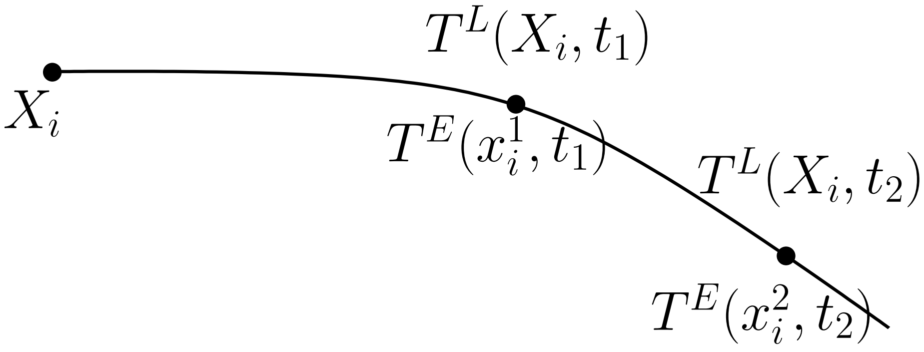
On-line Lecture 1

¶See Section 1.1, Eulerian, Lagrangian, material derivative



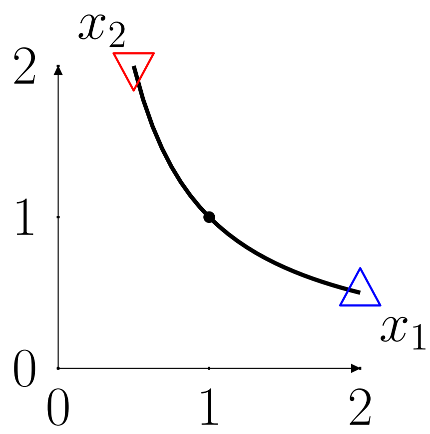
▶Lagrangian approach

- The (fluid) particle is described by its initial position, X_i, and time, t
- In other words we "label" a particle with X_i and then follow it.
- The variation of T^L is expressed as dT^L/dt.

▶Eulerian approach

- We look a point, x_i, and see what happens.
- Hence T^E depends on both x_i and t
- Chain rule: $\frac{dT^E}{dt} = \frac{\partial T^E}{\partial t} + \frac{dx_i}{dt}\frac{\partial T^E}{\partial x_i} = \frac{\partial T^E}{\partial t} + v_i \frac{\partial T^E}{\partial x_i}$ ▶ $\underbrace{\frac{dT^E}{dt}}_{\text{material change}} = \underbrace{\frac{\partial T^E}{\partial t}}_{\text{local change}} + \underbrace{v_j \frac{\partial T^E}{\partial x_j}}_{\text{convective change}}$

¶See Section 1.2, What is the difference between $\frac{dv_2}{dt}$ and $\frac{\partial v_2}{\partial t}$?



Flow path $x_2 = 1/x_1$. Filled circle: $(x_1, x_2) = (1, 1)$.

$$x_1 = \exp(t), \quad x_2 = \exp(-t), \quad \text{and hence } x_2 = 1/x_1 \tag{30.1}$$

$\Rightarrow$ ▽: start: $t = \ln(0.5)$ △: end: $t = \ln(2)$

▶The flow is steady (in Eulerian coordinates).

▶Equation 30.1 gives the velocities

$$v_1^L = \frac{dx_1}{dt} = \exp(t), \quad v_2^L = \frac{dx_2}{dt} = -\exp(-t) \tag{30.2}$$

▶Equations 30.1 and 30.2 give

$$v_1^E = x_1, \quad v_2^E = -x_2 \tag{30.3}$$

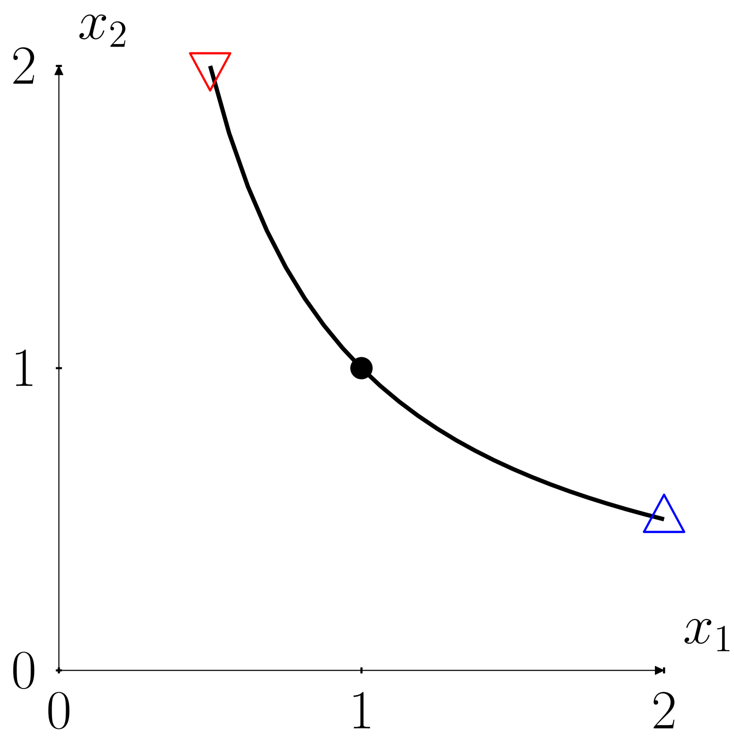
$$x_1 = \exp(t), \quad x_2 = \exp(-t), \quad v_1^L = \exp(t), \quad v_2^L = -\exp(-t), \quad v_1^E = x_1, \quad v_2^E = -x_2$$

► $\frac{dv_2}{dt} = \frac{dv_2^L}{dt}, \quad \frac{dv_2^E}{dt} = \frac{\partial v_2}{\partial t} + v_j \frac{\partial v_2}{\partial x_j}$ ►Let's compute them at point $(1, 1)$

$$\frac{dv_2^L}{dt} = \exp(-t), \quad t = \ln(1) = 0 \quad \Rightarrow \quad \frac{dv_2^L}{dt} = 1$$

$$\frac{dv_2^E}{dt} = \frac{\partial v_2^E}{\partial t} + v_1^E \frac{\partial v_2^E}{\partial x_1} + v_2^E \frac{\partial v_2^E}{\partial x_2} = 0 + x_1 \cdot 0 - x_2 \cdot (-1) = x_2 = 1$$

►Of course $\frac{dv_2}{dt} = \frac{dv_2^E}{dt} = \frac{dv_2^L}{dt} = 1$



Flow path $x_2 = 1/x_1$. Filled circle: $(x_1, x_2) = (1, 1)$.

► $\dfrac{dv_2^E}{dt} = x_2, \quad \dfrac{dv_2^L}{dt} = \exp(-t).$

►Consider the point $(x_1, x_2) = (1, 1)$. The velocity at this point does not change in time; hence $\dfrac{\partial v_2^E}{\partial t} = 0$.

►If we however travel with the particle then the v_2 velocity changes with time, i.e. $\dfrac{dv_2^L}{dt} = \dfrac{dv_2}{dt} = 1$ (it increases, i.e. it gets less negative with time).

¶See Section B, Introduction to tensor notation

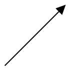
- ► a: A tensor of zeroth rank (scalar)
- ► a_i: A tensor of first rank (vector) → $a_i = (2, 1, 0)$
- ► a_{ij}: A tensor of second rank (tensor)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

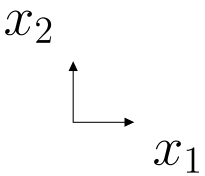
$\sigma_{ij} = \sigma_{ji}$

► What is a tensor?

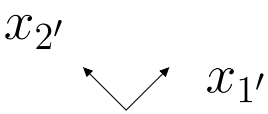
► A tensor is a *physical* quantity. It is
independent of the coordinate system. The tensor of rank one (vector) b_i below



is physically the same expressed in the coordinate system (x_1, x_2)



where $b_i = (1/\sqrt{2}, 1/\sqrt{2}, 0)^T$ and in the coordinate system $(x_{1'}, x_{2'})$



where $b_{i'} = (1, 0, 0)^T$. The tensor is the same even if its *components* are different.

► This also applied for σ_{ij}

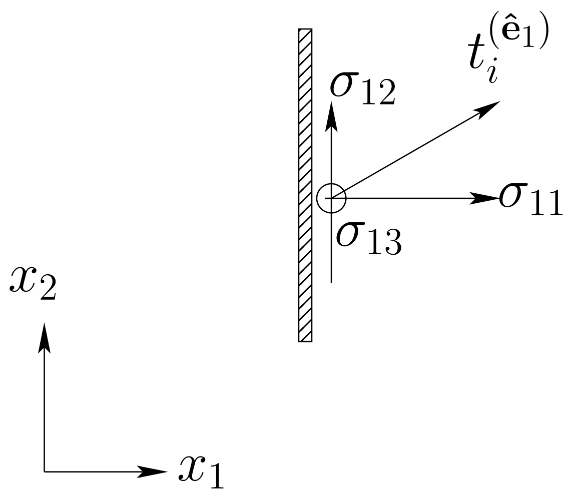
¶See Section 1.3, Viscous stress, pressure

►The momentum balance equation derived in the continuum mechanics lectures reads

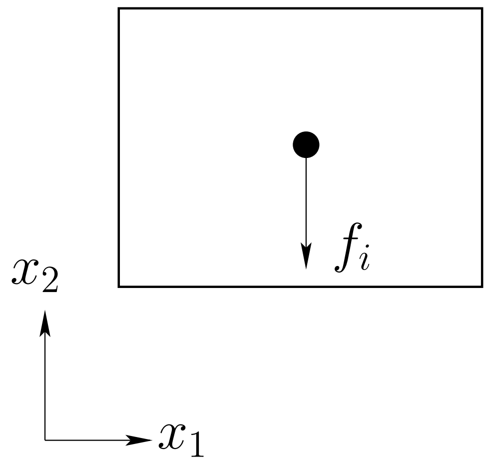
$$\boxed{\rho \dot{v}_i - \sigma_{ji,j} - \rho f_i = 0}$$

►We write it as

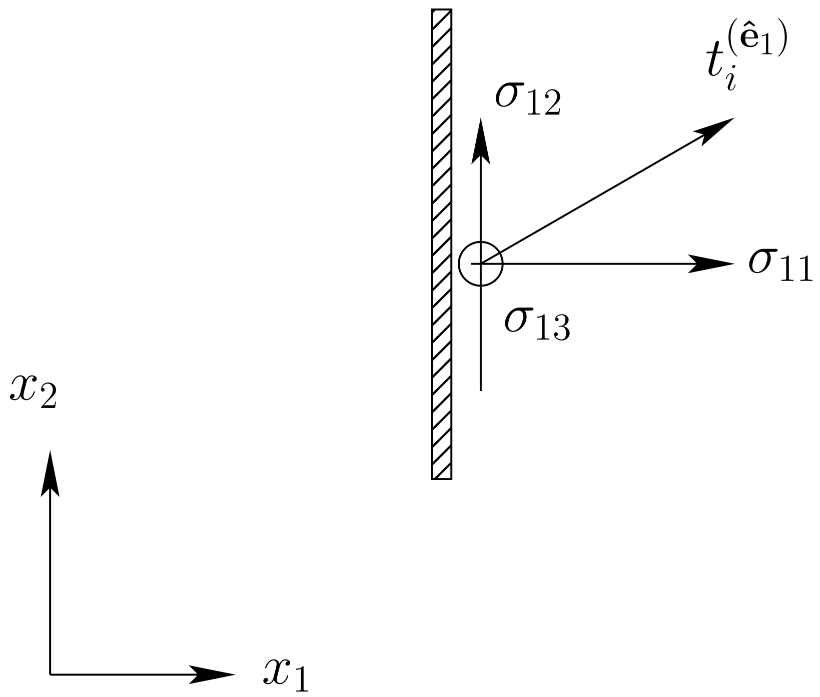
$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i \qquad (30.4)$$



Stress components and stress vector on a surface.



Volume force, $f_i = (0, -g, 0)$, acting in the middle of the fluid element.



Stress components and stress vector on a surface.

▶surface forces (σ_{ij} denotes the stress tensor). Stress is force per unit area. The surface stress vector is computed as

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ji} n_j$$

where $\hat{\mathbf{n}} = n_j$ is the unit normal vector of the surface.

▶The stress tensor, σ_{ij}, is split into one part which includes pressure, P, and one which includes viscous stresses (friction)

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

where $P = -\dfrac{1}{3}\sigma_{kk}$.

►A constitutive relation can be derived for the stress tensor which reads

$$\sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}, \quad \tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}, \quad S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{30.5}$$

►Before we insert Eq. 30.5 into Eq. 30.4, let's look at $\dfrac{\partial v_i}{\partial x_j}$, and S_{ij} in some detail.

¶See Section 1.4, Strain rate tensor, vorticity

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\underbrace{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}}_{2\partial v_i/\partial x_j} + \underbrace{\frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i}}_{=0}\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = S_{ij} + \Omega_{ij} \tag{30.6}$$

►The vorticity reads

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}, \quad \omega_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

►The vorticity represents rotation of a fluid particle. Inserting the expression for S_{ij} and Ω_{ij} gives

$$\omega_i = \varepsilon_{ijk}(S_{kj} + \Omega_{kj}) = \varepsilon_{ijk}\Omega_{kj} \tag{30.7}$$

the product of a symmetric, S_{kj}, and an antisymmetric tensor, ε_{ijk}, is zero.

$$\omega_i = \varepsilon_{ijk}(S_{kj} + \Omega_{kj}) = \varepsilon_{ijk}\Omega_{kj} \tag{30.7}$$

►Now let's invert Eq. 30.7. ►We start by multiplying it with $\varepsilon_{i\ell m}$ so that

$$\varepsilon_{i\ell m}\omega_i = \varepsilon_{i\ell m}\varepsilon_{ijk}\Omega_{kj} \tag{30.8}$$

- ε_{ijk} is the permutation tensor.
  - It is one if ijk is equal to 123 or any cyclic permutation ►$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$.
  - Switch two indices and it is equal to minus one ►i.e, $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{132} = -1$.
  - If two indices are equal, then ε_{ijk} is zero.
- δ_{ij} is the unit or identity tensor. It is one if ijk are equal and zero otherwise, i.e.

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$
$$\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$$

►Using the ε-δ-identity (see Section C) on the right side of Eq. 30.8 gives

$$\varepsilon_{i\ell m}\varepsilon_{ijk}\Omega_{kj} = (\delta_{\ell j}\delta_{mk} - \delta_{\ell k}\delta_{mj})\Omega_{kj} = \delta_{\ell j}\delta_{mk}\Omega_{kj} - \delta_{\ell k}\delta_{mj}\Omega_{kj} = \delta_{mk}\Omega_{k\ell} - \delta_{mj}\Omega_{\ell j} = \Omega_{m\ell} - \Omega_{\ell m} = 2\Omega_{m\ell}$$

$$\varepsilon_{i\ell m}\omega_i = \varepsilon_{i\ell m}\varepsilon_{ijk}\Omega_{kj} \tag{30.8}$$

$$\varepsilon_{i\ell m}\varepsilon_{ijk}\Omega_{kj} = 2\Omega_{m\ell}$$

▶Inserted in Eq. 30.8 we get

$$\Omega_{m\ell} = \frac{1}{2}\varepsilon_{i\ell m}\omega_i = \frac{1}{2}\varepsilon_{mi\ell}\omega_i = -\frac{1}{2}\varepsilon_{m\ell i}\omega_i$$

where we first used cyclic permutation of $\varepsilon_{i\ell m}$, ▶then used the fact that $\varepsilon_{i\ell m}$ is anti-symmetric.

▶Actually, it is easier to invert Eq. 30.7 $\omega_i = \varepsilon_{ijk}\Omega_{kj}$ component-by-component.

▶For example, for $i = 3$

$$\omega_3 = \varepsilon_{3jk}\Omega_{kj} = \varepsilon_{321}\Omega_{12} + \varepsilon_{312}\Omega_{21} = -1\cdot\Omega_{12} + 1\cdot\Omega_{21} = -2\Omega_{12}$$

▶ $\Rightarrow \quad \Omega_{12} = -\frac{1}{2}\omega_3$

¶See Section 1.5, Product of a symmetric and antisymmetric tensor

► The product of a symmetric, a_{ji}, and antisymmetric tensor, b_{ji}, is zero

$$a_{ij}b_{ij} = a_{ji}b_{ij} = -a_{ji}b_{ji},$$

where we used

2nd expression	$a_{ij} = a_{ji}$	(symmetric)
last expression	$b_{ij} = -b_{ji}$	(antisymmetric)

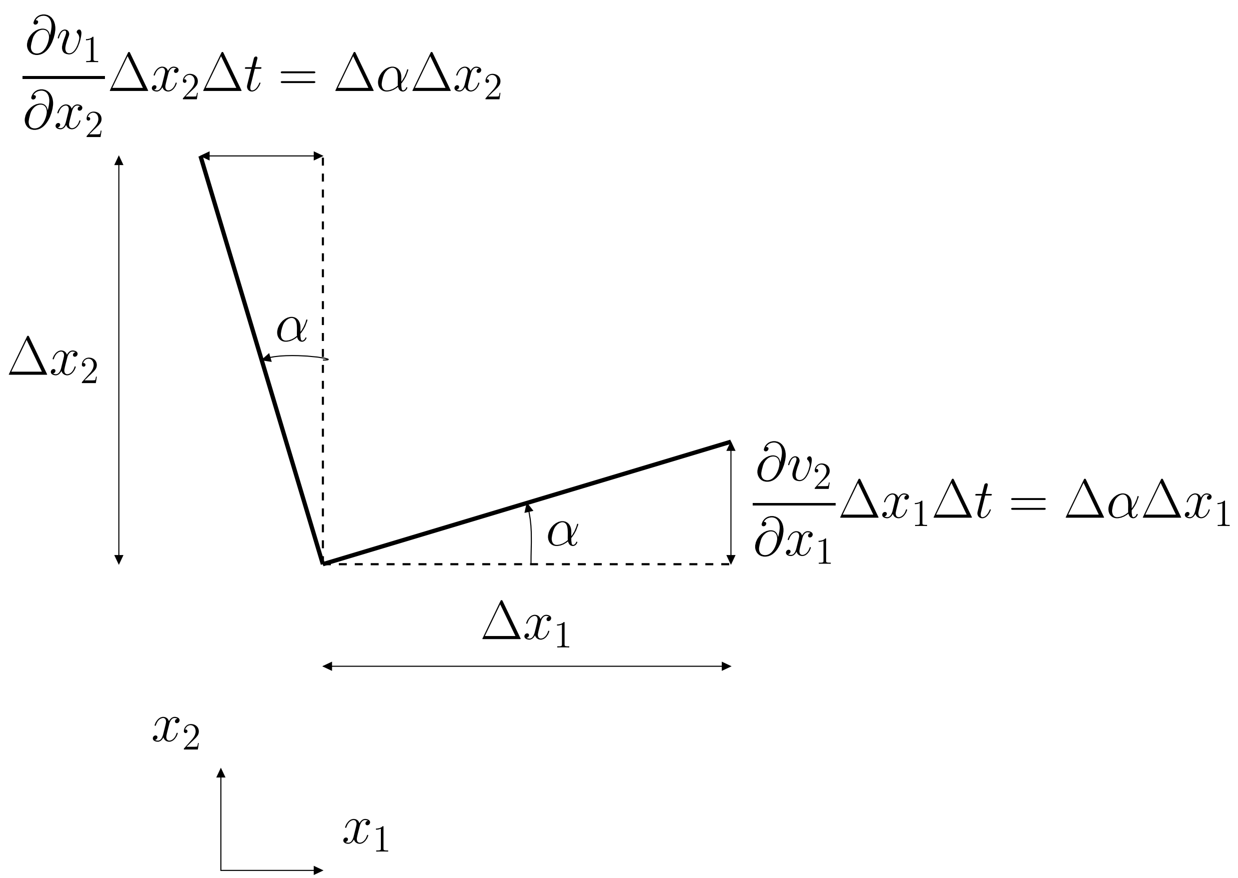
► Indices i and j are dummy indices $\Rightarrow$

$$a_{ij}b_{ij} = -a_{ij}b_{ij}$$

► This expression says that $A = -A$ which can be only true if $A = 0$ and hence $a_{ij}b_{ij} = 0$.

