31446 Mechanics of fluids: Formula sheet to be used at written examinations

▶The $\epsilon - \delta$ identity reads

$$\epsilon_{inm}\epsilon_{mjk} = \epsilon_{min}\epsilon_{mjk} = \epsilon_{nmi}\epsilon_{mjk} = \delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj}$$

▶Strain rate tensor, vorticity tensor

$$\begin{aligned}\frac{\partial v_i}{\partial x_j} &= \frac{1}{2}\left(\underbrace{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}}_{2\partial v_i/\partial x_j} + \underbrace{\frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i}}_{=0}\right)\\ &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = S_{ij} + \Omega_{ij}\end{aligned}$$

▶The vorticity vector is computed as

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$$
$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

▶The material derivative

$$\rho\frac{d\Psi}{dt} = \rho\frac{\partial \Psi}{\partial t} + \rho v_j \frac{\partial \Psi}{\partial x_j}$$

where $\Psi = v_i, u, T, k, \overline{v_i' v_j'} \ldots$

▶The balance equation for mass

$$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$$

▶The balance equation for linear momentum

$$\rho\frac{dv_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i$$

▶The balance equation for internal energy

$$\rho\frac{du}{dt} = \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

▶The equation for kinetic energy reads

$$\rho\frac{dk}{dt} = \frac{\partial v_i \sigma_{ji}}{\partial x_j} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} + \rho v_i f_i$$

▶The constitutive law for Newtonian viscous fluids

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}, \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$
$$q_i = -k\frac{\partial T}{\partial x_i}$$

▶Viscosity

μ: dynamic viscosity

ν: kinematic viscosity ($\nu = \mu/\rho$)

▶The continuity equation and the Navier-Stokes equation for incompressible flow with constant viscosity read (*conservative* form, p denotes the hydrostatic pressure, i.e. $p = 0$ if $v_i = 0$)

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\rho \frac{\partial v_i}{\partial t} + \rho \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

▶The Navier-Stokes equation for incompressible flow with constant viscosity read (*non-conservative* form)

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The viscous stress tensor then reads

$$\tau_{ij} = 2\mu S_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

▶The equation for internal energy reads

$$\rho \frac{du}{dt} = -p \frac{\partial v_i}{\partial x_i} + \underbrace{2\mu S_{ij} S_{ij} - \frac{2}{3} \mu S_{kk} S_{ii}}_{\Phi} + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right)$$

▶Streamfunction, Ψ; potential, Φ

$$v_1 = \frac{\partial \Psi}{\partial x_2}, \quad v_2 = -\frac{\partial \Psi}{\partial x_1}$$

$$v_k = \frac{\partial \Phi}{\partial x_k}$$

▶The Rayleigh problem

$$\eta = \frac{x_2}{2\sqrt{\nu t}}, \quad f = \frac{v_1}{V_0}$$

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0$$

▶Blasius solution

$$\xi = \left(\frac{V_{1,\infty}}{\nu x_1} \right)^{1/2} x_2, \quad \Psi = \left(\nu V_{1,\infty} x_1 \right)^{1/2} g$$

$$\frac{1}{2} g g'' + g''' = 0$$

▶The Navier-Stokes (different form of the convective term)

$$\frac{\partial v_i}{\partial t} + \frac{\partial k}{\partial x_i} - \varepsilon_{ijk} v_j \omega_k = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + f_i$$

▶Complex functions

$$f = u + iv, \quad z = x + iy, \quad z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

In fluid mechanics, $f = \Phi + i\Psi$, i.e. the streamfunction is the imaginary part and the velocity potential is the real part.

▶The Laplace operator in polar coordinates reads

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

▶Velocities from Ψ in polar coordinates

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \Psi}{\partial r}$$

▶The Bernoulli equation

$$p_1 + \frac{\rho V_1^2}{2} + \rho g h_1 = const$$

▶The transport equation for the vorticity reads

$$\frac{d\omega_p}{dt} \equiv \frac{\partial \omega_p}{\partial t} + v_k \frac{\partial \omega_p}{\partial x_k} = \omega_k \frac{\partial v_p}{\partial x_k} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

▶The Kolmogorov scales $v_\eta = (\nu\varepsilon)^{1/4}$, $\ell_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$, $\tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

