4.4.1 Complex variables for potential solutions of plane flows

Complex analysis is a suitable tool for studying potential flow. We start this section by repeating some basics of complex analysis. For real functions, the value of a partial derivative, $\partial f/\partial x$, at $x = x_0$ is defined by making x approach x_0 and then evaluating $(f(x+x_0)-f(x))/x_0$. The total derivative, df/dt, is defined by approaching the point $x_{10}, x_{20}, x_{30}, t$ as a linear combination of all independent variables (cf. Eq. 1.1).

A complex derivative of a complex variable is defined as $(f(z + z_0) - f(z))/z_0$ where $z = x+iy$ and $f = u+iv$. We can approach the point z_0 both in the real coordinate direction, x, and in the imaginary coordinate direction, y. The complex derivative is defined only if the value of the derivative is independent of how we approach the point z_0. Hence

$$
\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, iy_0) - f(x_0, iy_0)}{\Delta x} &= \lim_{\Delta y \to 0} \frac{f(x_0, iy_0 + i\Delta y) - f(x_0, iy_0)}{i\Delta y}.
\end{aligned}
\tag{4.35}
$$

The second line can be written as

$$
\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{i}{i^2}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y} \tag{4.36}
$$

since $i^2 = -1$. Inserting $f = u + iv$ and taking the partial derivative of f we get

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
i\frac{\partial f}{\partial y} &= i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}
\end{aligned}
\tag{4.37}
$$

Using Eq. 4.36 gives

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4.38}
$$

Equations 4.38 are called the *Cauchy-Riemann* equations. Another way to derive Eq. 4.38 is to require that f should depend only on z but not on $\bar{z}$ [7] ($\bar{z}$ is the complex conjugate of z, i.e. $\bar{z} = x - iy$).

So far the complex plane has been expressed as $z = x+iy$. It can also be expressed in polar coordinates (see Fig. 4.5)

$$
z = re^{i\theta} = r(\cos\theta + i\sin\theta) \tag{4.39}
$$

Now we return to fluid mechanics and potential flow. Let us introduce a complex potential, f, based on the streamfunction, Ψ (Eq. 3.43), and the velocity potential, Φ (Eq. 1.22)

$$
f = \Phi + i\Psi \tag{4.40}
$$

Recall that for two-dimensional, incompressible flow, the velocity potential satisfies the Laplace equation, see Eq. 4.28. The streamfunction also satisfies the Laplace equation in potential flow where the vorticity, ω_i, is zero. This is easily seen by taking the divergence of the streamfunction, Eq. 3.43

$$
\frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2} = -\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} = -\omega_3 = 0 \tag{4.41}
$$