¶See Section 1.6, Deformation, rotation

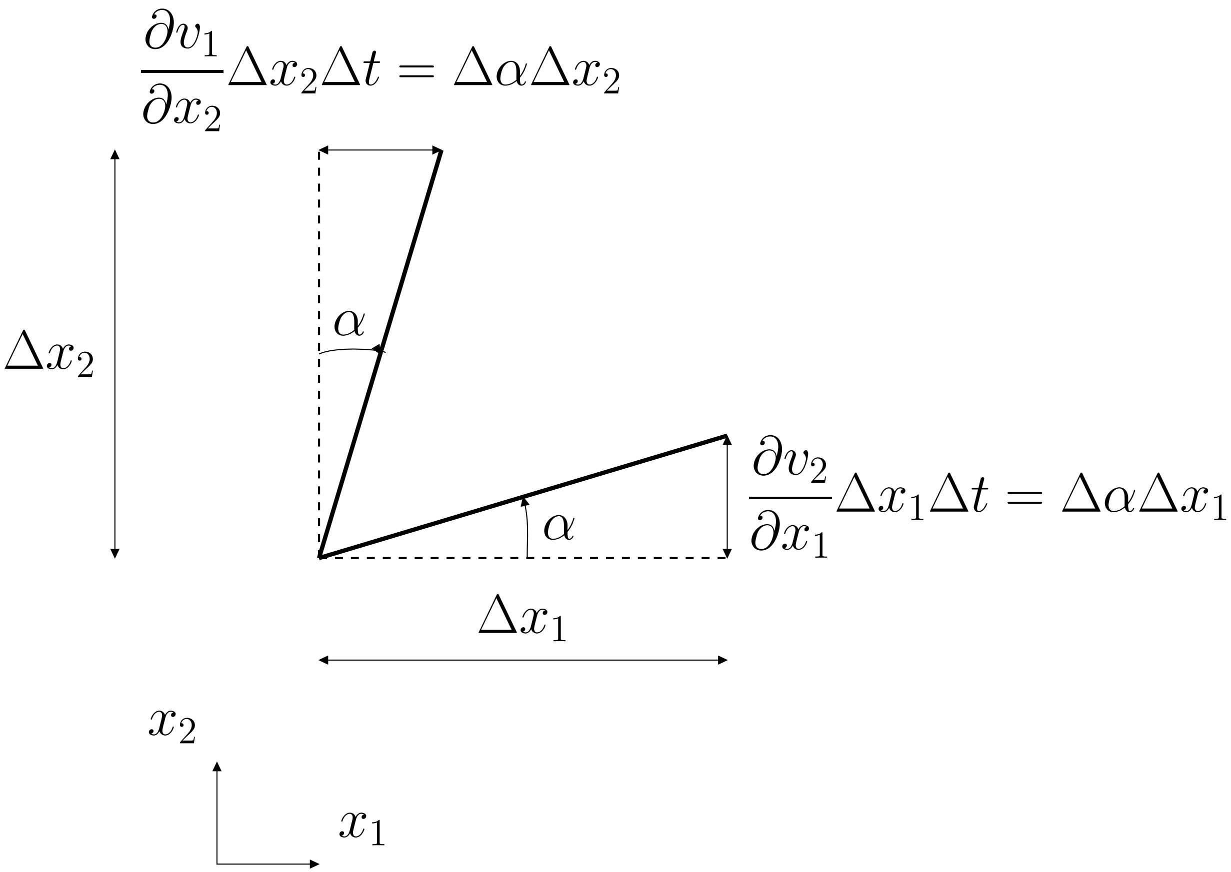
▶Rotation of a fluid particle ($\omega_3 > 0$, $\Omega_{12} < 0$) during time Δt



$\frac{\partial v_2}{\partial x_1}\Delta x_1$ ▶= the v_2 vel. at right end of horizonatal edge ▶$\frac{\partial v_2}{\partial x_1}\Delta x_1 \Delta t$ is the vertical displacement

- angle rotation per unit time: $\frac{\Delta\alpha}{\Delta t} \simeq d\alpha/dt = \partial v_2/\partial x_1$ = $-\partial v_1/\partial x_2$
- If not solid body, we take the average $d\alpha/dt = (\partial v_2/\partial x_1 - \partial v_1/\partial x_2)/2$.
- $\omega_3 = \partial v_2/\partial x_1 - \partial v_1/\partial x_2 = -2\Omega_{12}$ is twice the average rotation of the horizontal and vertical edge

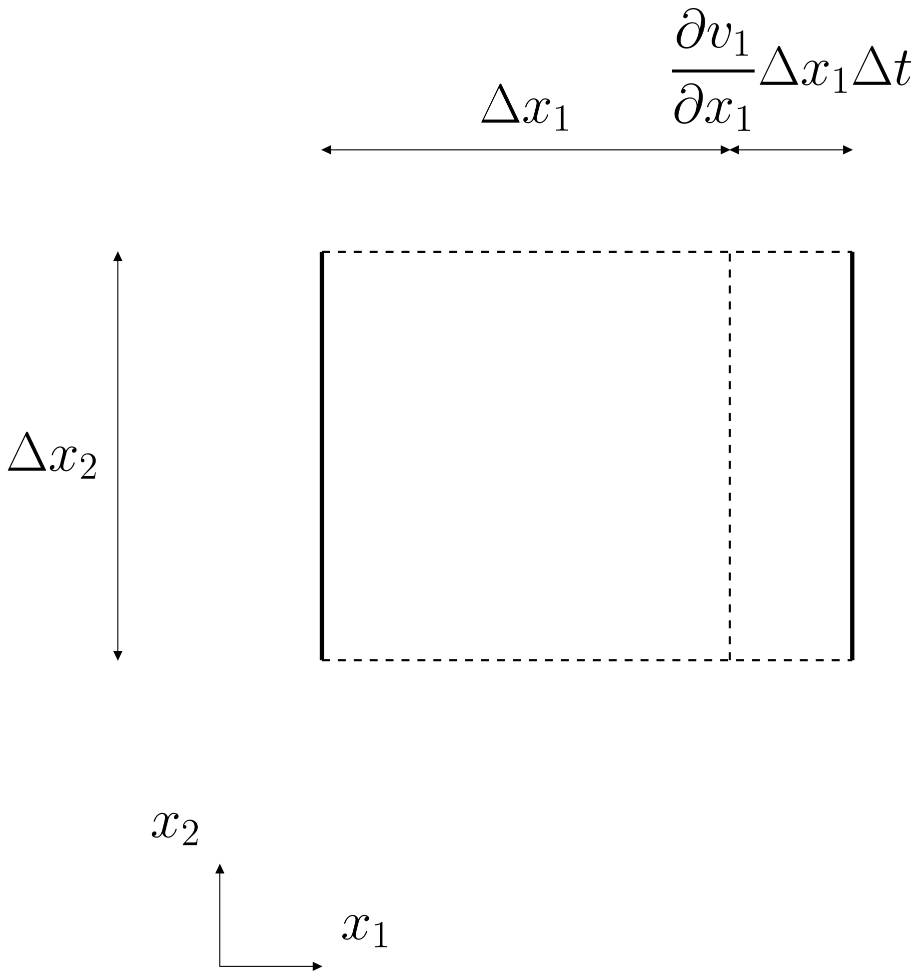
►Deformation of a fluid particle by shear during time Δt.



►Here, we take the average of $\partial v_2/\partial x_1$ and $\partial v_1/\partial x_2$

$S_{12} = (\partial v_1/\partial x_2 + \partial v_2/\partial x_1)/2.$

▶Elongation of a fluid particle during time Δt.



$\frac{\partial v_1}{\partial x_1}\Delta x_1$ ▶This is the v_1 vel. of the right edge ▶$\frac{\partial v_1}{\partial x_1}\Delta x_1 \Delta t$ is the horizontal elongation

▶Summary:

- Rotation of a fluid element is described by Ω_{ij}
- Shear corresponds to the off-diagonal elements of S_{ij}

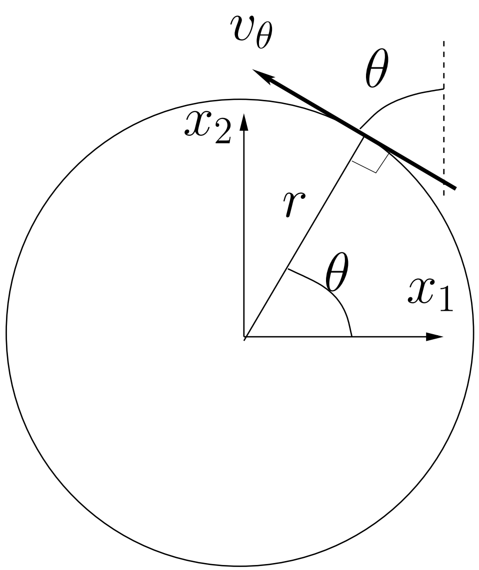
- The diagonal elements of S_{ij} represents elongation of a fluid eleent

¶See Section 1.7, Irrotational and rotational flow

►Flows are often classified based on rotation: they are *rotational* ($\omega_i \neq 0$) or *irrotational* ($\omega_i = 0$)

¶See Section 1.7.1, Ideal vortex line

$$\Phi = \frac{\Gamma\theta}{2\pi}, \quad v_k = \frac{\partial\Phi}{\partial x_k}, \quad v_\theta = \frac{\Gamma}{2\pi r}, \quad v_r = 0$$

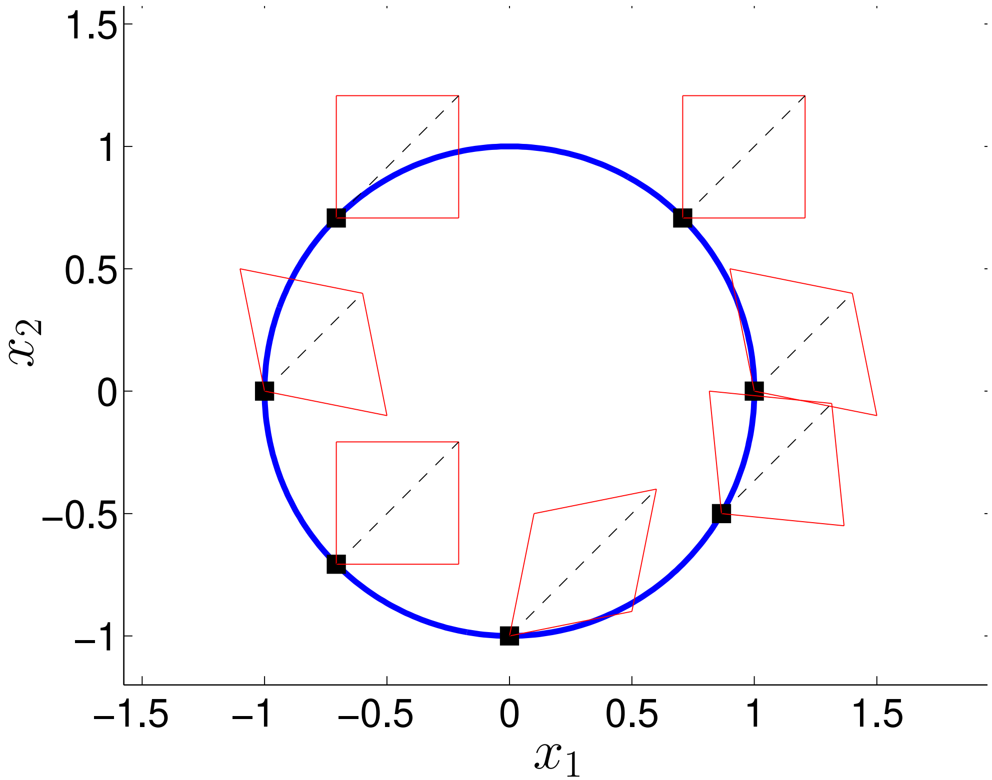


Transform v_θ into Cartesian components.

$$v_1 = -\frac{\Gamma x_2}{2\pi(x_1^2 + x_2^2)}, \quad v_2 = \frac{\Gamma x_1}{2\pi(x_1^2 + x_2^2)}.$$

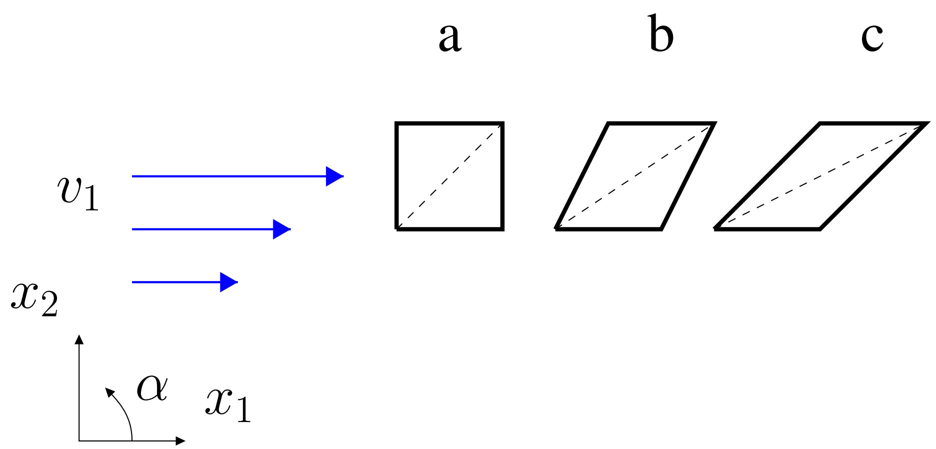
$$\frac{\partial v_1}{\partial x_2} = -\frac{\Gamma}{2\pi}\frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}, \quad \frac{\partial v_2}{\partial x_1} = \frac{\Gamma}{2\pi}\frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$$

$$\Rightarrow \omega_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \quad \Rightarrow$$ irrotational flow ►N.B.: vortex vs. vorticity



- The locations of the fluid particle is indicated by filled squares.
- The diagonals are shown as dashed lines.
- The fluid particle is shown at $\theta = 0$, $\pi/4$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $-\pi/6$.
- The fluid particle (i.e. its diagonal) does not rotate.
- Or, rather, the velocity field does not try to rotate the fluid elements

¶See Section 1.7.2, Shear flow



►Consider shear flow with $v_1 = cx_2^2$, $v_2 = 0$, see figure above.

►The vorticity is computed as

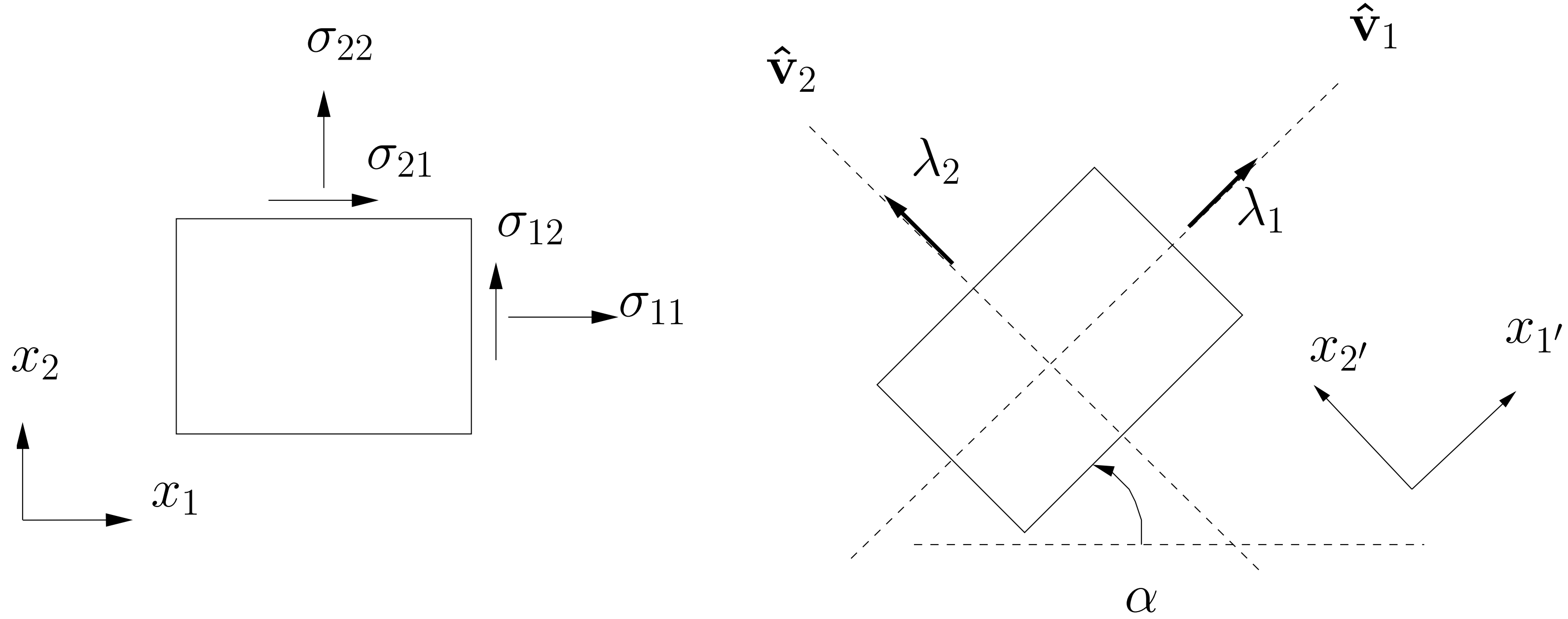
$$\omega_1 = \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} = 0, \quad \omega_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} = 0, \quad \omega_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = -2cx_2 \neq 0$$

Hence the flow is rotational

►The fluid particles rotate clock-wise (see figure above); i.e. they rotate in negative α direction.

►Finally: $\omega_i \neq 0$ does not really mean that a fluid element rotates.
  ►It only means that he velocity field tries to do that.

¶See Section 1.8, Eigenvalues and eigenvectors: physical interpretation



- A two-dimensional fluid element.
  - Left: in original state; σ_{ij} is symmetric;
  - right: rotated to principal coordinate directions.
  - $\lambda_1 = \sigma_{1'1'}$ and $\lambda_2 = \sigma_{2'2'}$ denote eigenvalues;
  - $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ denote unit eigenvectors.

Lecture 2

¶See Section 2.1.1, The continuity equation

$$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 \quad \text{incompressible flow gives} \quad \frac{\partial v_i}{\partial x_i} = 0$$

¶See Section 2.1.2, The momentum equation

$$\begin{aligned}
\sigma_{ij} &= -P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij} \\
\rho\frac{dv_i}{dt} &= \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i \\
&= -\frac{\partial P}{\partial x_j}\delta_{ij} + \frac{\partial}{\partial x_j}\left(2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}\right) + \rho f_i \\
&= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ij}) - \frac{2}{3}\frac{\partial}{\partial x_j}(\mu S_{kk}\delta_{ij}) + \rho f_i \\
&= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ij}) - \frac{2}{3}\frac{\partial}{\partial x_i}(\mu S_{kk}) + \rho f_i
\end{aligned}$$

$$\rho\frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ij}\right) - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu S_{kk}\right) + \rho f_i$$

►Note that the right-side (stress tensor, σ_{ij}) depends only on S_{ij} (deformation), not on Ω_{ij} (rotation).

►Incompressible flow:

$$\frac{\partial v_i}{\partial x_i} = 0 = S_{ii} \quad \text{continuity equation}$$

$$\rho\frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ij}\right) + \rho f_i$$

►Incompressible and constant μ:

$$\begin{aligned}
\rho\frac{dv_i}{dt} &= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\} + \rho f_i \\
&= -\frac{\partial P}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left\{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\} + \rho f_i \\
&= -\frac{\partial P}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}\right) + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right) + \rho f_i \\
&= -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) + \rho f_i \\
&= -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \rho f_i
\end{aligned}$$

¶See Section 2.2, The energy equation

▶First law of thermodynamics:

$$\underbrace{\rho\frac{du}{dt}}_{\text{internal energy change}} = \underbrace{\sigma_{ji}\frac{\partial v_i}{\partial x_j}}_{\text{exchange of work}} \underbrace{-\frac{\partial q_i}{\partial x_i}}_{\text{exchange of heat}} \tag{31.1}$$

$$q_i = -k\frac{\partial T}{\partial x_i} \quad \text{constitutive law}$$

$$\sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij} \quad \text{constitutive law}$$

▶Recall:

$$\frac{\partial v_i}{\partial x_j} = S_{ij} + \Omega_{ij}, \quad S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = \sigma_{ij}(S_{ij} + \Omega_{ij}) = \sigma_{ij}S_{ij}$$

▶This gives

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = \left\{-P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}\right\} S_{ij} =$$

$$= -P\delta_{ij}S_{ij} + 2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ij}\delta_{ij} = -PS_{ii} + 2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}$$

$$\underbrace{\rho\frac{du}{dt}}_{\text{internal energy change}} = \underbrace{\sigma_{ji}\frac{\partial v_i}{\partial x_j}}_{\text{exchange of work}} \underbrace{-\frac{\partial q_i}{\partial x_i}}_{\text{exchange of heat}} \tag{31.1}$$

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = -P\frac{\partial v_i}{\partial x_i} + 2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}, \quad q_i = -k\frac{\partial T}{\partial x_i}$$

Insert the constitutive relations above into Eq. 31.1 gives

$$\underbrace{\rho\frac{du}{dt}}_{\Delta U} = \underbrace{-P\frac{\partial v_i}{\partial x_i}}_{Rev} + \underbrace{2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}}_{\Phi} + \underbrace{\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)}_{Q}$$

▶During time, dt, the following happens:

▶<u>ΔU:</u> Change of internal energy of the fluid

▶<u>Rev:</u> Reversible work done by the fluid (compression or expansion)

▶<u>Φ:</u> Irreversible work (dissipation) done by the fluid

▶<u>Q</u>: Exchange of heat to the surrounding fluid

►Incompressible flow (low speed, $|v_i| < \frac{1}{3}$speed of sound)

$$\underbrace{\rho\frac{du}{dt}}_{\Delta U} = \underbrace{-P\frac{\partial v_i}{\partial x_i}}_{Rev} + \underbrace{2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}}_{\Phi} + \underbrace{\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)}_{Q}$$

$$du = c_p dT \quad \Rightarrow \quad \rho c_p \frac{dT}{dt} = \Phi + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$$

►Φ important for lubricant oils, See paper of Erwin Adi Hartono at course www page.