▶The $-5/3$ law

$$E(\kappa) = \text{const.}\,\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$$

▶The *time averaged* continuity equation and Navier-Stokes equation for incompressible flow with constant viscosity read

$$\begin{aligned} \frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\ \rho \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho \overline{v_i' v_j'}\right) \end{aligned} \quad (1)$$

▶Units

$$\begin{array}{ll} \nu = \frac{\mu}{\rho} & m^2/s \\ \varepsilon & m^2/s^3 \\ k & m^2/s^2 \end{array}$$

▶The wall region

Friction velocity: $\tau_w = \rho u_\tau^2$, $x_2^+ = \frac{x_2 u_\tau}{\nu}$

The linear law: $\dfrac{\bar v_1}{u_\tau} = \dfrac{u_\tau x_2}{\nu}$ or $\bar v_1^+ = x_2^+$

The log-law: $\dfrac{\bar v_1}{u_\tau} = \dfrac{1}{\kappa}\ln\left(\dfrac{x_2 u_\tau}{\nu}\right) + B$ or $\bar v_1^+ = \dfrac{1}{\kappa}\ln\left(x_2^+\right) + B$

▶The exact k equation reads

$$\frac{\partial \bar v_j k}{\partial x_j} = -\overline{v_i' v_j'}\frac{\partial \bar v_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶The exact K equation reads

$$\begin{aligned}\frac{\partial \bar v_j K}{\partial x_j} &= \nu\frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \bar v_i \bar p}{\partial x_i} - \nu\frac{\partial \bar v_i}{\partial x_j}\frac{\partial \bar v_i}{\partial x_j}\\ &- \frac{\partial \bar v_i \overline{v_i' v_j'}}{\partial x_j} + \overline{v_i' v_j'}\frac{\partial \bar v_i}{\partial x_j}.\end{aligned}$$

▶The modelled k and ε equations

$$\begin{aligned}\frac{\partial k}{\partial t} + \bar v_j\frac{\partial k}{\partial x_j} &= \nu_t\left(\frac{\partial \bar v_i}{\partial x_j} + \frac{\partial \bar v_j}{\partial x_i}\right)\frac{\partial \bar v_i}{\partial x_j} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar\theta}{\partial x_i}\\ &-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]\\ \frac{\partial \varepsilon}{\partial t} + \bar v_j\frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k}c_{\varepsilon 1}\nu_t\left(\frac{\partial \bar v_i}{\partial x_j} + \frac{\partial \bar v_j}{\partial x_i}\right)\frac{\partial \bar v_i}{\partial x_j}\\ &+ c_{\varepsilon 1} g_i\frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar\theta}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]\\ \nu_t &= C_\mu\frac{k^2}{\varepsilon}\end{aligned}$$

▶Wall functions

$$\begin{aligned}u_\tau &= \frac{\kappa \bar v_{1,P}}{\ln(E u_\tau \delta x_2/\nu)}\\ k_P &= C_\mu^{-1/2}u_\tau^2, \quad C_\mu = 0.09\\ \varepsilon_P &= P^k = \frac{u_\tau^3}{\kappa\delta x_2}\end{aligned}$$

▶Low-Reynolds number model: different wall boundary conditions for ε:

$$\begin{aligned}\varepsilon_{wall} &= \nu\frac{\partial^2 k}{\partial x_2^2}\\ \varepsilon_{wall} &= 2\nu\left(\frac{\partial\sqrt{k}}{\partial x_2}\right)^2\\ \varepsilon_{wall} &= \frac{2\nu k}{x_2^2}\end{aligned}$$

▶In the Boussinesq assumption an eddy (i.e. a *turbulent*) viscosity is introduced to model the unknown Reynolds stresses in Eq. 1. The stresses are modelled as

$$\overline{v_i' v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) + \frac{2}{3}\delta_{ij} k = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij} k$$

Trick 1:

$$A_i \frac{\partial B_j}{\partial x_k} = \frac{\partial A_i B_j}{\partial x_k} - B_j \frac{\partial A_i}{\partial x_k}$$

Trick 2:

$$A_i \frac{\partial A_i}{\partial x_j} = \frac{1}{2} \frac{\partial A_i A_i}{\partial x_j}$$