2015-01-05, Exam in

Mechanics of fluids, TME225

- **Time:** 14.00–18.00 **Location:** M
- **Teacher:** Lars Davidson, phone 772 1404, 0730-791 161
- **Aids**
  - Formula sheet, 4 pages, appended to the exam
- **Checking the evaluation and results of your written exam**: contact the assistant Hamid Abedi, hamidreza.abedi@chalmers.se
- **Grading**: 24-35p: 3, 36-47: 4, 48-60: 5.

1

a) The relation between the viscous stress tensor, τ_{ij}, and the viscous stress vector, $t_i^{\hat{n}}$ reads $t_i^{\hat{n}} = \tau_{ji} n_j$. Consider fully developed, incompressible, laminar channel flow. Give the expressions for the stress vector on two surfaces with the normal vectors

1. $(0, 1, 0)$
2. $(1/\sqrt{2}, 1/\sqrt{2}, 0)$

Illustrate also the stress vectors with figures.

(5p)

b) Consider the permutation tensor ϵ_{ijk} and the strain-rate tensor, S_{ij}. Show that their inner product is zero, *i.e.* $\epsilon_{ijk} S_{ij} = 0$.

(5p)

2

a) Show that the continuity equation is automatically satisfied in 2D when the velocity is expressed in the streamfunction, Ψ.

(5p)

b) Explain the energy transfer between kinetic energy, k, and internal energy, u. Which part is reversible and which part is irreversible?

(5p)

3

a) Derive the flow equations for fully developed flow between two parallel plates, i.e. fully developed channel flow. The flow is steady, incompressible and two-dimensional.

(5p)

b) Show the similarities between the vorticity and temperature transport equations in fully developed flow between two parallel plates. The flow is steady, incompressible and two-dimensional.
Hint: if the boundary is adiabatic, the temperature in the channel will be zero (or, more correctly, stay constant).

(5p)

4

a) What does isotropic turbulence mean? How is homogeneous turbulence defined?

(5p)

b) Derive the transport equation for turbulent, kinetic energy, k.

(5p)

5

a) In fully developed turbulent channel flow (incompressible, steady and two-dimensional), the time-averaged Navier-Stokes consists only of three terms. Show how the three terms vary across the channel. Which two terms balance each other in the outer region? Which term drives ("pushes") the flow in the x_1 direction? Which two terms are large in the inner region? Which term drives the flow?

(5p)

b) Where is the production term, $P^k = -\overline{v_1' v_2'}\partial\bar{v}_1/\partial x_2$, largest? Show how $-\overline{v_1' v_2'}$ and $\partial\bar{v}_1/\partial x_2$ vary near the wall.

(5p)

6

a) Consider the dissipation term, ε_{12}, for the shear stress: how large is it? Explain why.

(5p)

b) Consider fully developed channel flow: how are the expressions for the production terms in the $\overline{v_i' v_j'}$ equations simplified? Which production terms are zero and non-zero, respectively?

(5p)

TME225 Mechanics of fluids: Formula sheet to be used at written examinations

▶The $\epsilon - \delta$ identity reads

$$\epsilon_{inm}\epsilon_{mjk} = \epsilon_{min}\epsilon_{mjk} = \epsilon_{nmi}\epsilon_{mjk} = \delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj}$$

▶The vorticity vector is computed as

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$$
$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

▶The material derivative

$$\rho\frac{d\Psi}{dt} = \rho\frac{\partial \Psi}{\partial t} + \rho v_j \frac{\partial \Psi}{\partial x_j}$$

where $\Psi = v_i, u, T, k, \overline{v_i' v_j'} \ldots$

▶The balance equation for mass

$$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$$

▶The balance equation for linear momentum

$$\rho\frac{dv_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i$$

▶The balance equation for internal energy

$$\rho\frac{du}{dt} = \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

▶The equation for kinetic energy reads

$$\rho\frac{dk}{dt} = \frac{\partial v_i \sigma_{ji}}{\partial x_j} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} + \rho v_i f_i$$

▶The constitutive law for Newtonian viscous fluids

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}, \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$
$$q_i = -k\frac{\partial T}{\partial x_i}$$

▶The continuity equation and the Navier-Stokes equation for incompressible flow with constant viscosity read (*conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\frac{\partial v_i}{\partial x_i} = 0$$
$$\rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