►For gases and "usual" liquids (i.e. not lubricant oils) we get ($\Phi \simeq 0$, k is constant)

$$\frac{dT}{dt} = \alpha\frac{\partial^2 T}{\partial x_i \partial x_i}, \quad \alpha = \frac{k}{\rho c_p}, \quad Pr = \frac{\nu}{\alpha}$$

¶See Section 2.3, Transformation of energy

▶$k = v_i v_i/2$ equation (multiply the momentum equation by v_i)

$$v_i\left(\rho\frac{dv_i}{dt} - \frac{\partial \sigma_{ji}}{\partial x_j} - \rho f_i\right) = v_i\rho\frac{dv_i}{dt} - v_i\frac{\partial \sigma_{ji}}{\partial x_j} - v_i\rho f_i = 0 \tag{31.2}$$

▶The first term on the left side can be re-written (Trick 2)

$$\rho v_i\frac{dv_i}{dt} = \frac{1}{2}\rho\frac{d(v_i v_i)}{dt} = \rho\frac{dk}{dt}, \quad v_i v_i/2 = v^2/2 = k \tag{31.3}$$

▶Trick 2: the product rule $\frac{1}{2}\frac{\partial A_i A_i}{\partial x_j} = \frac{1}{2}\left(A_i\frac{\partial A_i}{\partial x_j} + A_i\frac{\partial A_i}{\partial x_j}\right) = A_i\frac{\partial A_i}{\partial x_j}$

Using it backwards $A_i\frac{\partial A_i}{\partial x_j} = \frac{1}{2}\frac{\partial A_i A_i}{\partial x_j}$

▶Eqs. 31.2 and 31.3 give

$$\rho\frac{dk}{dt} = v_i\frac{\partial \sigma_{ji}}{\partial x_j} + \rho v_i f_i$$

Re-write the stress-velocity term so that (Trick 1)

$$\rho\frac{dk}{dt} = \frac{\partial v_i \sigma_{ji}}{\partial x_j} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} + \rho v_i f_i$$

▶Compare with the equation for innternal energy (Eq. 31.1)

$$\rho\frac{du}{dt} = \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

▶Trick 1: the product rule $\dfrac{\partial A_i B_i}{\partial x_j} = \left(A_i \dfrac{\partial B_i}{\partial x_j} + B_i \dfrac{\partial A_i}{\partial x_j} \right)$

Using it backwards $A_i \dfrac{\partial B_i}{\partial x_j} = \dfrac{\partial A_i B_i}{\partial x_j} - B_i \dfrac{\partial A_i}{\partial x_j}$

¶See Section 2.4, Left side of the transport equations

▶Left-hand side ($\Psi = v_i, u, T \ldots$)

$$\rho\frac{d\Psi}{dt} = \rho\frac{\partial\Psi}{\partial t} + \rho v_j\frac{\partial\Psi}{\partial x_j}$$

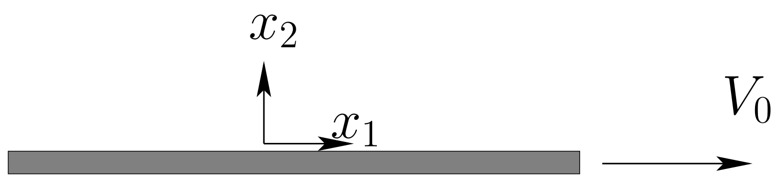
$$\rho\frac{\partial\Psi}{\partial t} + \rho v_j\frac{\partial\Psi}{\partial x_j} + \Psi\underbrace{\left(\frac{d\rho}{dt} + \rho\frac{\partial v_j}{\partial x_j}\right)}_{=0}$$

$$\underline{\rho\frac{\partial\Psi}{\partial t}} + \underbrace{\rho v_j\frac{\partial\Psi}{\partial x_j}} + \Psi\left(\underline{\frac{\partial\rho}{\partial t}} + \underbrace{v_j\frac{\partial\rho}{\partial x_j}} + \underbrace{\rho\frac{\partial v_j}{\partial x_j}}\right) = \frac{\partial\rho\Psi}{\partial t} + \frac{\partial\rho v_j\Psi}{\partial x_j}$$

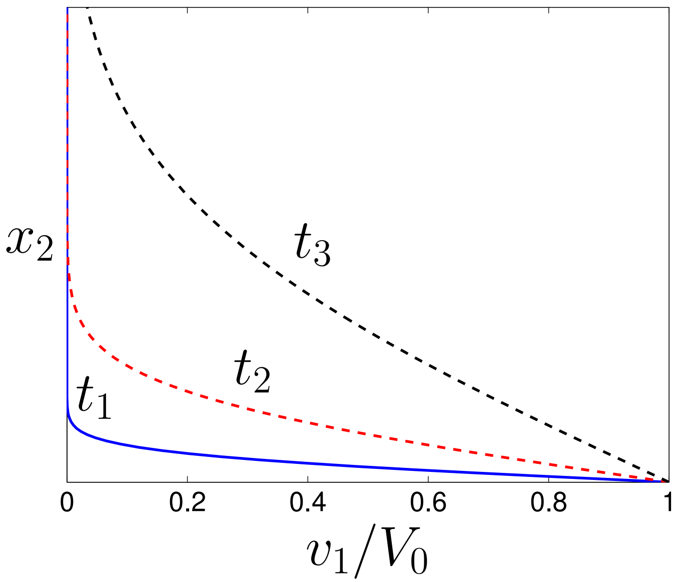
$$\rho\frac{\partial\Psi}{\partial t} + \rho v_j\frac{\partial\Psi}{\partial x_j} \quad \text{non-conservative}$$

$$\frac{\partial\rho\Psi}{\partial t} + \frac{\partial\rho v_j\Psi}{\partial x_j} \quad \text{conservative}$$

¶See Section 3.1, The Rayleigh problem



The plate moves to the right with speed V_0 for $t > 0$.



▶The v_1 velocity at three different times. $t_3 > t_2 > t_1$.

$$\rho \frac{dv_1}{dt} \equiv \rho \frac{\partial v_1}{\partial t} + \rho v_j \frac{\partial v_1}{\partial x_j} = -\frac{\partial P}{\partial x_1} + \mu \frac{\partial^2 v_1}{\partial x_j \partial x_j} + \rho f_1$$

▶Simplificatons: ▶ $\frac{\partial v_1}{\partial x_1} = \frac{\partial v_3}{\partial x_3} = 0 \quad \Rightarrow \frac{\partial v_2}{\partial x_2} = 0 \quad \Rightarrow v_2 = C_1$ b.c. $\Rightarrow v_2 \equiv 0$

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_2^2}, \quad \nu = \frac{\mu}{\rho}$$

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_2^2}$$

►Similarity solution: the number of independent variables is reduced by one: from two(x_2 and t) to one (η).

$$\eta = \frac{x_2}{2\sqrt{\nu t}}$$

$$\frac{\partial v_1}{\partial t} = \frac{dv_1}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{x_2 t^{-3/2}}{4\sqrt{\nu}}\frac{dv_1}{d\eta} = -\frac{1}{2}\frac{\eta}{t}\frac{dv_1}{d\eta}$$

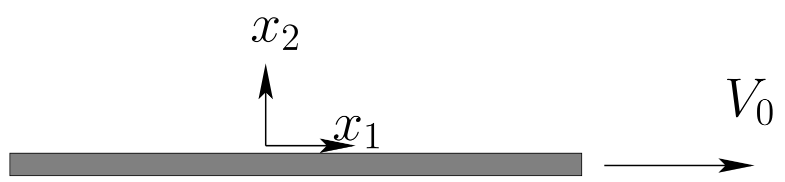
$$\frac{\partial v_1}{\partial x_2} = \frac{dv_1}{d\eta}\frac{\partial \eta}{\partial x_2} = \frac{1}{2\sqrt{\nu t}}\frac{dv_1}{d\eta}$$

$$\frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{1}{2\sqrt{\nu t}}\frac{dv_1}{d\eta}\right) = \frac{1}{2\sqrt{\nu t}}\frac{\partial}{\partial x_2}\left(\frac{dv_1}{d\eta}\right)$$

$$= \frac{1}{2\sqrt{\nu t}}\frac{dv_1}{d\eta}\frac{\partial}{\partial \eta}\left(\frac{dv_1}{d\eta}\right) = \frac{1}{4\nu t}\frac{d^2 v_1}{d\eta^2}$$

►We get

$$f = \frac{v_1}{V_0}, \quad \frac{d^2 f}{d\eta^2} + 2\eta\frac{df}{d\eta} = 0$$



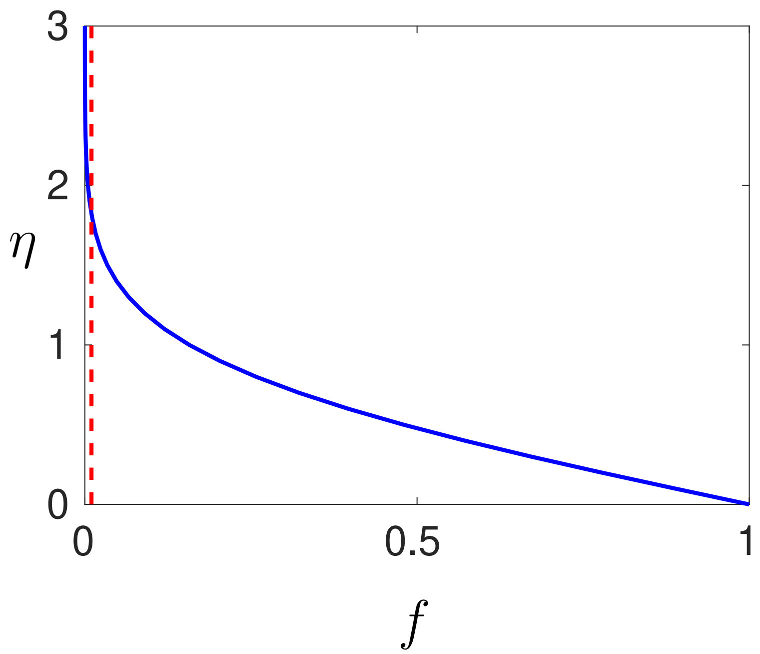
Boundary conditions

$$\eta = \frac{x_2}{2\sqrt{\nu t}}, \quad f = \frac{v_1}{V_0}, \quad \frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0$$

$$\begin{aligned} v_1(x_2, t=0) &= 0 \quad \Rightarrow \quad f(\eta \to \infty) = 0 \\ v_1(x_2 = 0, t) &= V_0 \quad \Rightarrow \quad f(\eta = 0) = 1 \\ v_1(x_2 \to \infty, t) &= 0 \quad \Rightarrow \quad f(\eta \to \infty) = 0 \end{aligned}$$

▶The solution reads

$$f(\eta) = 1 - \mathrm{erf}(\eta), \quad \eta = \frac{x_2}{2\sqrt{\nu t}}, \quad f = \frac{v_1}{V_0} \tag{31.4}$$



The velocity, $f = v_1/V_0$, given by Eq. 31.4.

▶$v_1 = 0.99V_0$, usual boundary layer. ▶Boundary layer thickness: $f = v_1/V_0 = 0.01$ (dashed line)

▶The figure above gives (for $f = 0.01$) $\eta = 1.8$ so that

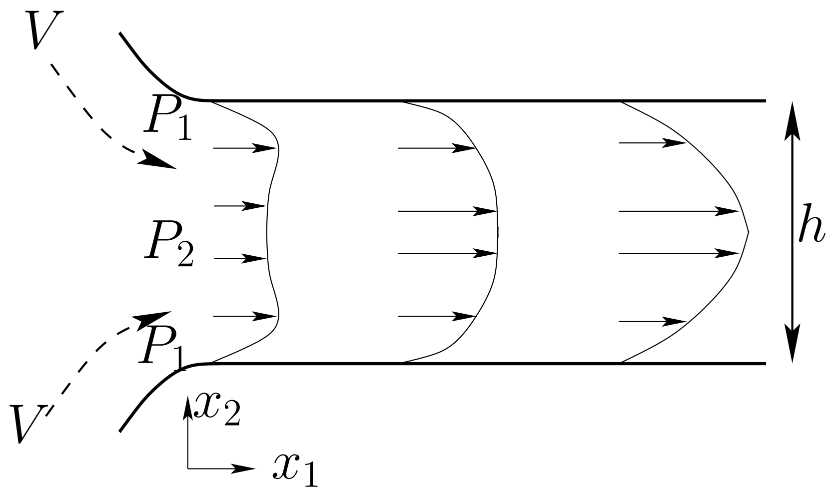
$$\eta = 1.8 = \frac{\delta}{2\sqrt{\nu t}} \quad \Rightarrow \quad \delta = 3.6\sqrt{\nu t}, \quad t = \frac{\delta^2}{3.6^2\nu}$$

$$\begin{aligned}\delta &= 10.8cm \quad \text{air, after 10 minutes}\\ \delta &= 2.8cm \quad \text{water, after 10 minutes}\\ \delta &= 1m \quad \text{air} \quad \Rightarrow \quad \text{84 minutes}\end{aligned}$$

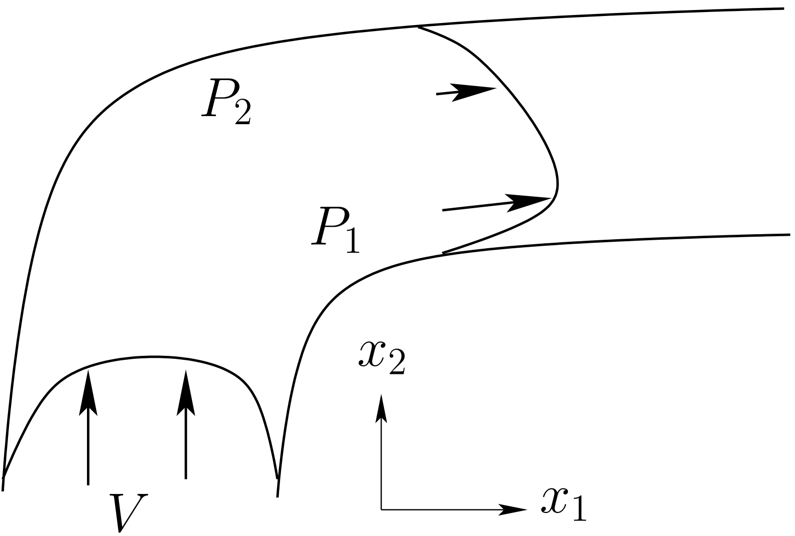
Lecture 3

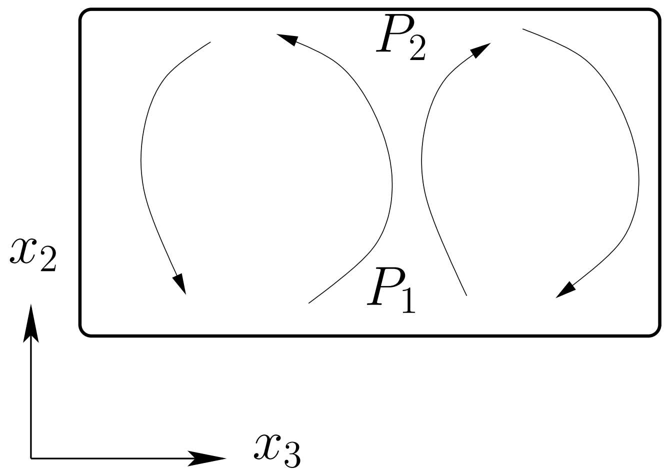
¶See Section 3.2.1, Curved plates

▶The inlet part of a channel. $P_2 > P_1$



▶Flow in a channel bend. $P_2 > P_1$





¶See Section 3.2.2, Flat plates

▶Fully developed incompressible flow in a channel. 2D and steady. o ▶$0 = \frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_1} = \frac{\partial v_3}{\partial x_3} = v_3.$

$$\frac{\partial v_i}{\partial x_i} = 0 \quad \Rightarrow \frac{\partial v_2}{\partial x_2} = 0 \quad \Rightarrow \quad v_2 = C_1(x_1) \quad \Rightarrow v_2 \equiv 0$$

▶The Navier-Stokes for v_1 ($g_i = (0, -g, 0)$)

$$\frac{dv_1}{dt} \equiv \frac{\partial v_1}{\partial t} + v_j \frac{\partial v_1}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_1} + \nu \frac{\partial^2 v_1}{\partial x_j \partial x_j} + f_1$$

$$\cancelto{0}{\frac{\partial v_1}{\partial t}} + v_1 \cancelto{0}{\frac{\partial v_1}{\partial x_1}} + v_2 \cancelto{0}{\frac{\partial v_1}{\partial x_2}} = -\frac{\partial P}{\partial x_1} + \mu \left(\cancelto{0}{\frac{\partial^2 v_1}{\partial x_1^2}} + \frac{\partial^2 v_1}{\partial x_2^2} \right) + \cancelto{0}{f_1}$$

$$\Rightarrow \mu \frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial P}{\partial x_1} \tag{32.1}$$

▶The Navier-Stokes for v_2 gives

$$\frac{\partial v_2}{\partial t} + v_j \frac{\partial v_2}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_2} + \nu \frac{\partial^2 v_2}{\partial x_j \partial x_j} + f_2$$

$$\cancelto{0}{\frac{\partial v_2}{\partial t}} + v_1 \cancelto{0}{\frac{\partial v_2}{\partial x_1}} + v_2 \cancelto{0}{\frac{\partial v_2}{\partial x_2}} = -\frac{\partial P}{\partial x_2} + \mu \left(\cancelto{0}{\frac{\partial^2 v_2}{\partial x_1^2}} + \cancelto{0}{\frac{\partial^2 v_2}{\partial x_2^2}} \right) - \rho g \quad \Rightarrow \quad 0 = -\frac{\partial P}{\partial x_2} - \rho g$$

$$0 = -\frac{\partial P}{\partial x_2} - \rho g \quad \Rightarrow P = -\rho g x_2 + C_1(x_1) = -\rho g x_2 + p(x_1)$$

$$\Rightarrow -\frac{\partial P}{\partial x_1} = -\frac{\partial p}{\partial x_1} \quad (p = p(x_1) \text{ is pressure at lower wall})$$

$$\Rightarrow \mu \frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial P}{\partial x_1} \tag{32.1}$$

►The Navier-Stokes for v_1 (replacing $\partial P/\partial x_1$ by $\partial p/\partial x_1$ in Eq. 32.1)

$$\Rightarrow \mu \underbrace{\frac{\partial^2 v_1}{\partial x_2^2}}_{f(x_2)} = \underbrace{\frac{\partial p}{\partial x_1}}_{f(x_1)} = \text{const}$$

►Integrate twice gives $v_1 = -\frac{h}{2\mu}\frac{dp}{dx_1} x_2 \left(1 - \frac{x_2}{h}\right)$

¶See Section 3.3, Two-dimensional boundary layer flow over flat plate

$$v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}=\nu\frac{\partial^2 v_1}{\partial x_2^2},\quad \frac{\partial p}{\partial x_1}=0,\quad \frac{\partial p}{\partial x_2}=0,\quad \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0$$

(note that both terms on the left side are retained)

▶Streamfunction Ψ: $\quad v_1=\frac{\partial \Psi}{\partial x_2},\quad v_2=-\frac{\partial \Psi}{\partial x_1}$ ▶The continuity equation is automatically satisfied

$$0=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=\frac{\partial^2 \Psi}{\partial x_1 \partial x_2}-\frac{\partial^2 \Psi}{\partial x_2 \partial x_1}=0$$

▶Inserting the equation above into the streamwise momentum equation

$$\frac{\partial \Psi}{\partial x_2}\frac{\partial^2 \Psi}{\partial x_1 \partial x_2}-\frac{\partial \Psi}{\partial x_1}\frac{\partial^2 \Psi}{\partial x_2^2}=\nu\frac{\partial^3 \Psi}{\partial x_2^3}$$

▶Similarity solution: $x_1, x_2 \quad \Rightarrow \quad \xi; \Psi \Rightarrow g(\xi)$.

$$\xi=\left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} x_2,\quad \Psi=\left(\nu V_{1,\infty} x_1\right)^{1/2} g$$

$$\frac{\partial \Psi}{\partial x_2}\frac{\partial^2 \Psi}{\partial x_1 \partial x_2} - \frac{\partial \Psi}{\partial x_1}\frac{\partial^2 \Psi}{\partial x_2^2} = \nu \frac{\partial^3 \Psi}{\partial x_2^3}, \quad \xi = \left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} x_2, \quad \Psi = \left(\nu V_{1,\infty} x_1\right)^{1/2} g$$

► First we need the derivatives $\partial \xi / \partial x_1$ and $\partial \xi / \partial x_2$

$$\frac{\partial \xi}{\partial x_1} = -\frac{1}{2}\left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} \frac{x_2}{x_1} = -\frac{\xi}{2x_1}, \quad \frac{\partial \xi}{\partial x_2} = \left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} = \frac{\xi}{x_2}$$

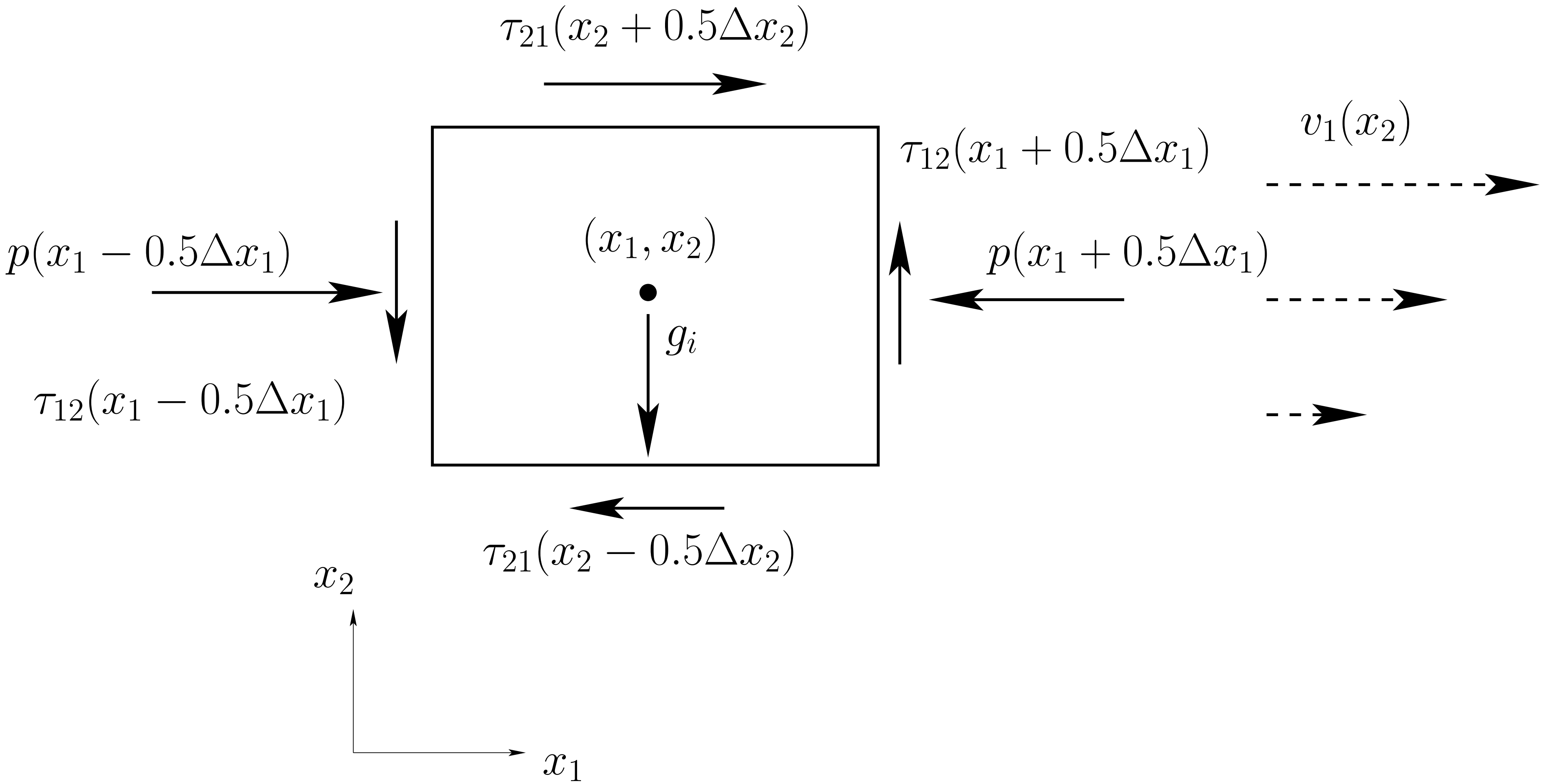
$$\begin{aligned}
\frac{\partial \Psi}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\left(\nu V_{1,\infty} x_1\right)^{1/2}\right) g + \left(\nu V_{1,\infty} x_1\right)^{1/2} \frac{dg}{d\xi}\frac{\partial \xi}{\partial x_1} \\
&= \frac{\partial}{\partial x_1}\left(\left(\nu V_{1,\infty} x_1\right)^{1/2}\right) g + \left(\nu V_{1,\infty} x_1\right)^{1/2} g' \frac{\partial \xi}{\partial x_1} \\
&= \frac{1}{2}\left(\frac{\nu V_{1,\infty}}{x_1}\right)^{1/2} g - \left(\nu V_{1,\infty} x_1\right)^{1/2} g' \frac{\xi}{2x_1} \\
\ldots &\Rightarrow -\frac{1}{2} g g'' + g''' = 0
\end{aligned}$$

► This is Blasius solution (from his PhD thesis in 1907).

