_j'}$ equation. The *mean* normal stress can be defined as $\overline{v_{av}'^2} = \overline{v_i' v_i'}/3$; what sign will the pressure-strain term have for the normal stresses which are, respectively, larger and smaller than $\overline{v_{av}'^2}$?

(5p)

b) What is Taylors hypothesis? Show that the integral length scale can be estimated from the integral time scale

(5p)

TME225 Mechanics of fluids: Formula sheet to be used at written examinations

▶The $\epsilon - \delta$ identity reads

$$\epsilon_{inm}\epsilon_{mjk} = \epsilon_{min}\epsilon_{mjk} = \epsilon_{nmi}\epsilon_{mjk} = \delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj}$$

▶The vorticity vector is computed as

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$$
$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

▶Strain rate tensor, vorticity tensor

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\underbrace{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}}_{2\partial v_i/\partial x_j} + \underbrace{\frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i}}_{=0}\right)$$
$$= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = S_{ij} + \Omega_{ij}$$

▶The material derivative

$$\rho\frac{d\Psi}{dt} = \rho\frac{\partial \Psi}{\partial t} + \rho v_j\frac{\partial \Psi}{\partial x_j}$$

where $\Psi = v_i, u, T, k, \overline{v_i' v_j'} \ldots$

▶The balance equation for mass

$$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$$

▶The balance equation for linear momentum

$$\rho\frac{dv_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i$$

▶The balance equation for internal energy

$$\rho\frac{du}{dt} = \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

▶The equation for kinetic energy reads

$$\rho\frac{dk}{dt} = \frac{\partial v_i \sigma_{ji}}{\partial x_j} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} + \rho v_i f_i$$

▶The constitutive law for Newtonian viscous fluids

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}, \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$
$$q_i = -k\frac{\partial T}{\partial x_i}$$

▶The continuity equation and the Navier-Stokes equation for incompressible flow with constant viscosity read (*conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\frac{\partial v_i}{\partial x_i} = 0$$
$$\rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

▶The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form)

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

▶The equation for internal energy reads

$$\rho\frac{du}{dt} = -p\frac{\partial v_i}{\partial x_i} + \underbrace{2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}}_{\Phi} + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$$

▶Streamfunction, Ψ; potential, Φ

$$v_1 = \frac{\partial \Psi}{\partial x_2}, \quad v_2 = -\frac{\partial \Psi}{\partial x_1}$$
$$v_k = \frac{\partial \Phi}{\partial x_k}$$

▶The Rayleigh problem

$$\eta = \frac{x_2}{2\sqrt{\nu t}}, \quad f = \frac{v_1}{V_0}$$
$$\frac{d^2 f}{d\eta^2} + 2\eta\frac{df}{d\eta} = 0$$

▶Blasius solution

$$\xi = \left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} x_2, \quad \Psi = \left(\nu V_{1,\infty} x_1\right)^{1/2} g$$
$$\frac{1}{2}gg'' + g''' = 0$$

▶The transport equation for the vorticity reads

$$\frac{d\omega_p}{dt} \equiv \frac{\partial \omega_p}{\partial t} + v_k \frac{\partial \omega_p}{\partial x_k} = \omega_k \frac{\partial v_p}{\partial x_k} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

▶The Kolmogorov scales $v_\eta = (\nu\varepsilon)^{1/4}$, $\ell_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$, $\tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

▶The $-5/3$ law

$$E(\kappa) = \text{const.}\, \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}}$$

▶The *time averaged* continuity equation and Navier-Stokes equation for incompressible flow with constant viscosity read

$$\begin{aligned} \frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\ \rho \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho \overline{v_i' v_j'}\right) \end{aligned} \tag{1}$$

▶Units

$\nu = \frac{\mu}{\rho}$	m^2/s
ε	m^2/s^3
k	m^2/s^2

▶The wall region

Friction velocity: $\tau_w = \rho u_\tau^2$, $x_2^+ = \frac{x_2 u_\tau}{\nu}$

The linear law: $\frac{\bar{v}_1}{u_\tau} = \frac{u_\tau x_2}{\nu}$ or $\bar{v}_1^+ = x_2^+$

The log-law: $\frac{\bar{v}_1}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{x_2 u_\tau}{\nu}\right) + B$ or $\bar{v}_1^+ = \frac{1}{\kappa} \ln\left(x_2^+\right) + B$

▶The exact k equation reads

$$\frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu \frac{\partial k}{\partial x_j}\right] - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶The exact K equation reads

$$\begin{aligned} \frac{\partial \bar{v}_j K}{\partial x_j} &= \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \bar{v}_i \bar{p}}{\partial x_i} - \nu \frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j} \\ &\quad - \frac{\partial \bar{v}_i \overline{v_i' v_j'}}{\partial x_j} + \overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j}. \end{aligned}$$

►The exact $\overline{v_i'v_j'}$ equation reads

$$\frac{\partial}{\partial x_k}(\bar{v}_k\overline{v_i'v_j'}) = -\overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}\left(\overline{v_i'v_j'v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i'p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j'p'} - \nu\frac{\partial \overline{v_i'v_j'}}{\partial x_k}\right)$$
$$+\frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}$$

►In the Boussinesq assumption an eddy (i.e. a *turbulent*) viscosity is introduced to model the unknown Reynolds stresses in Eq. 1. The stresses are modeled as

$$\overline{v_i'v_j'} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}k = -2\nu_t\bar{s}_{ij} + \frac{2}{3}\delta_{ij}k$$

Trick 1:

$$A_i\frac{\partial B_j}{\partial x_k} = \frac{\partial A_iB_j}{\partial x_k} - B_j\frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i\frac{\partial A_i}{\partial x_j} = \frac{1}{2}\frac{\partial A_iA_i}{\partial x_j}$$