▶The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form)

$$\rho\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}$$

▶The equation for internal energy reads

$$\rho\frac{du}{dt}=-p\frac{\partial v_i}{\partial x_i}+\underbrace{2\mu S_{ij}S_{ij}-\frac{2}{3}\mu S_{kk}S_{ii}}_{\Phi}+\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$$

▶Streamfunction, Ψ; potential, Φ

$$v_1=\frac{\partial\Psi}{\partial x_2},\quad v_2=-\frac{\partial\Psi}{\partial x_1}$$

$$v_k=\frac{\partial\Phi}{\partial x_k}$$

▶The Rayleigh problem

$$\eta=\frac{x_2}{2\sqrt{\nu t}},\quad f=\frac{v_1}{V_0}$$

$$\frac{d^2f}{d\eta^2}+2\eta\frac{df}{d\eta}=0$$

▶Blasius solution

$$\xi=\left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2}x_2,\quad \Psi=\left(\nu V_{1,\infty}x_1\right)^{1/2}g$$

$$\frac{1}{2}gg''+g'''=0$$

▶The transport equation for the vorticity reads

$$\frac{d\omega_p}{dt}\equiv\frac{\partial\omega_p}{\partial t}+v_k\frac{\partial\omega_p}{\partial x_k}=\omega_k\frac{\partial v_p}{\partial x_k}+\nu\frac{\partial^2\omega_p}{\partial x_j\partial x_j}$$

▶The Kolmogorov scales $v_\eta=(\nu\varepsilon)^{1/4}$, $\ell_\eta=\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$, $\tau_\eta=\left(\frac{\nu}{\varepsilon}\right)^{1/2}$

▶The $-5/3$ law

$$E(\kappa)=\text{const.}\,\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$$

▶The *time averaged* continuity equation and Navier-Stokes equation for incompressible flow with constant viscosity read

$$\begin{aligned} \frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\ \rho \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho \overline{v_i' v_j'}\right) \end{aligned} \tag{1}$$

▶Units

$\nu = \frac{\mu}{\rho}$	m^2/s
ε	m^2/s^3
k	m^2/s^2

▶The wall region

Friction velocity: $\tau_w = \rho u_\tau^2$, $x_2^+ = \frac{x_2 u_\tau}{\nu}$

The linear law: $\frac{\bar{v}_1}{u_\tau} = \frac{u_\tau x_2}{\nu}$ or $\bar{v}_1^+ = x_2^+$

The log-law: $\frac{\bar{v}_1}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{x_2 u_\tau}{\nu}\right) + B$ or $\bar{v}_1^+ = \frac{1}{\kappa}\ln\left(x_2^+\right) + B$

▶The exact k equation reads

$$\frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶The exact K equation reads

$$\begin{aligned} \frac{\partial \bar{v}_j K}{\partial x_j} = \nu\frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \bar{v}_i \bar{p}}{\partial x_i} - \nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j} \\ - \frac{\partial \bar{v}_i \overline{v_i' v_j'}}{\partial x_j} + \overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j}. \end{aligned}$$

▶The exact $\overline{v_i' v_j'}$ equation reads

$$\begin{aligned} \frac{\partial}{\partial x_k}(\bar{v}_k \overline{v_i' v_j'}) = -\overline{v_j' v_k'}\frac{\partial \bar{v}_i}{\partial x_k} - \overline{v_i' v_k'}\frac{\partial \bar{v}_j}{\partial x_k} \\ -\frac{\partial}{\partial x_k}\left(\overline{v_i' v_j' v_k'} + \frac{1}{\rho}\delta_{jk}\overline{v_i' p'} + \frac{1}{\rho}\delta_{ik}\overline{v_j' p'} - \nu\frac{\partial \overline{v_i' v_j'}}{\partial x_k}\right) \\ + \frac{1}{\rho}\overline{p'\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}} \end{aligned}$$

▶In the Boussinesq assumption an eddy (i.e. a *turbulent*) viscosity is introduced to model the unknown Reynolds stresses in Eq. 1. The stresses are modeled as

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) + \frac{2}{3}\delta_{ij}k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij}k$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2} \frac{\partial A_i A_i}{\partial x_j}$$