► The numerical solution is given in Table 3.1.

Lecture 4

¶See Section 4.1, Vorticity and rotation



►Surface forces. $\partial\tau_{12}/\partial x_1=0$, $\partial\tau_{21}/\partial x_2>0$.

$$\frac{\partial \tau_{ji}}{\partial x_j} = \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

► The right side can be re-written as

$$\begin{aligned}\frac{\partial^2 v_i}{\partial x_j \partial x_j} &= \frac{\partial^2 v_j}{\partial x_j \partial x_i} - \left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} - \frac{\partial^2 v_i}{\partial x_j \partial x_j}\right) \\ &= \underbrace{\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)}_{=0} - \varepsilon_{inm}\varepsilon_{mjk}\frac{\partial^2 v_k}{\partial x_j \partial x_n} = -\varepsilon_{inm}\varepsilon_{mjk}\frac{\partial^2 v_k}{\partial x_j \partial x_n}\end{aligned}$$

► Let's verify that

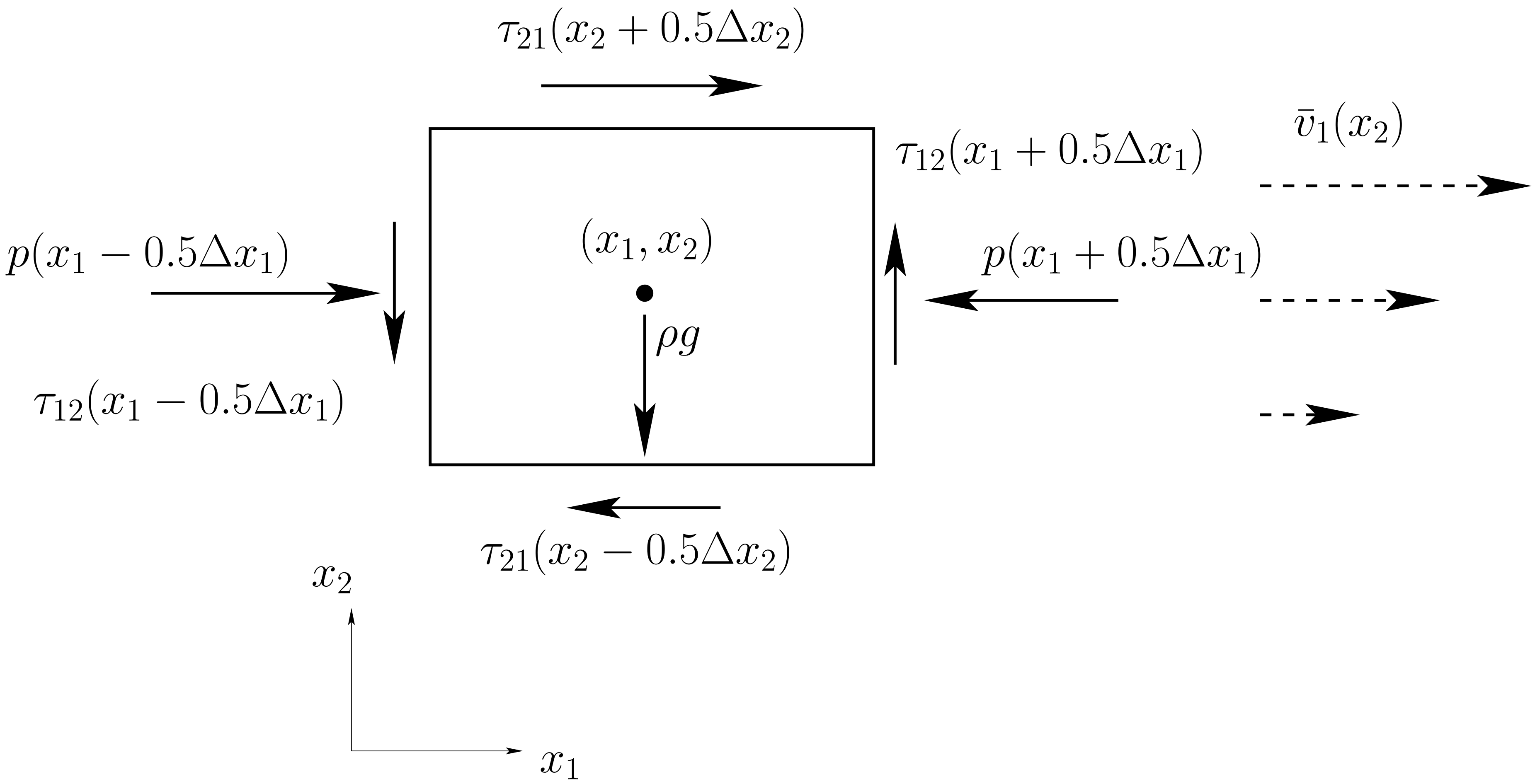
$$\left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} - \frac{\partial^2 v_i}{\partial x_j \partial x_j}\right) = \varepsilon_{inm}\varepsilon_{mjk}\frac{\partial^2 v_k}{\partial x_j \partial x_n}$$

► Use the $\varepsilon - \delta$-identity

$$\varepsilon_{inm}\varepsilon_{mjk}\frac{\partial^2 v_k}{\partial x_j \partial x_n} = (\delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj})\frac{\partial^2 v_k}{\partial x_j \partial x_n} = \frac{\partial^2 v_k}{\partial x_i \partial x_k} - \frac{\partial^2 v_i}{\partial x_j \partial x_j} \quad \text{verified!}$$

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} = -\varepsilon_{inm}\varepsilon_{mjk}\frac{\partial^2 v_k}{\partial x_j \partial x_n} = -\varepsilon_{inm}\frac{\partial}{\partial x_n}\left(\varepsilon_{mjk}\frac{\partial v_k}{\partial x_j}\right) = -\varepsilon_{inm}\frac{\partial \omega_m}{\partial x_n}$$

$$\rho\frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} + \boxed{\frac{\partial \tau_{ji}}{\partial x_j}} = -\frac{\partial P}{\partial x_i}\boxed{-\mu\varepsilon_{inm}\frac{\partial \omega_m}{\partial x_n}} \tag{33.1}$$



▶change in vorticity ⇔ change in shear stresses ▶irrotational flow ⇔ ▶potential flow ⇔

▶no change in ω_i (often $\omega_i = 0$)

▶As a first step for deriving the ω_i transport equation, let's re-write the left-side of N-S:

$$v_j \frac{\partial v_i}{\partial x_j} = v_j(S_{ij} + \Omega_{ij}) = v_j \left(S_{ij} - \frac{1}{2}\varepsilon_{ijk}\omega_k \right) \tag{33.2}$$

Inserting $S_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ and multiplying by two gives

$$2v_j \frac{\partial v_i}{\partial x_j} = v_j \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \varepsilon_{ijk} v_j \omega_k$$

$$\Rightarrow \quad v_j \frac{\partial v_i}{\partial x_j} = v_j \frac{\partial v_j}{\partial x_i} - \varepsilon_{ijk} v_j \omega_k \tag{33.3}$$

▶The first term on the right side can be written as (Trick 2)

$$v_j \frac{\partial v_j}{\partial x_i} = \frac{1}{2} \frac{\partial (v_j v_j)}{\partial x_i} \tag{33.4}$$

▶With Eqs. 33.3, 33.4 and Eq. 33.2 N-S can be written

$$\frac{\partial v_i}{\partial t} + \underbrace{\frac{\partial \frac{1}{2} v^2}{\partial x_i}}_{\text{no rotation}} - \underbrace{\varepsilon_{ijk} v_j \omega_k}_{\text{rotation}} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + f_i \tag{33.5}$$

$$\frac{\partial v_i}{\partial t}+\underbrace{\frac{\partial\frac{1}{2}v^2}{\partial x_i}}_{\text{no rotation}}-\underbrace{\varepsilon_{ijk}v_j\omega_k}_{\text{rotation}}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 v_i}{\partial x_j\partial x_j}+f_i \tag{33.5}$$

▶Now we will derive the transport equation for $\omega_p=\varepsilon_{pqi}\partial v_i/\partial x_q$. $\quad\omega_p=(\varepsilon_{pqi}\partial/\partial x_q)(v_i)$.

▶Multiply the Navier-Stokes equation by $\varepsilon_{pqi}\partial/\partial x_q$ so that

$$\varepsilon_{pqi}\frac{\partial^2 v_i}{\partial t\partial x_q}+\cancelto{0}{\varepsilon_{pqi}\frac{\partial^2\frac{1}{2}v^2}{\partial x_i\partial x_q}}-\varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j\omega_k}{\partial x_q}=\cancelto{0}{-\varepsilon_{pqi}\frac{1}{\rho}\frac{\partial^2 P}{\partial x_i\partial x_q}}+\nu\varepsilon_{pqi}\frac{\partial^3 v_i}{\partial x_j\partial x_j\partial x_q}+\cancelto{0}{\varepsilon_{pqi}\frac{\partial g_i}{\partial x_q}} \tag{33.6}$$

- Term 2 on left side: zero because product of anti-symmetric & symmetric tensor
- Term 1 in right side: zero because product of anti-symmetric & symmetric tensor
- last term: zero because g_i is constant

▶Re-write unsteady and viscous terms in Eq. 33.6:

$$\varepsilon_{pqi}\frac{\partial^2 v_i}{\partial t\partial x_q}=\frac{\partial}{\partial t}\left(\varepsilon_{pqi}\frac{\partial v_i}{\partial x_q}\right)=\frac{\partial\omega_p}{\partial t},\quad \nu\varepsilon_{pqi}\frac{\partial^3 v_i}{\partial x_j\partial x_j\partial x_q}=\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\varepsilon_{pqi}\frac{\partial v_i}{\partial x_q}\right)=\nu\frac{\partial^2\omega_p}{\partial x_j\partial x_j}$$

▶Inserted in Eq. 33.6 gives

$$\frac{\partial\omega_p}{\partial t}-\varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j\omega_k}{\partial x_q}=\nu\frac{\partial^2\omega_p}{\partial x_j\partial x_j}$$

$$\frac{\partial \omega_p}{\partial t} - \varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j\omega_k}{\partial x_q} = \nu\frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

►The second term on the left side is re-written using the ε-δ identity

$$\begin{aligned}\varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j\omega_k}{\partial x_q} &= (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\frac{\partial v_j\omega_k}{\partial x_q} = \frac{\partial v_p\omega_k}{\partial x_k} - \frac{\partial v_q\omega_p}{\partial x_q} \\ &= v_p\frac{\partial \omega_k}{\partial x_k} + \omega_k\frac{\partial v_p}{\partial x_k} - v_q\frac{\partial \omega_p}{\partial x_q} - \omega_p\underbrace{\frac{\partial v_q}{\partial x_q}}_{0}\end{aligned} \tag{33.7}$$

►Term 1: Using the definition of ω_i we find that

$$\frac{\partial \omega_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}\right) = \varepsilon_{ijk}\frac{\partial^2 v_k}{\partial x_j \partial x_i} = 0 \tag{33.8}$$

(product of symmetric and anti-symmetric tensor).

►Using Eq. 33.8, Eq. 33.7 can be written

$$\varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j\omega_k}{\partial x_q} = \omega_k\frac{\partial v_p}{\partial x_k} - v_k\frac{\partial \omega_p}{\partial x_k} \tag{33.9}$$

►Finally, we can write the transport equation for the vorticity

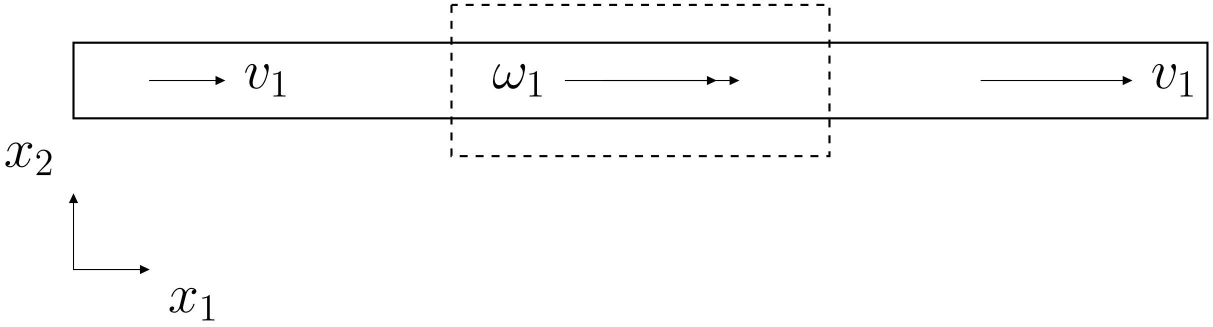
$$\frac{\partial \omega_p}{\partial t} + v_k\frac{\partial \omega_p}{\partial x_k} = \underline{\omega_k\frac{\partial v_p}{\partial x_k}} + \nu\frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

$$\frac{\partial \omega_p}{\partial t} + v_k \frac{\partial \omega_p}{\partial x_k} = \underline{\omega_k \frac{\partial v_p}{\partial x_k}} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

▶Underlined term: Vortex stretching and vortex tilting

▶Vortex stretching

$\frac{\partial v_1}{\partial x_1} > 0$

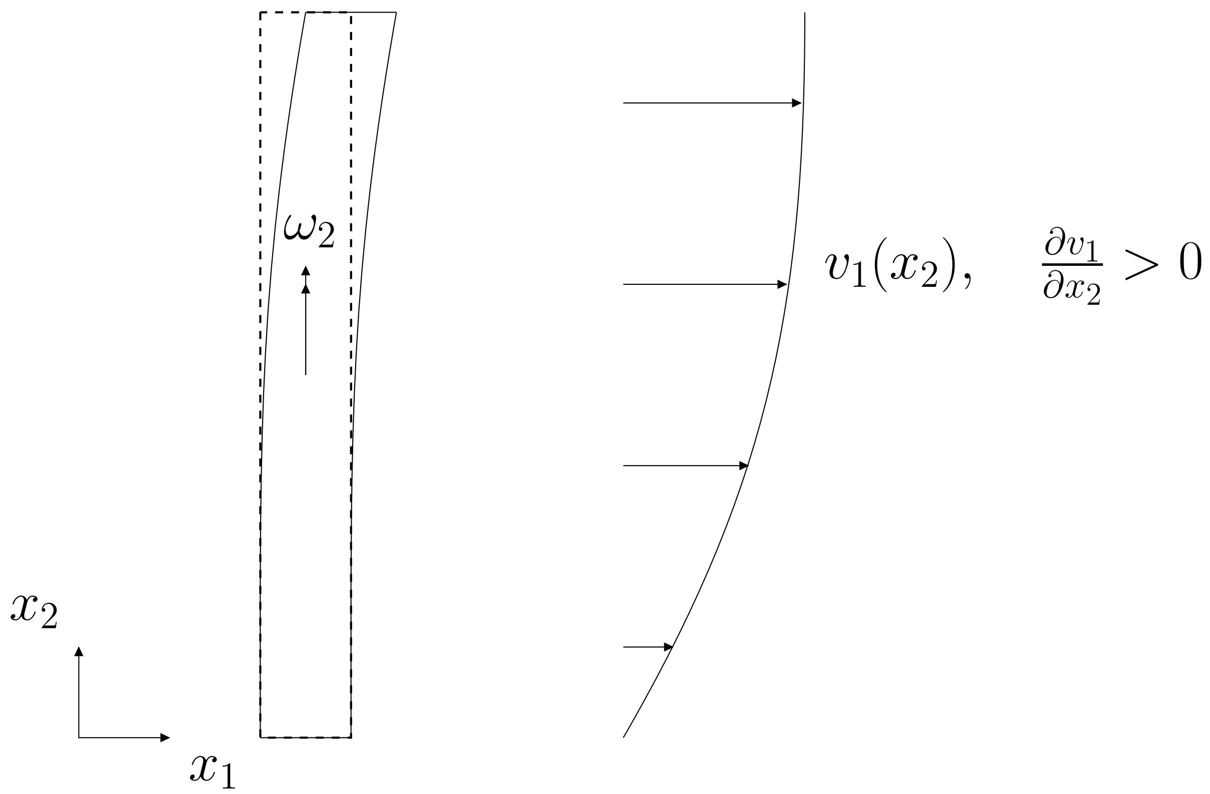


$$\omega_k \frac{\partial v_p}{\partial x_k} = \begin{cases} \omega_1 \frac{\partial v_1}{\partial x_1} + \omega_2 \frac{\partial v_1}{\partial x_2} + \omega_3 \frac{\partial v_1}{\partial x_3}, & p = 1 \\ \omega_1 \frac{\partial v_2}{\partial x_1} + \omega_2 \frac{\partial v_2}{\partial x_2} + \omega_3 \frac{\partial v_2}{\partial x_3}, & p = 2 \\ \omega_1 \frac{\partial v_3}{\partial x_1} + \omega_2 \frac{\partial v_3}{\partial x_2} + \omega_3 \frac{\partial v_3}{\partial x_3}, & p = 3 \end{cases}$$

$\frac{\partial v_1}{\partial x_1} > 0$: the term $\omega_1 \frac{\partial v_1}{\partial x_1}$ will increase ω_1

$$\omega_k \frac{\partial v_p}{\partial x_k} = \begin{cases} \omega_1 \frac{\partial v_1}{\partial x_1} + \omega_2 \frac{\partial v_1}{\partial x_2} + \omega_3 \frac{\partial v_1}{\partial x_3}, & p = 1 \\ \omega_1 \frac{\partial v_2}{\partial x_1} + \omega_2 \frac{\partial v_2}{\partial x_2} + \omega_3 \frac{\partial v_2}{\partial x_3}, & p = 2 \\ \omega_1 \frac{\partial v_3}{\partial x_1} + \omega_2 \frac{\partial v_3}{\partial x_2} + \omega_3 \frac{\partial v_3}{\partial x_3}, & p = 3 \end{cases}$$

- Vortex tilting/deflection
- Assume $\frac{\partial v_1}{\partial x_2} > 0$



- $\frac{\partial v_1}{\partial x_2} > 0$: the term $\omega_2 \frac{\partial v_1}{\partial x_2}$ will increase ω_1

¶See Section 4.3, The vorticity transport equation in two dimensions

▶3D flow: $\frac{\partial \omega_p}{\partial t} + v_k \frac{\partial \omega_p}{\partial x_k} = \underline{\omega_k \frac{\partial v_p}{\partial x_k}} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$

▶2D flow: $v_i = (v_1, v_2, 0)$, $\quad \frac{\partial}{\partial x_3} = v_3 = 0$

▶The vorticity: $\omega_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$

$$\omega_1 = \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} = 0, \quad \omega_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} = 0, \quad \omega_3 \neq 0$$

▶Hence, the vortex stretching/tilting term $\omega_k \frac{\partial v_p}{\partial x_k} = \omega_3 \frac{\partial v_p}{\partial x_3} = 0$

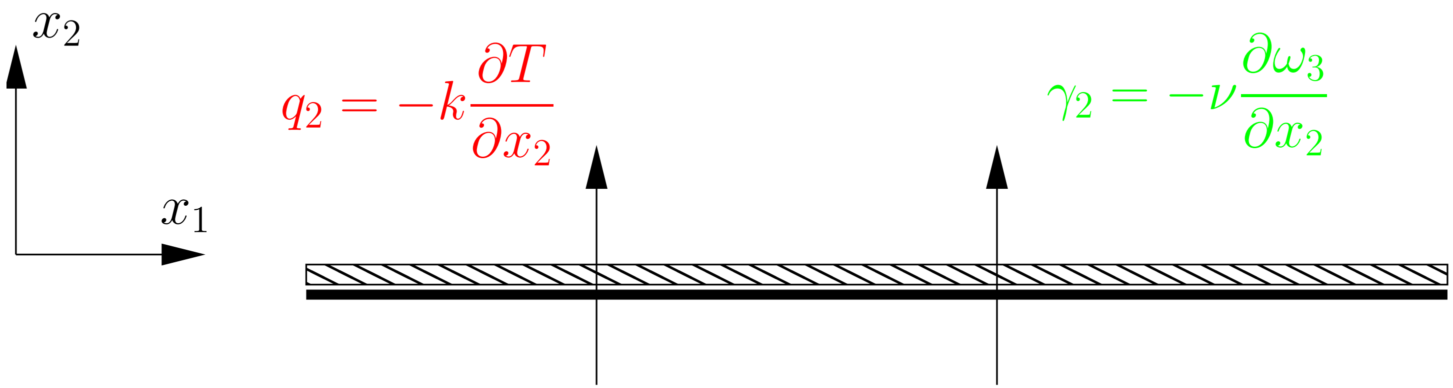
▶The 2D ω_3 equation reads

$$\frac{\partial \omega_3}{\partial t} + v_k \frac{\partial \omega_3}{\partial x_k} = \nu \frac{\partial^2 \omega_3}{\partial x_j \partial x_j}$$

$$\frac{\partial \omega_3}{\partial t} + v_k \frac{\partial \omega_3}{\partial x_k} = \nu \frac{\partial^2 \omega_3}{\partial x_j \partial x_j}$$

▶ Consider fully developed channel flow

▶ **heat conduction:** $0 = k\frac{\partial^2 T}{\partial x_2^2}$ ▶ **vorticity diffusion** $0 = \nu\frac{\partial^2 \omega_3}{\partial x_2^2}$



▶ Temperature: $q_2 = 0$ $\Rightarrow$ no temperature (increase)

▶ Vorticity: $\gamma_2 = 0$ $\Rightarrow$ no vorticity (increase)

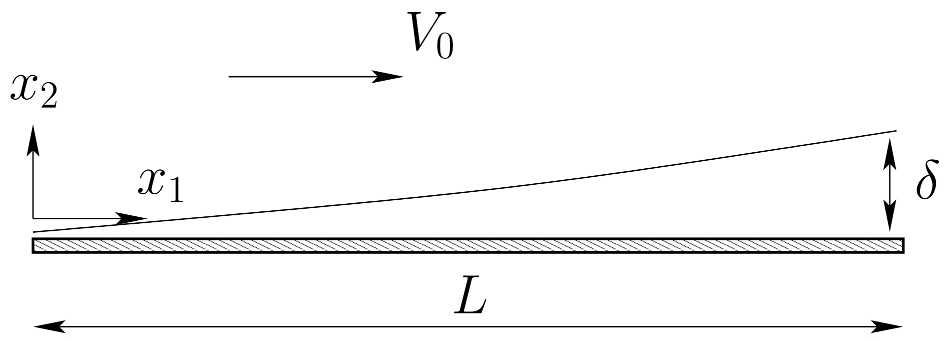
– In a boundary layer this is true because $\frac{\partial p}{\partial x_1} = 0 \quad \Leftrightarrow \quad \left(\frac{\partial^2 v_1}{\partial^2 x_2^2}\right)_{wall} = 0$

– for channel flow, $\frac{\partial p}{\partial x_1} \neq 0 \quad \Rightarrow \quad \gamma_2 \neq 0$

¶See Section 4.3.1, Boundary layer thickness from the Rayleigh problem

▶Rayleigh problem: $\delta(t) = 3.6\sqrt{\nu t}$ was presented for the v_1 equation.
Also for the concentration/temperature equation.

▶Here we will use it for the vorticity equation.



▶Boundary layer thickness: $\delta \propto \sqrt{\nu t} = \sqrt{\dfrac{L\nu}{V_0}} = L\sqrt{\dfrac{\nu}{V_0 L}} \quad \Rightarrow \quad \dfrac{\delta}{L} \propto \sqrt{\dfrac{1}{Re_L}}$

Lecture 5

¶See Section 4.4, Potential flow

▶Define a potential

$$v_i = \frac{\partial \Phi}{\partial x_i} \tag{34.1}$$

If it exists, the vorticity is zero

$$\omega_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} = \varepsilon_{ijk}\frac{\partial^2 \Phi}{\partial x_j \partial x_k} = 0$$

The continuity eq. reads

$$0 = \frac{\partial v_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial x_i}\right) = \frac{\partial^2 \Phi}{\partial x_i \partial x_i} \tag{34.2}$$

▶Derive the Bernoulli eq. ▶The N-S reads (see Eqs. 33.1 and 33.5)

$$\frac{\partial v_i}{\partial t} + \underbrace{\frac{\partial \frac{1}{2}v^2}{\partial x_i}}_{\text{no rotation}} - \underbrace{\varepsilon_{ijk}v_j\omega_k}_{\text{rotation}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu\varepsilon_{inm}\frac{\partial \omega_m}{\partial x_n} + g_i$$

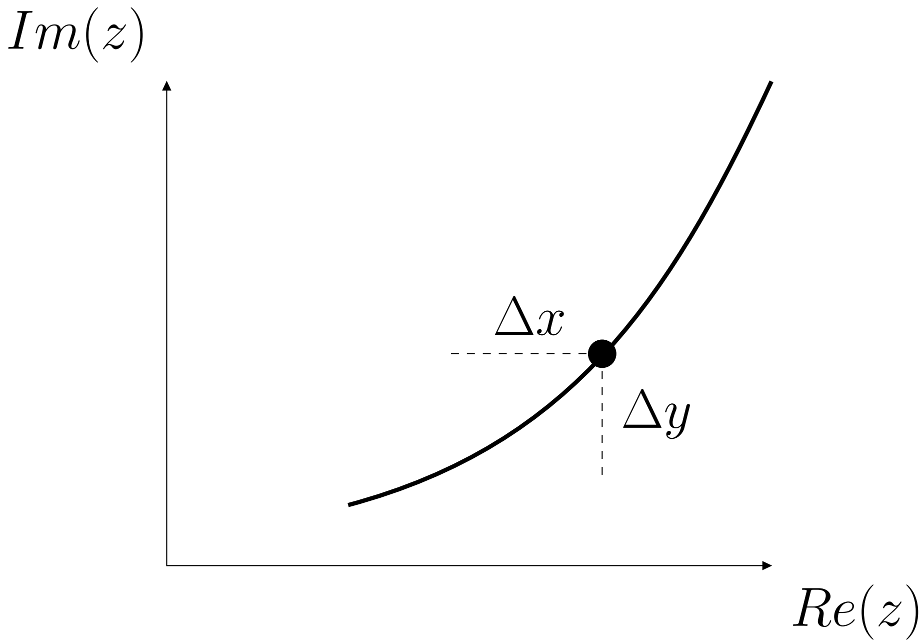
In potential flow, $\omega_i = 0$. Insert Φ (Eq. 34.1) and a gravitation potential ($g_i = -\partial \mathcal{X}/\partial x_i$)

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial t}\right) + \frac{\partial \frac{1}{2}v^2}{\partial x_i} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial \mathcal{X}}{\partial x_i} = 0$$ Integrate: $$\frac{\partial \Phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \mathcal{X} = C_1 \quad \mathcal{X} = -g_3x_3 = gh$$ **Bernoulli**

¶See Section 4.4.2, Complex variables for potential solutions of plane flows

▶Complex functions.

▶The derivative of a complex function, f, by a complex variable, z ($f = u + iv$ and $z = x + iy$) is defined only if the derivatives in the real and imaginary directions are the same, i.e.



$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{34.3}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, iy_0) - f(x_0, iy_0)}{\Delta x} = \lim_{\Delta y \to 0} \frac{f(x_0, iy_0 + i\Delta y) - f(x_0, iy_0)}{i\Delta y}. \tag{34.4}$$

▶This means that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{i \partial f}{i^2 \partial y} = -i\frac{\partial f}{\partial y} \tag{34.5}$$

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{34.5}$$

►Inserting $f = u + iv$ in Eq. 34.5 gives

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} - i^2\frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

We get

$$\begin{aligned} \text{real part:} \quad \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \text{imaginary part:} \quad \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned} \tag{34.6}$$

They are called the *Cauchy-Riemann* equations.

►A complex function in polar coordinates: $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$\text{Cauchy-Riemann} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{34.6}$$

▶Fluid dynamics: define a complex potential $f = \Phi + i\Psi$ where Ψ is the streamfunction: recall

$$v_1 = \frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad v_2 = -\frac{\partial \Psi}{\partial x} = \frac{\partial \Phi}{\partial y} \tag{34.7}$$

▶We want f to be differentiable: hence Eq. 34.6 must hold (replace u and v with Φ and Ψ)

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \quad \text{which is satisfied, see Eq. 34.7} \tag{34.8}$$

▶Φ satisfies Laplace eq. (see Eq. 34.2). Since $\omega_3 = 0$ (potential flow), this applies also for Ψ

$$\frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2} = -\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} = -\omega_3 = 0 \tag{34.9}$$

▶Φ and Ψ satisfy Laplace equation. ▶Hence, f also satisfies Laplace equation

▶Furthermore, f has a physical meaning in fluid dynamics: it describes potential flow

▶namely, continuity equation, i.e. $\frac{\partial^2 \Phi}{\partial x_i \partial x_i} = 0$ ▶and $\omega_3 = 0$, i.e. $\frac{\partial^2 \Psi}{\partial x_i \partial x_i} = 0$

¶See Section 4.4.3, $f \propto z^n$

1. Now we "guess"/dream up a complex function $f = \Phi + i\Psi$
2. then we check if it satisfies the Laplace equation (i.e. the continuity equation, 34.2 and that the flow is inviscid, $\omega_3 = 0$, Eq. 34.9)
3. then we find out if f corresponds to a meaningful fluid flow situation

►We guess $f = C_1 z^n = C_1 r^n e^{in\theta} = C_1 r^n(\cos(n\theta) + i\sin(n\theta))$

Check that it satisfies Laplace equation

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

►It does, see eBook

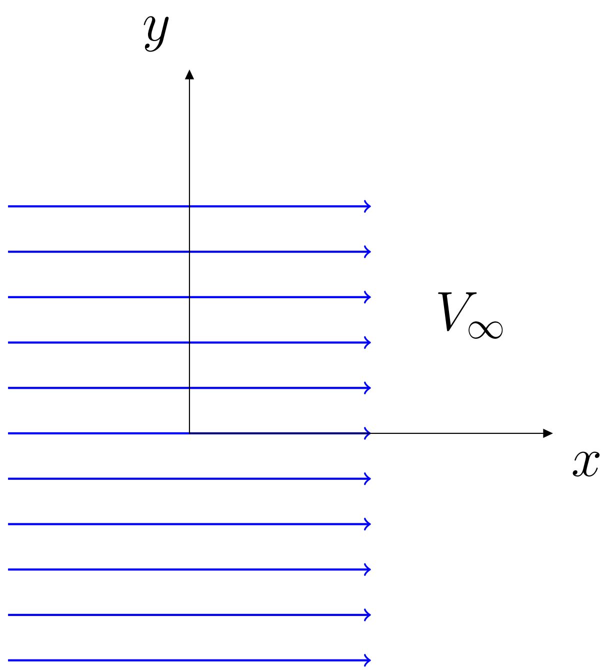
¶See Section 4.4.3.1, *Parallel flow*

▶Our guess $f = C_1 r^n(\cos(n\theta) + i\sin(n\theta))$

▶Parallel flow, $n = 1$. $f = C_1 z = V_\infty z = V_\infty(x + iy)$

The streamfunction, Ψ, is the imaginary part of f, i.e. $\Psi = V_\infty y$ which gives

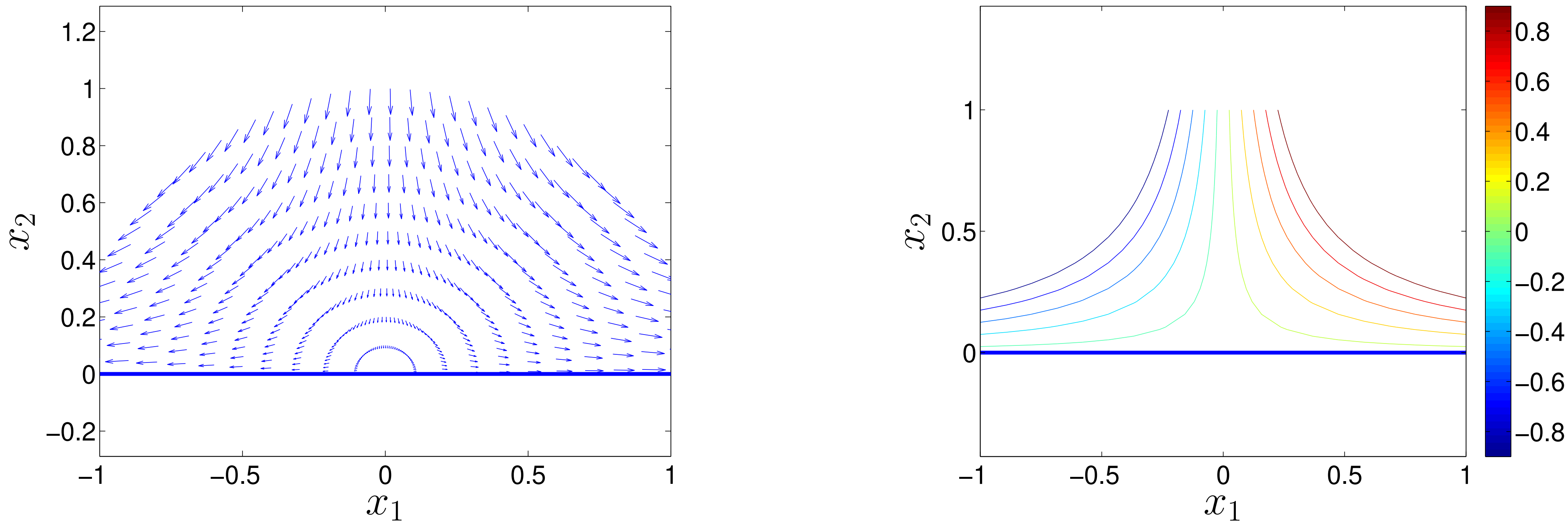
$$v_1 = \frac{\partial \Psi}{\partial y} = V_\infty, \quad v_2 = -\frac{\partial \Psi}{\partial x} = 0$$



¶See Section 4.4.3.2, *Stagnation flow*

- Our guess with $n=2$: $f = r^2(\cos(2\theta) + i\sin(2\theta))$
- The streamfunction, Ψ, is the imaginary part of f, i.e. $\Psi = r^2\sin(2\theta)$
- The polar and Cartesian velocity components are obtained as

$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = 2r\cos(2\theta), \quad v_\theta = -\frac{\partial\Psi}{\partial r} = -2r\sin(2\theta)$$
$$v_1 = 2x_1, \quad v_2 = -2x_2$$



Vector plot. Streamlines. Colors represent Ψ.

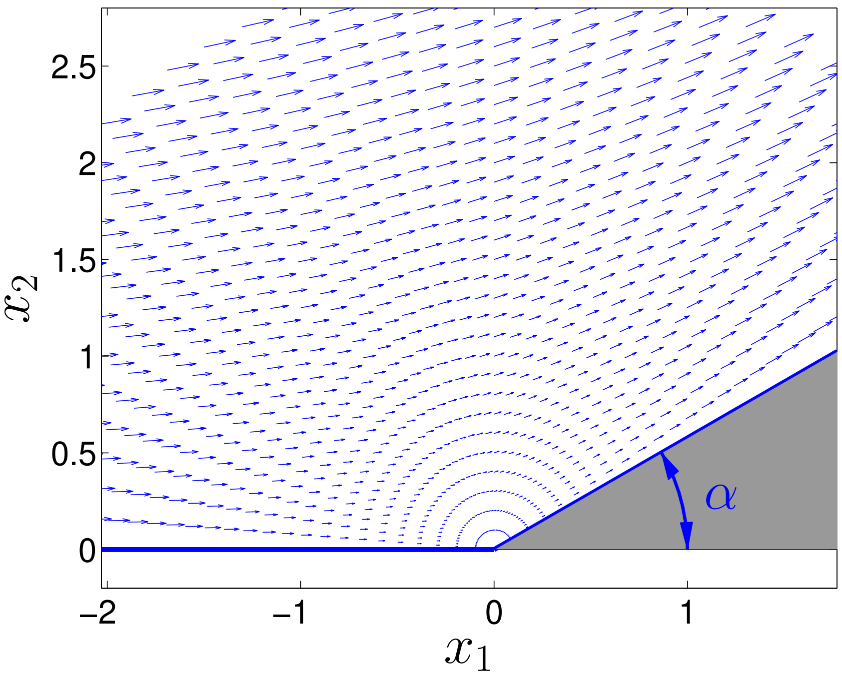
- The flow impinges at the wall at $x_2 = 0$ where $v_2 = 0$.
- Ψ=0 along the symmetry line, $x_1 = 0$, and it is negative to the left and positive to the right.

¶See Section 4.4.3.3, *Flow over a wedge and flow in a concave corner*

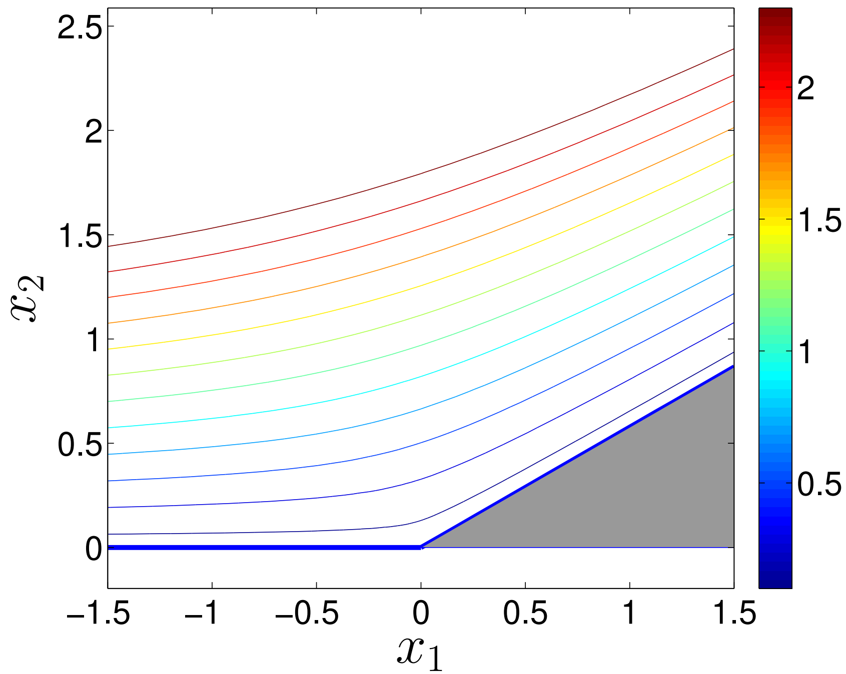
▶Our guess with $n = 6/5$: $f = r^{6/5}(\cos(6\theta/5) + i\sin(6\theta/5))$

▶The streamfunction, Ψ, is the imaginary part of f, i.e. $\Psi = r^{6/5}\sin(6\theta/5)$

$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{6}{5} r\cos(6\theta/5), \quad v_\theta = -\frac{\partial\Psi}{\partial r} = -\frac{6}{5} r\sin(6\theta/5)$$



Vector plot.



Streamlines. Colors represent Ψ.

▶The lower boundary for $x_1 < 0$ can either be a wall (concave corner) or symmetry line (wedge).

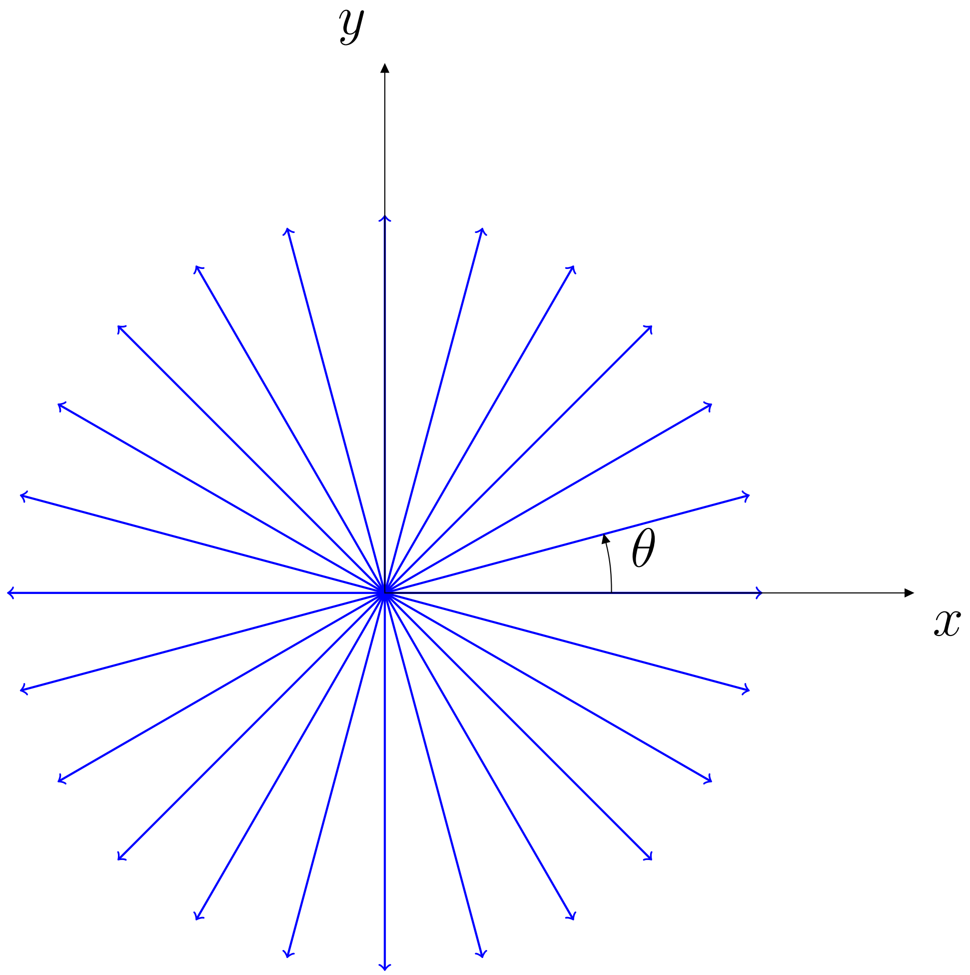
▶Ψ=0 along the lower boundary. The angle, α, in the figure above is given by $\alpha = \frac{(n-1)\pi}{n} = \frac{\pi}{6}$

¶See Section 4.4.4, Analytical solutions for a line source

$$f = \frac{\dot{m}}{2\pi} \ln z = \frac{\dot{m}}{2\pi} \ln \left(r e^{i\theta} \right) = \frac{\dot{m}}{2\pi} \left(\ln r + \ln \left(e^{i\theta} \right) \right) = \frac{\dot{m}}{2\pi} (\ln r + i\theta)$$

Check that it satisfies Laplace equation (it does, see eBook)

$$\Psi = \frac{\dot{m}\theta}{2\pi}, \quad v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\dot{m}}{2\pi r}, \quad v_\theta = -\frac{\partial \Psi}{\partial r} = 0$$

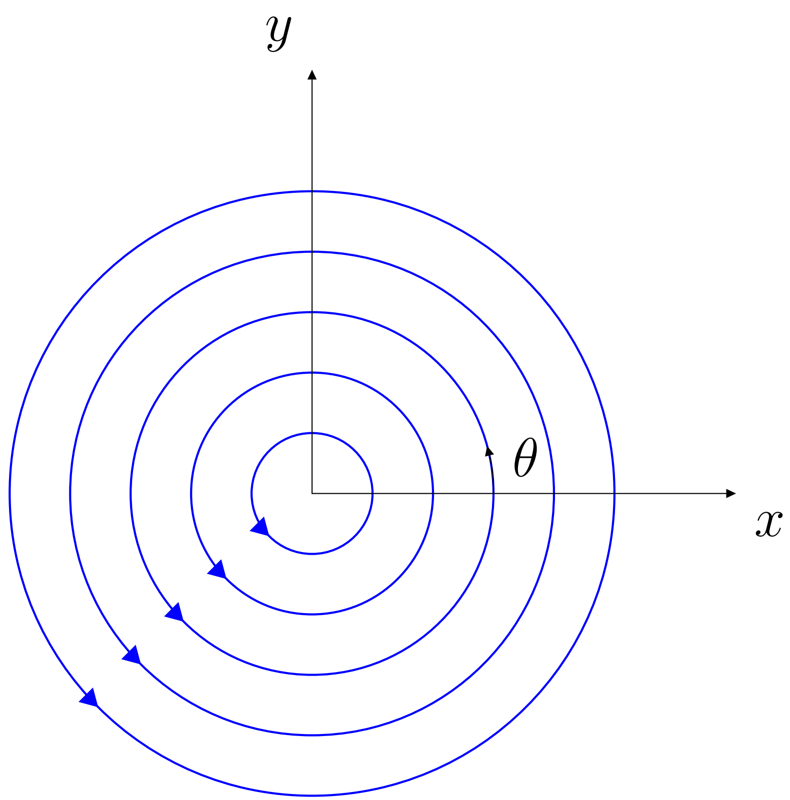


¶See Section 4.4.5, Analytical solutions for a vortex line

$$
\begin{aligned}
f &= -i\frac{\Gamma}{2\pi}\ln z = -i\frac{\Gamma}{2\pi}\ln\left(re^{i\theta}\right) = -i\frac{\Gamma}{2\pi}\left(\ln r + \ln\left(e^{i\theta}\right)\right) = \frac{\Gamma}{2\pi}\left(-i\ln r - i\ln\left(e^{i\theta}\right)\right) \\
&= \frac{\Gamma}{2\pi}\left(-i\ln r - i^2\theta\right) = \frac{\Gamma}{2\pi}\left(-i\ln r + \theta\right)
\end{aligned}
$$

►Check if it satisfies Laplace equation (it does, see eBook)

$$
\Psi = \frac{\Gamma}{2\pi}\ln r, \quad v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = 0, \quad v_\theta = -\frac{\partial \Psi}{\partial r} = \frac{\Gamma}{2\pi r}
$$



¶See Section 4.4.6, Analytical solutions for flow around a cylinder

▶Doublet: take a line source ($\dot{m} > 0$) a line sink ($\dot{m} < 0$) with a separation ε: let $\varepsilon \to 0$ which gives

$$f = \frac{\mu}{\pi z} = \frac{V_\infty r_0^2}{z} \quad \text{where} \quad r_0^2 = \mu/(\pi V_\infty)$$

▶Recall: $\dfrac{\partial^2 \Psi}{\partial x_i \partial x_i} = 0$ is linear $\quad \Rightarrow \Psi_{sol} = \Psi_{sol,1} + \Psi_{sol,2} + \ldots$

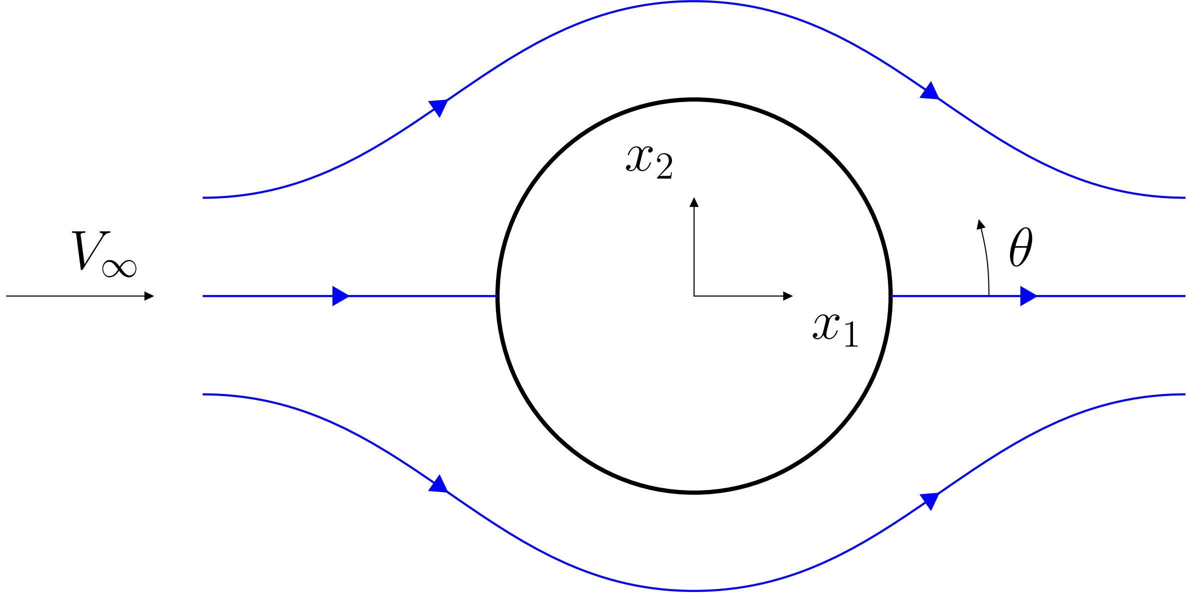
▶Add parallel flow ($f = V_\infty z$) gives cylinder flow

$$\begin{aligned} f = \frac{V_\infty r_0^2}{z} + V_\infty z &= \frac{V_\infty r_0^2}{re^{i\theta}} + V_\infty r e^{i\theta} = V_\infty \left(\frac{r_0^2}{r} e^{-i\theta} + r e^{i\theta} \right) \\ &= \frac{V_\infty r_0^2}{r} (\cos\theta - i \sin\theta) + V_\infty r \left(\cos\theta + i \sin\theta \right) \end{aligned}$$

▶The streamfunction reads (imaginary part) ▶$\Psi = V_\infty \left(r - \dfrac{r_0^2}{r} \right) \sin\theta$

and we get the velocity components

$$v_r = \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = V_\infty \left(1 - \frac{r_0^2}{r^2} \right) \cos\theta, \quad v_\theta = -\frac{\partial \Psi}{\partial r} = -V_\infty \left(1 + \frac{r_0^2}{r^2} \right) \sin\theta$$



$$v_r = V_\infty \left(1 - \frac{r_0^2}{r^2}\right) \cos\theta, \quad v_\theta = -V_\infty \left(1 + \frac{r_0^2}{r^2}\right) \sin\theta$$

$$\begin{aligned} r \to r_0 &: \quad v_r \to 0 \\ \theta = 0, r \to \infty &: \quad \Rightarrow v_r \to V_\infty \\ \theta = \pi, r \to \infty &: \quad \Rightarrow v_r \to -V_\infty \end{aligned}$$

¶See Section 4.4.7, Analytical solutions for flow around a cylinder with circulation

►We have f for a cylinder. Now we add f for a vortex line

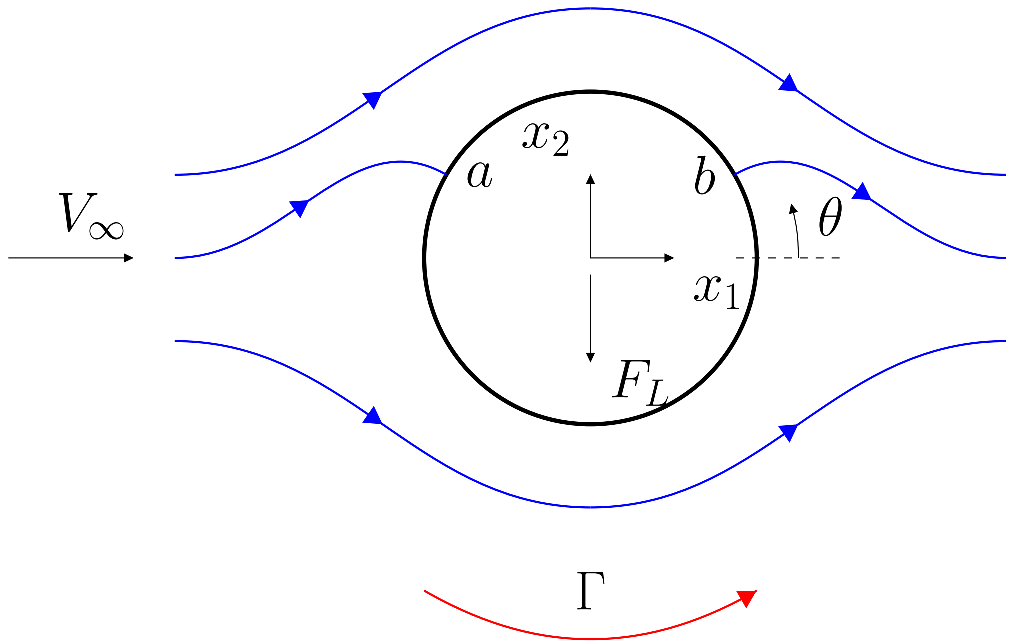
$$\begin{aligned} f &= V_\infty \left(\frac{r_0^2}{r} (\cos\theta - i\sin\theta) + r(\cos\theta + i\sin\theta) \right) - i\frac{\Gamma}{2\pi} \ln z \\ \Gamma &= \oint v_m t_m d\ell = \int_S \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} n_i dS = \int_S \omega_i n_i dS = \int_S \omega_3 dS \end{aligned}$$

►The imaginary part gives the streamfunction

$$\Psi = V_\infty \left(r - \frac{r_0^2}{r} \right) \sin\theta - \frac{\Gamma}{2\pi} \ln r$$

We get the velocity components as

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = V_\infty \left(1 - \frac{r_0^2}{r^2} \right) \cos\theta, \quad v_\theta = -\frac{\partial \Psi}{\partial r} = -V_\infty \left(1 + \frac{r_0^2}{r^2} \right) \sin\theta + \frac{\Gamma}{2\pi r}$$



$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = V_\infty\left(1-\frac{r_0^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{\partial\Psi}{\partial r} = -V_\infty\left(1+\frac{r_0^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

- The velocity at the surface, $r = r_0$: ▶ $v_{r,s} = 0, \quad v_{\theta,s} = -2V_\infty \sin\theta + \frac{\Gamma}{2\pi r_0}$
- Location of the stagnation points, i.e. where $v_{\theta,s} = 0$. We get

$$2V_\infty \sin\theta_{stag} = \frac{\Gamma}{2\pi r_0} \Rightarrow \theta_{stag} = \arcsin\left(\frac{\Gamma}{4\pi r_0 V_\infty}\right)$$

- The surface pressure is obtained from Bernoulli equation

$$\frac{V_\infty^2}{2} + \frac{P_\infty}{\rho} = \frac{v_{\theta,s}^2}{2} + \frac{p_s}{\rho} \Rightarrow p_s = P_\infty + \rho\frac{V_\infty^2 - v_{\theta,s}^2}{2}$$

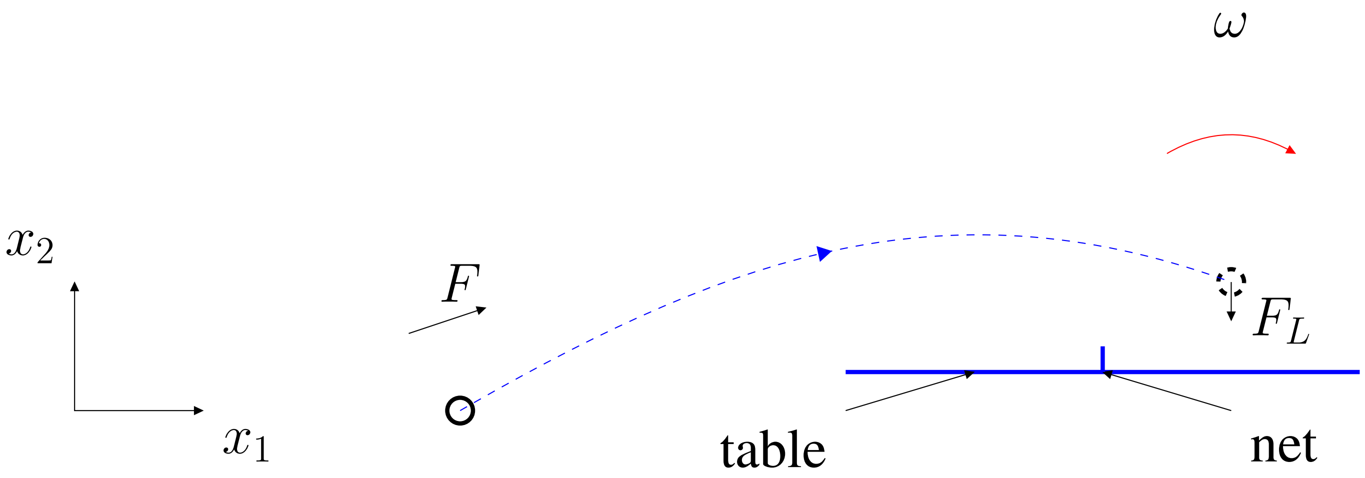
We get: ▶ $C_p = 1 - \frac{v_{\theta,s}^2}{V_\infty^2} = 1 - \left(-2\sin\theta + \frac{\Gamma}{2\pi r_0 V_\infty}\right)^2$

- Integration of C_p gives drag $F_D = 0$ and lift $F_L = -\rho V_\infty \Gamma$.

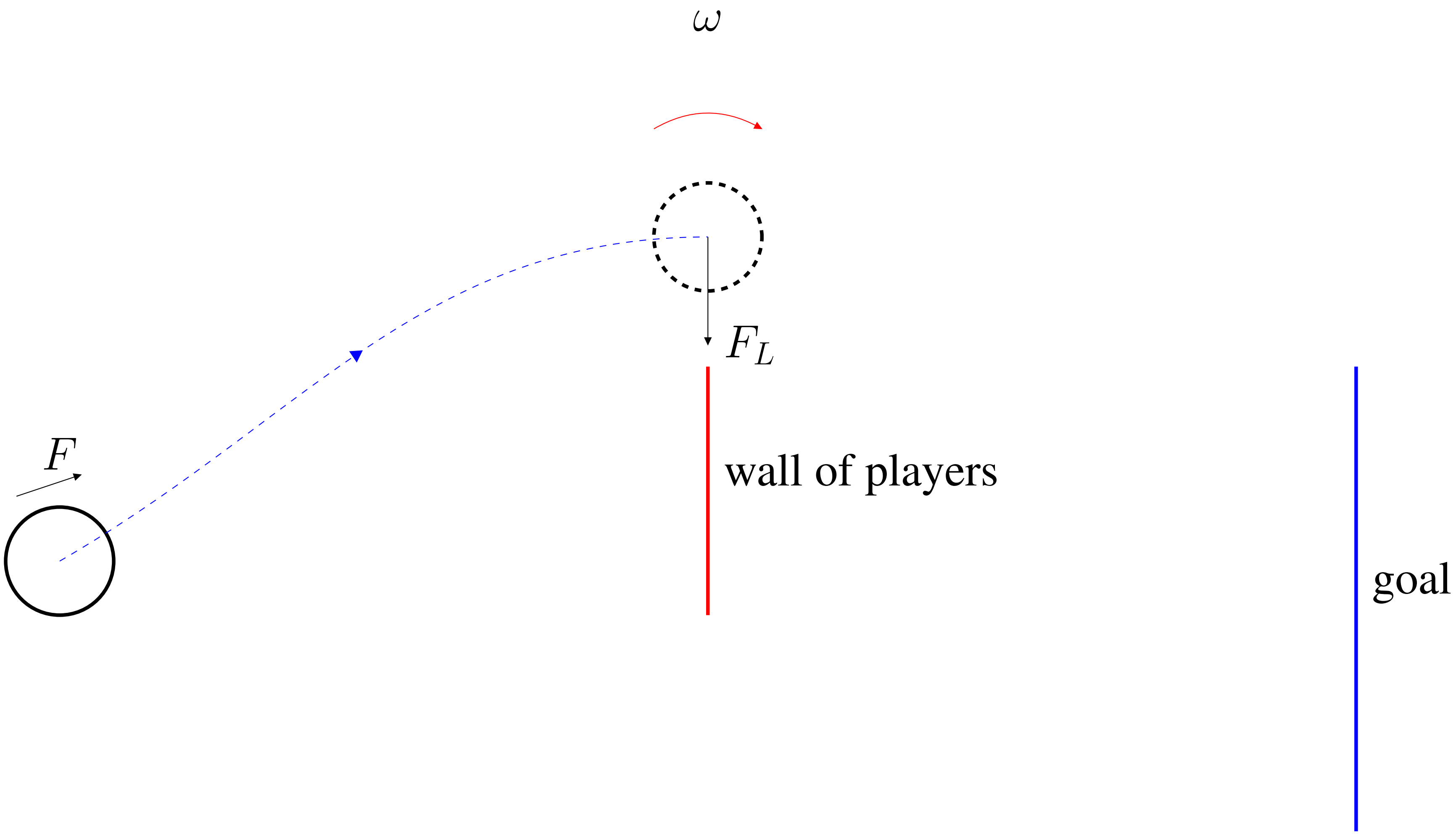
¶See Section 4.4.7.1, *The Magnus effect*

▶The Magnus' effect: three applications

▶Table tennis: loop or top-spin

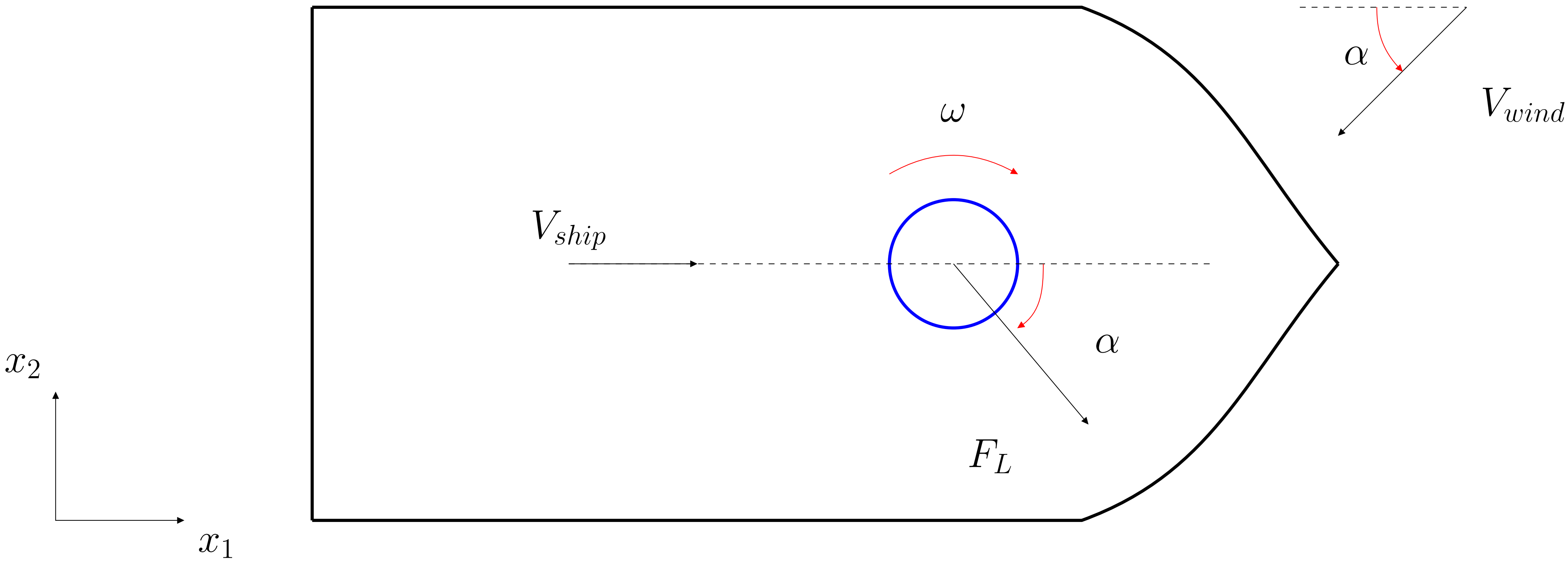


▶Footboll: Free-kick



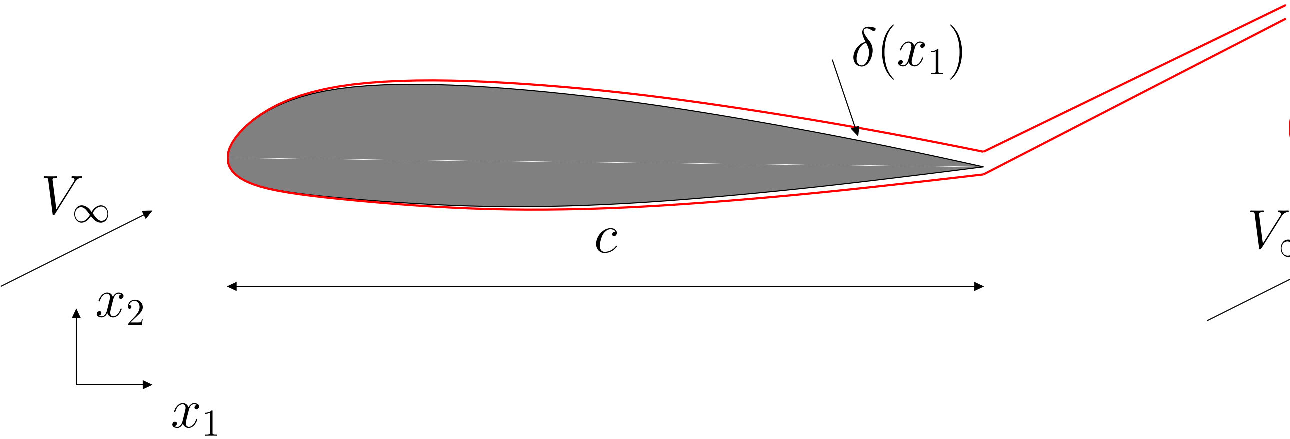
▶Course www page: ”Effect of panel shape of soccer ball on its flight characteristics”, rotation, ball trajectories for free-kicks in worldcup in football

▶Flettner rotors: the Magus effect ⇒ propulsion force of $F_L \cos(\alpha)$

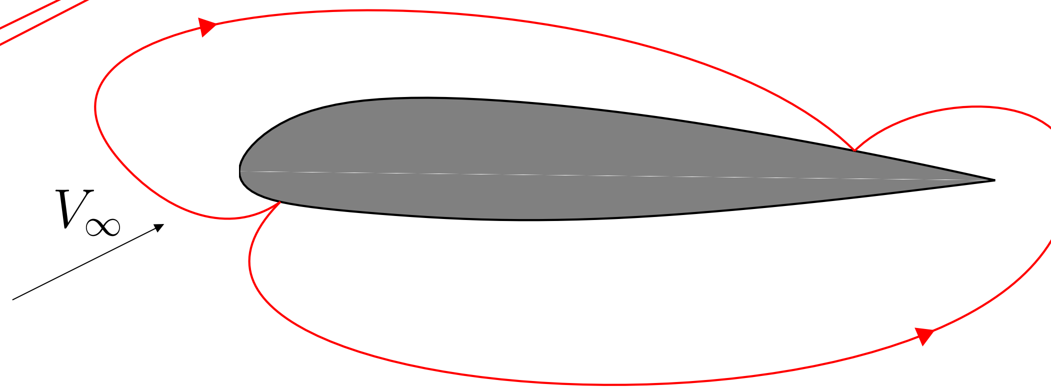


¶See Section 4.4.8, The flow around an airfoil

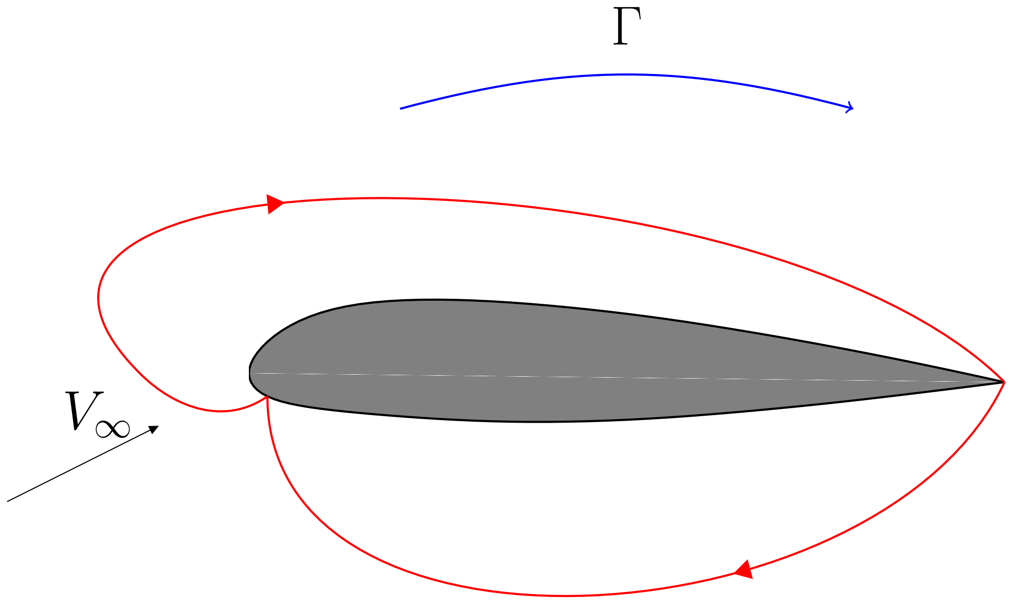
▶The boundary layers, $\delta(x_1)$, and the wake illustrated by the colored lines.



Real flow.



Unphysical rear stagnation point (suction side).

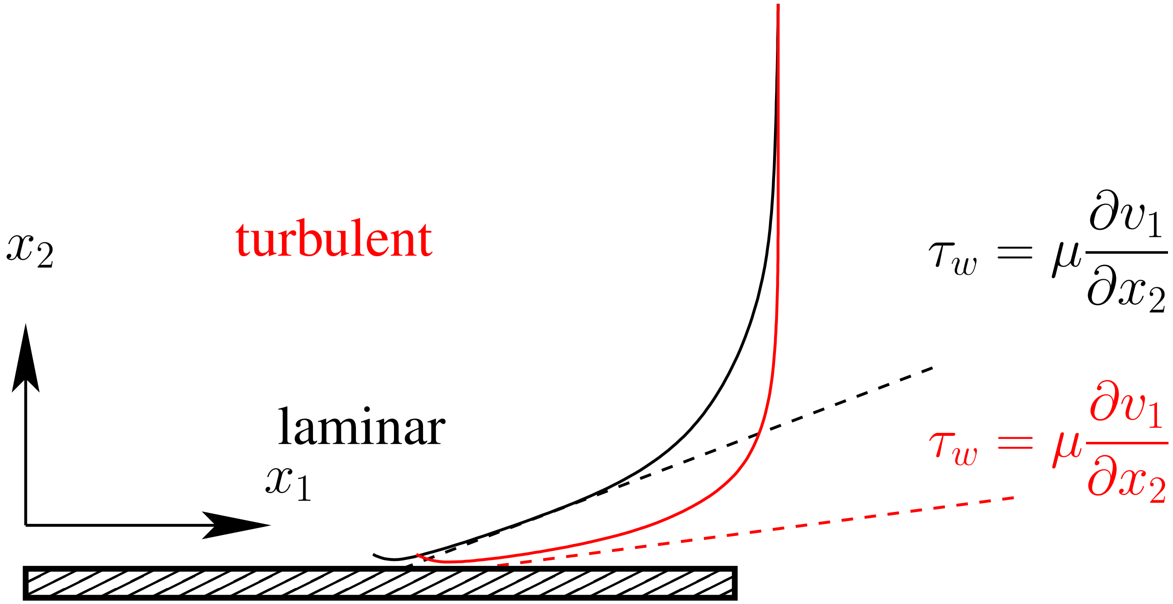


Potential flow, added Γ. Rear stagnation point at trailing edge. Lift force $F_L = -\rho V_\infty \Gamma$

Lecture 6

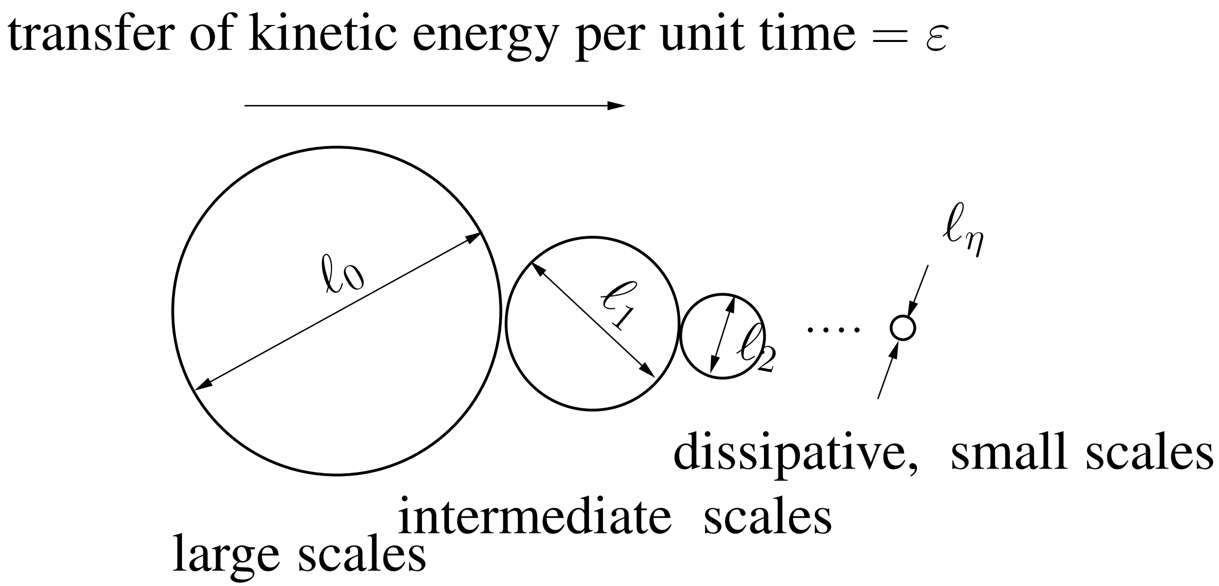
¶See Section 5, Turbulence

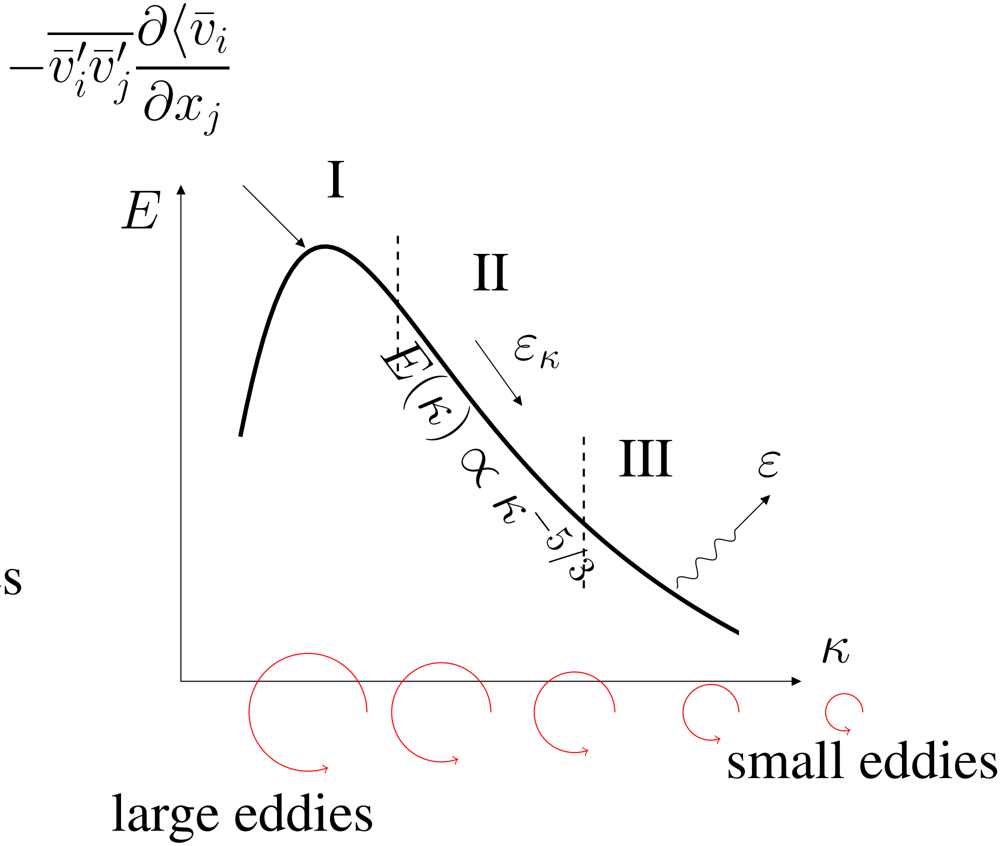
- $v_i = \bar{v}_i + v'_i$, is irregular and consists of eddies of different size
- increases diffusivity



Difference between a laminar and turbulent boundary later

- occurs at large Reynolds numbers. Pipes: $Re_D = \dfrac{VD}{\nu} \simeq 2300$; boundary layers: $Re_x = \dfrac{Vx}{\nu} \simeq 5E5$.
- is three-dimensional
- is dissipative. Kinetic energy, $v'_i v'_i/2$, in the small (dissipative) eddies $\rightarrow$ thermal energy, ΔT
- Almost all flow are turbulent. Exceptions: blood flow in your veins.





▶Dissipation $\varepsilon = \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$ ▶All dissipation (say 90%) takes place at the small scales.

▶This is called the cascade process

▶ Characterize the dissipation of kinetic energy at small scales in two relevant quantities: ε, ν

$$
\begin{aligned}
v_\eta &= \nu^a \qquad \varepsilon^b \\
[m/s] &= [m^2/s] \quad [m^2/s^3]
\end{aligned}
$$

$$
\begin{aligned}
[m] \qquad 1 &= 2a + 2b \\
[s] \qquad -1 &= -a - 3b
\end{aligned}
$$

▶ This gives the Kolmogorov scales, $a = b = 1/4$

$$
v_\eta = (\nu\varepsilon)^{1/4}, \; \ell_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \; \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}
$$

▶In a previous slide, we looked at energy spectra. It is based on Fourier series.

▶Any periodic function, f, can be expressed as a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(\kappa_n x) + b_n \sin(\kappa_n x)), \quad f = v', \quad \kappa_n = \frac{n\pi}{L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\kappa_n x)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\kappa_n x)dx$$

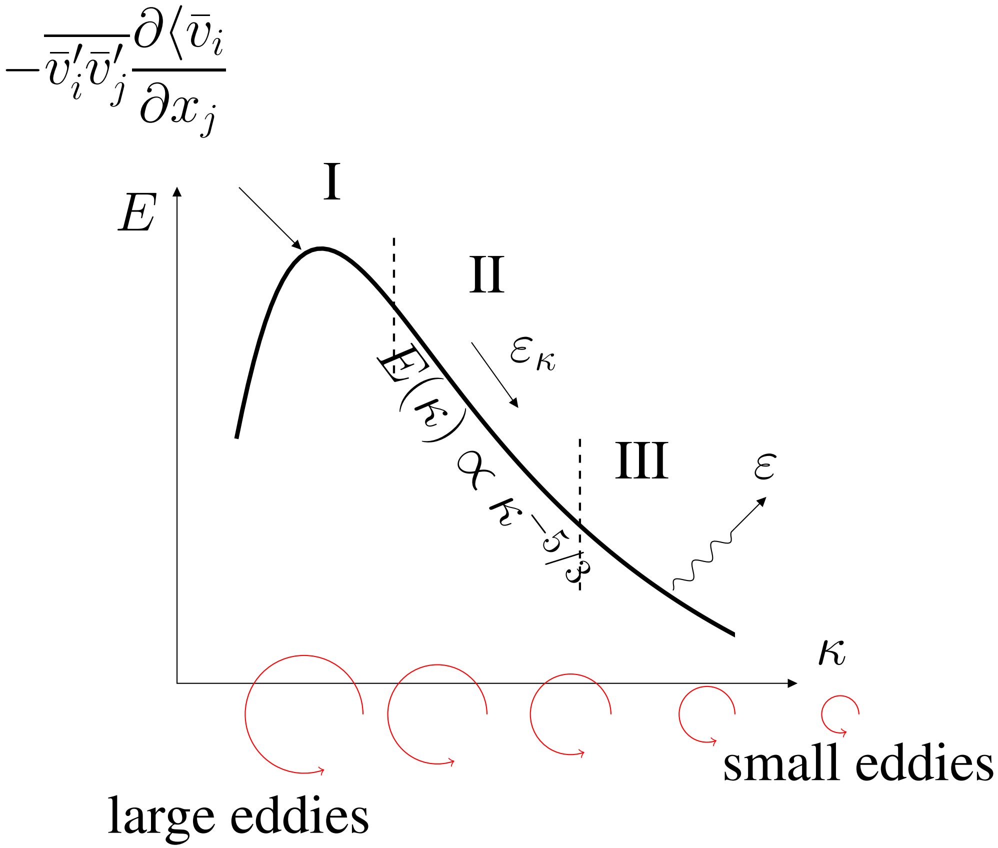
▶Parseval's formula states that the kinetic energy can be computed as

$$\int_{-L}^{L} v'^2(x)dx = \frac{L}{2}a_0^2 + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{35.1}$$

▶Hence, you can compute the kinetic energy by:

- integrating in Fourier (wavenumber) space (right-hand side)
- or integrating in physical space over all fluctuations (left-hand side)

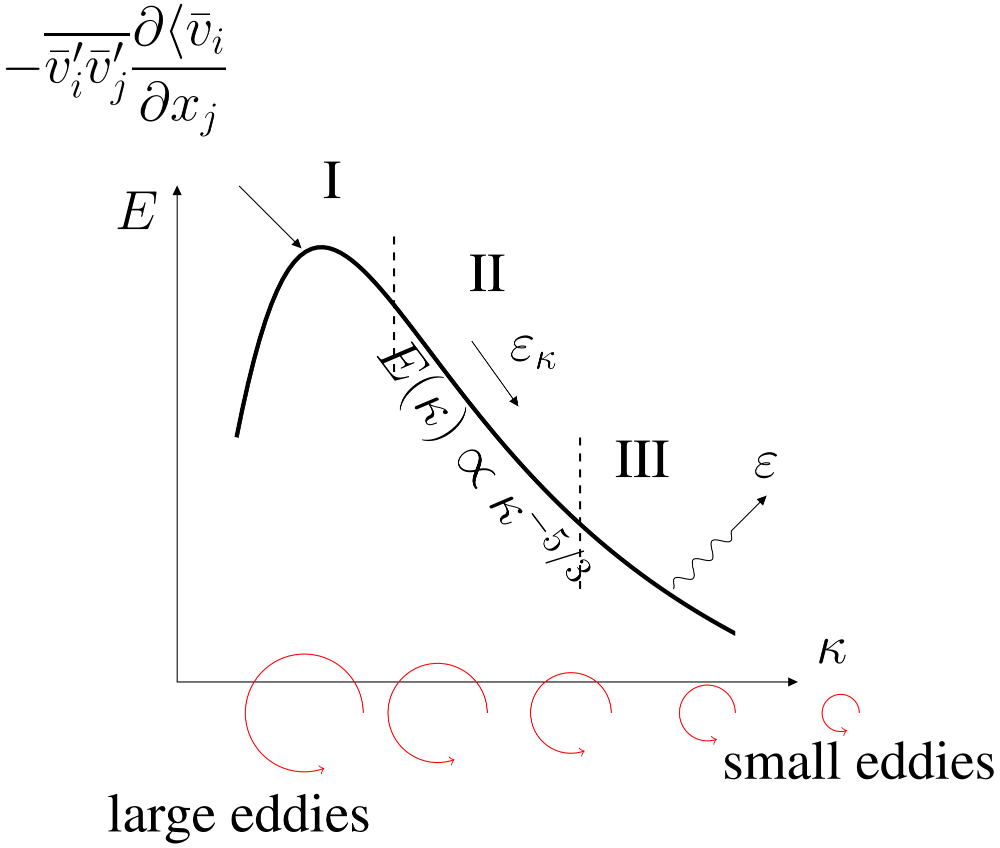
▶ Spectrum for turbulent kinetic energy, k



▶ $E(\kappa_n) \propto a_n^2 + b_n^2$, see the Fourier series on the previous slide ▶

$$k = \int_0^\infty E(\kappa)d\kappa = \sum_0^\infty E(\kappa_n)\Delta\kappa_n \tag{35.2}$$

▶ which corresponds to Parseval's formula



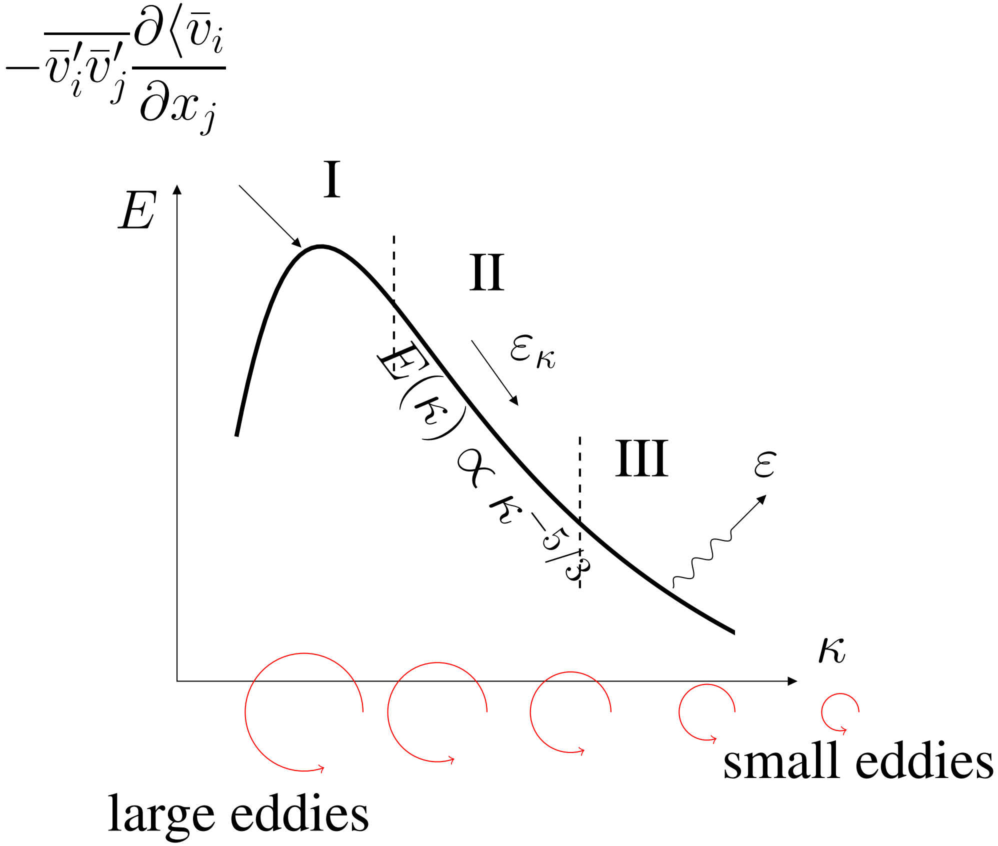
▶The turbulence spectrum is divided into three regions:

I. Large eddies carry most of the turb. kinetic energy. They extract energy from the mean flow, P^k.

II. Inertial subrange. Independent of both large eddies (mean flow) and viscosity. Isotropic eddies.

III. Dissipation range. Isotropic eddies ($\overline{v_i'v_j'} = c_1\delta_{ij}$) described by the Kolmogorov scales.

▶Turn. kinetic energy in Region II

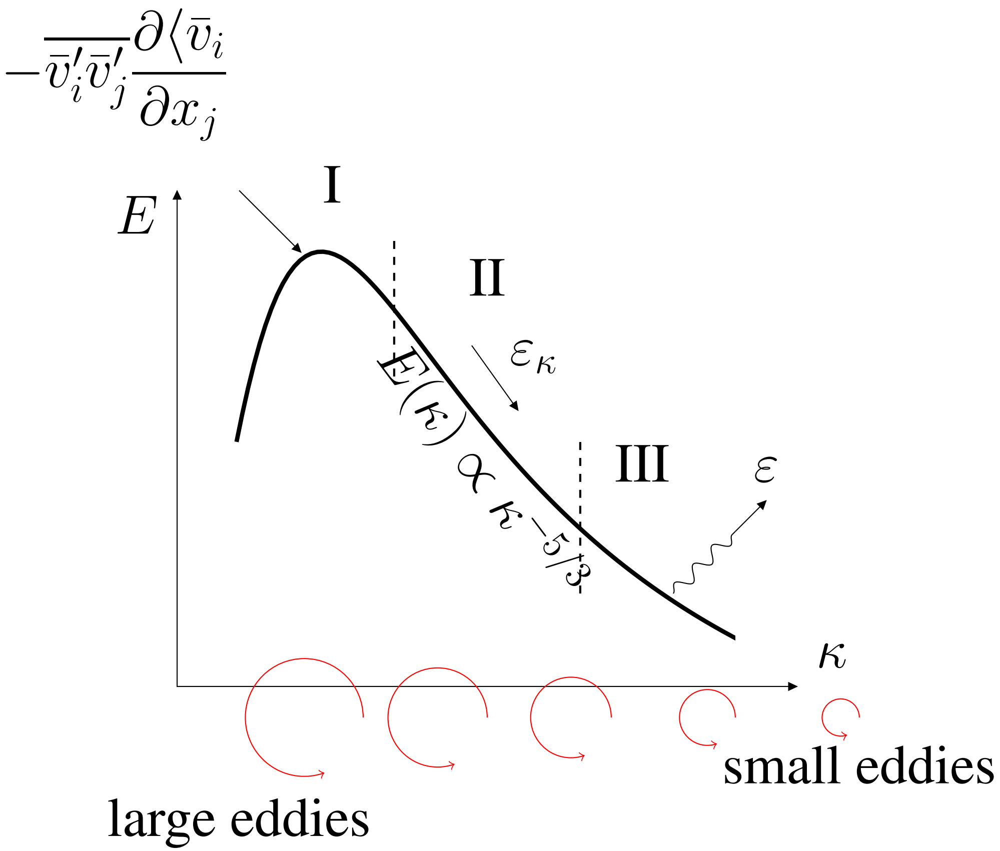


▶Turb. kinetic energy in Region II depends on: ▶ε and ▶eddy size $1/\kappa$ Recall: ▶$k=\int_0^\infty E(\kappa)d\kappa$

$$\begin{array}{lrcll} & E & = & \kappa^a & \varepsilon^b \\ & [m^3/s^2] & = & [1/m] & [m^2/s^3] \\ [m] & 3 & = & -a+2b & \\ [s] & -2 & = & -3b & \end{array}$$

$b=2/3, a=-5/3$ so that ▶$E(\kappa)=C_K\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$ ▶This is Kolmogorov spectrum law or $-5/3$ law

▶Turb. kinetic energy in Region III



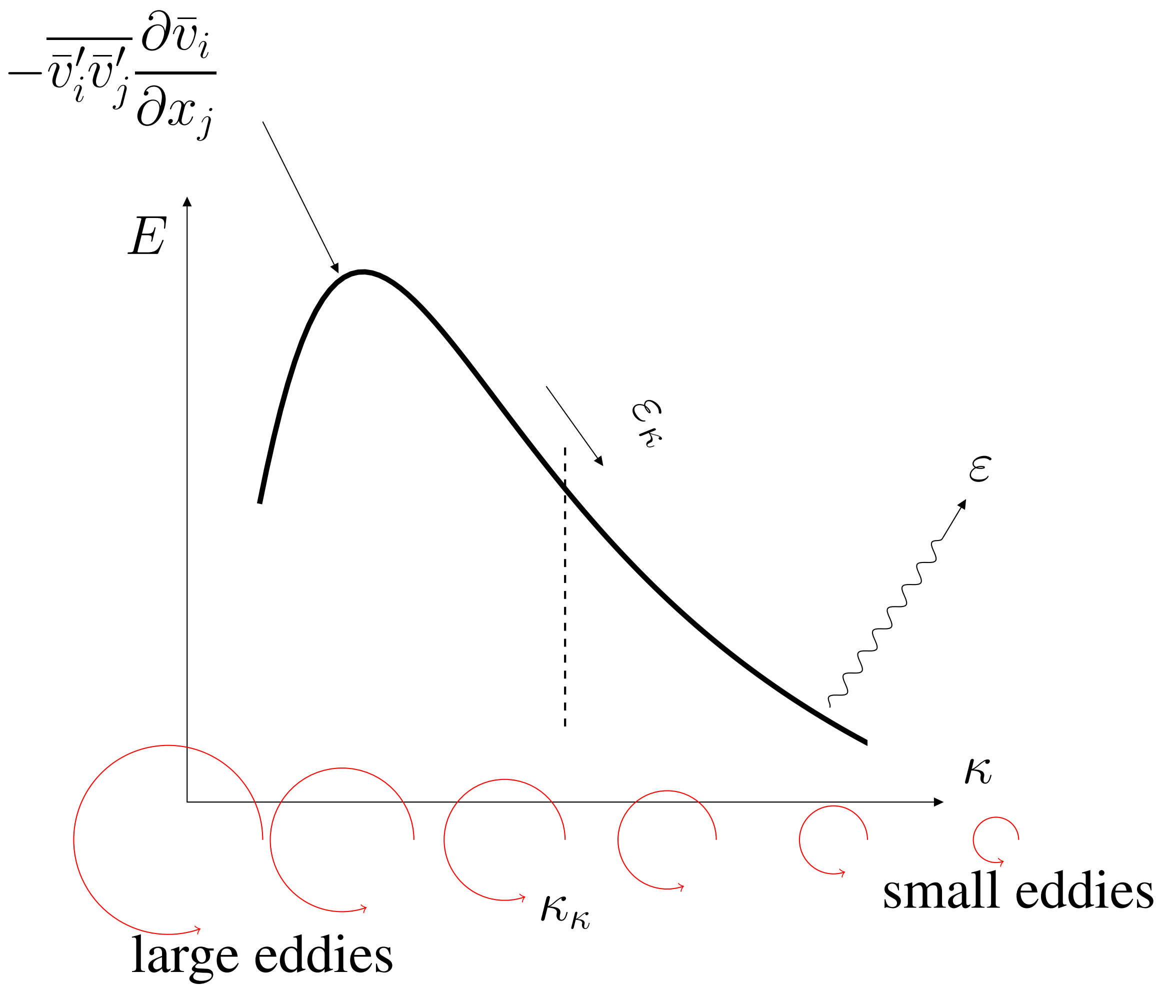
▶Small-scale turbulence is isotropic (see Section 5.3):

$\overline{v_1'^2} = \overline{v_2'^2} = \overline{v_3'^2}$. Not true instantaneously , i.e. $v_1' \neq v_2' \neq v_3'$.

Isotropy: if a coordinate direction is switched (i.e. rotated 180^o), nothing should change.
$\Rightarrow \overline{v_1'v_2'}$ in both coordinate directions must be the same. $\Rightarrow \overline{v_1'v_2'} = (\overline{v_1'v_2'})_{180^o} = -\overline{v_1'v_2'} = 0$.

▶On tensor form: $\overline{v_i'v_j'} = \text{const.}\delta_{ij}$

▶Energy transfer from eddy-to-eddy, κ_n



$$\varepsilon_\kappa \propto v_\kappa^2/t_\kappa \propto v_\kappa^2/(\ell_\kappa/v_\kappa) \propto \frac{v_\kappa^3}{\ell_\kappa} \propto \frac{v_0^3}{\ell_0}$$

- Find relation between largest and smallest scales: $Re = v_0\ell_0/\nu$, $v_\eta = (\nu\varepsilon)^{1/4}$, $\varepsilon = v_0^3/\ell_0$

$$\frac{v_0}{v_\eta} = (\nu\varepsilon)^{-1/4} v_0 = \left(\nu v_0^3/\ell_0\right)^{-1/4} v_0 = (v_0\ell_0/\nu)^{1/4} = Re^{1/4}$$

$$\frac{\ell_0}{\ell_\eta} = \left(\frac{\nu^3}{\varepsilon}\right)^{-1/4} \ell_0 = \left(\frac{\nu^3\ell_0}{v_0^3}\right)^{-1/4} \ell_0 = \left(\frac{\nu^3}{v_0^3\ell_0^3}\right)^{-1/4} = Re^{3/4}$$

$$\frac{\tau_o}{\tau_\eta} = \left(\frac{\nu\ell_0}{v_0^3}\right)^{-1/2} \tau_0 = \left(\frac{v_0^3}{\nu\ell_0}\right)^{1/2} \frac{\ell_0}{v_0} = \left(\frac{v_0\ell_0}{\nu}\right)^{1/2} = Re^{1/2}$$

- We do a DNS (Direct Numerical Simulation) at a certain Reynolds number.

- Now if we double the Re number, how much finer must the grid be?

$$\underbrace{2^{3/4}}_{x_1\,dir} \times \underbrace{2^{3/4}}_{x_2\,dir} \times \underbrace{2^{3/4}}_{x_3\,dir} \times \underbrace{2^{1/2}}_{time} = 2^{11/4} \simeq 7$$

- Hence, doubling the Re number requires 7 times more computational effort

- This explains why DNS (Direct Numerical Simulation) is too expensive at high Re numbers.

Lecture 7

¶See Section 6, Turbulent mean flow

▶The continuity and the Navier-Stokes for incompressible flow with constant μ read

$$\frac{\partial v_i}{\partial x_i}=0,\quad \rho\frac{\partial v_i}{\partial t}+\rho\frac{\partial v_i v_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}$$

▶Decompose the variables and time average

$$\begin{aligned}
\bar v &= \frac{1}{2T}\int_{-T}^{T} v dt,\quad v_i=\bar v_i+v_i',\quad \overline{v'}=0,\quad p=\bar p+p'\\
\frac{\partial\overline{\bar v_i+v_i'}}{\partial x_i} &= \frac{\partial\overline{\bar v_i}}{\partial x_i}+\frac{\partial\overline{v_i'}}{\partial x_i}=\frac{\partial\bar v_i}{\partial x_i}\quad (\overline{v_i'}=0)\\
\frac{\partial\overline{v_i v_j}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left\{\overline{(\bar v_i+v_i')(\bar v_j+v_j')}\right\}=\frac{\partial}{\partial x_j}\left(\overline{\bar v_i\bar v_j+\bar v_i v_j'+\bar v_j v_i'+v_i'v_j'}\right)\\
&= \frac{\partial}{\partial x_j}\left(\overline{\bar v_i\bar v_j}+\bar v_i\overline{v'_j}+\bar v_j\overline{v'_i}+\overline{v_i'v_j'}\right)=\frac{\partial\bar v_i\bar v_j}{\partial x_j}+\frac{\partial\overline{v_i'v_j'}}{\partial x_j}\quad (\overline{v'_j}=0,\ \overline{v'_i}=0)
\end{aligned}$$

▶The steady RANS (Reynolds-Averaged Navier-Stokes) equations

$$\frac{\partial\bar v_i}{\partial x_i}=0,\quad \rho\frac{\partial\bar v_i\bar v_j}{\partial x_j}=-\frac{\partial\bar p}{\partial x_i}+\frac{\partial}{\partial x_j}\underbrace{\left(\mu\frac{\partial\bar v_i}{\partial x_j}-\rho\overline{v_i'v_j'}\right)}_{\tau_{ij,tot}}\tag{36.1}$$

¶See Section 6.1.1, Boundary-layer approximation

▶RANS in developing boundary layer flow

$$\bar{v}_2 \ll \bar{v}_1, \ \frac{\partial \bar{v}_1}{\partial x_1} \ll \frac{\partial \bar{v}_1}{\partial x_2}$$

$$\rho \bar{v}_1 \frac{\partial \bar{v}_1}{\partial x_1} + \rho \bar{v}_2 \frac{\partial \bar{v}_1}{\partial x_2} = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2} \underbrace{\left[\mu \frac{\partial \bar{v}_1}{\partial x_2} - \rho \overline{v_1' v_2'}\right]}_{\tau_{tot}}$$

▶Left side: each term include one large ($\bar{v}_1$ and $\partial/\partial x_2$) and one small ($\bar{v}_2$ and $\partial/\partial x_1$) part

¶See Section 6.2, Wall region in fully developed channel flow

▶RANS in fully developed channel flow

$$0 = \underbrace{-\frac{\partial \bar{p}}{\partial x_1}}_{f(x_1)} + \underbrace{\frac{\partial}{\partial x_2} \left(\mu \frac{\partial \bar{v}_1}{\partial x_2} - \rho \overline{v_1' v_2'}\right)}_{g(x_2)} = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial \tau_{tot}}{\partial x_2}$$

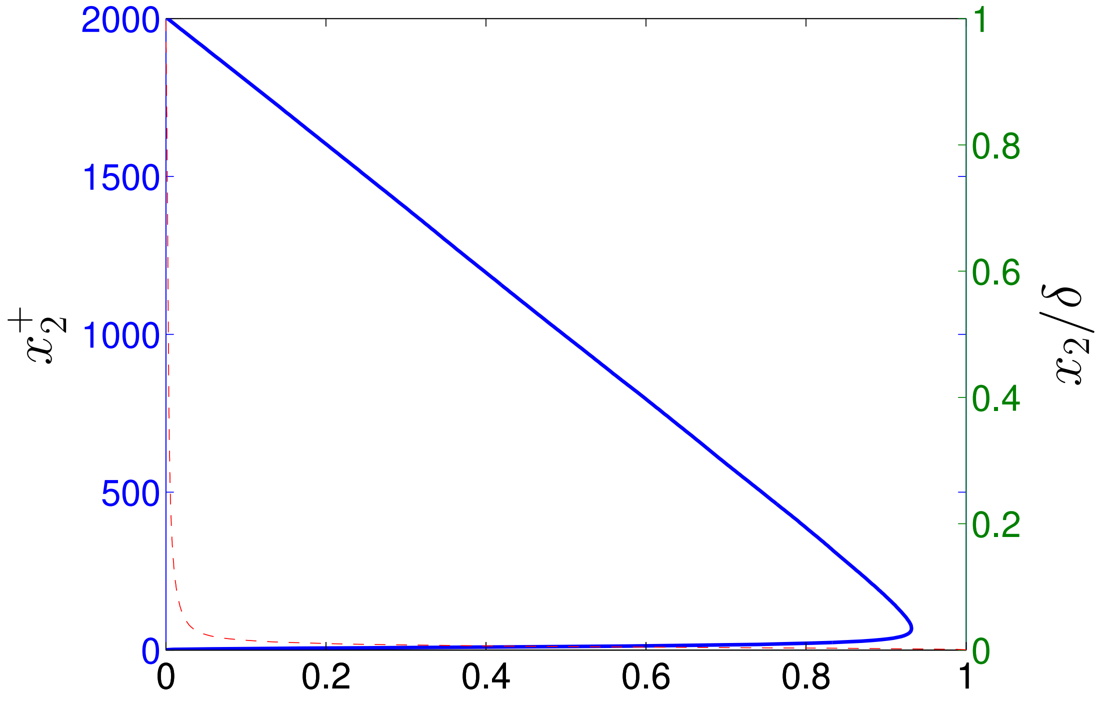
▶Integration from $x_2 = 0$ to x_2 gives

$$\tau_{tot}(x_2) - \tau_w = \frac{\partial \bar{p}}{\partial x_1} x_2 \quad \Rightarrow \quad \tau_{tot} = \tau_w + \frac{\partial \bar{p}}{\partial x_1} x_2 = \tau_w \left(1 - \frac{x_2}{\delta}\right)$$

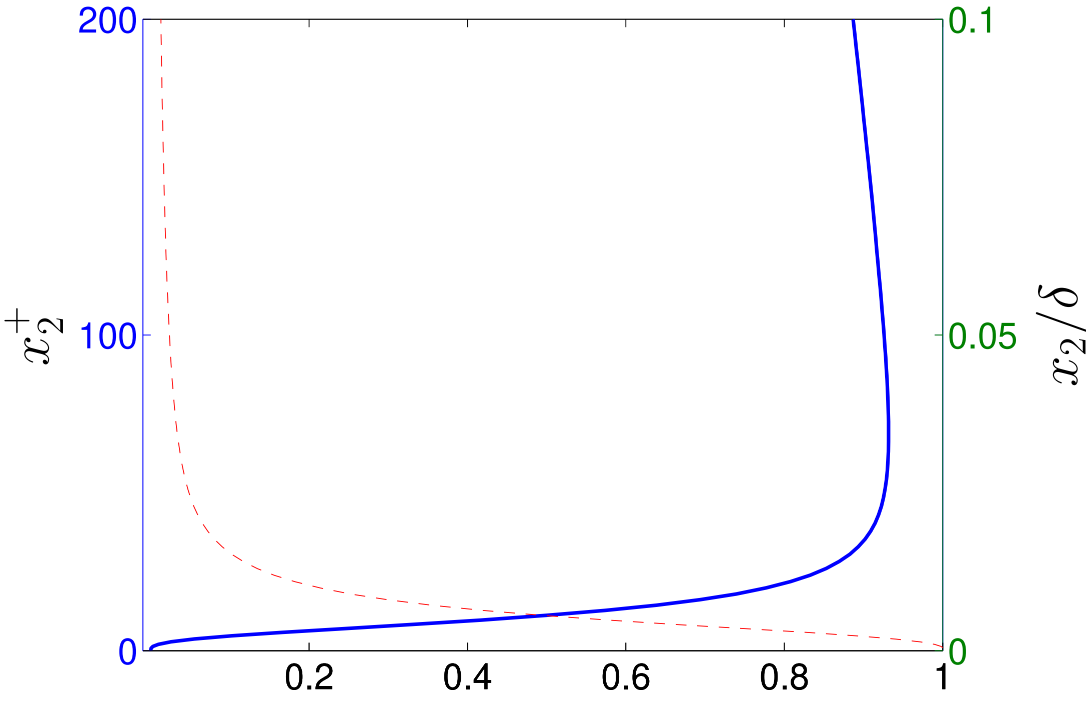
▶Last equality: $-\frac{\partial \bar{p}}{\partial x_1} = \frac{\tau_w}{\delta}$ (force balance)

$$0 = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial \bar{v}_1}{\partial x_2} - \rho \overline{v_1' v_2'}\right)$$

▶lower half of channel



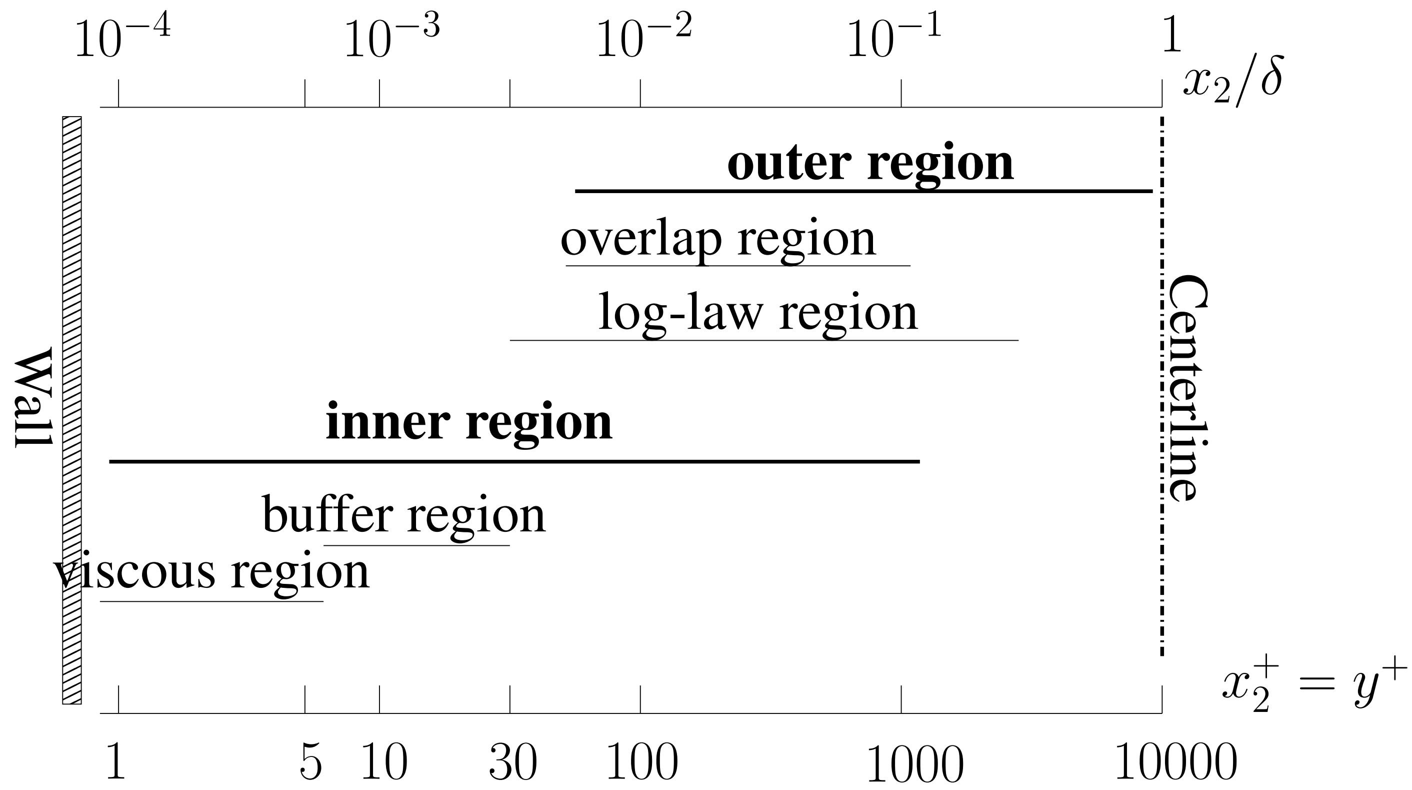
Full view



Zoom

—: $-\rho \overline{v_1' v_2'}/\tau_w$; - - -: $\mu(\partial \bar{v}_1/\partial x_2)/\tau_w$.

▶The different wall regions



▶Wall shear stress

$$\tau_w = \mu \frac{\partial \bar{v}_1}{\partial x_2}\bigg|_w \equiv \rho u_\tau^2 \quad \Rightarrow \quad u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2}, \quad x_2^+ = \frac{x_2 u_\tau}{\nu}$$

▶The linear velocity law

$$\left.\frac{\partial \bar{v}_1}{\partial x_2}\right|_w = \frac{\tau_w}{\mu} = \frac{\rho u_\tau^2}{\mu} = \frac{u_\tau^2}{\nu}$$

▶Integration gives (recall that both ν and u_τ^2 are constant)

$$\bar{v}_1 = \frac{1}{\nu} u_\tau^2 x_2 + C_1 = \frac{1}{\nu} u_\tau^2 x_2$$

Divide by u_τ:

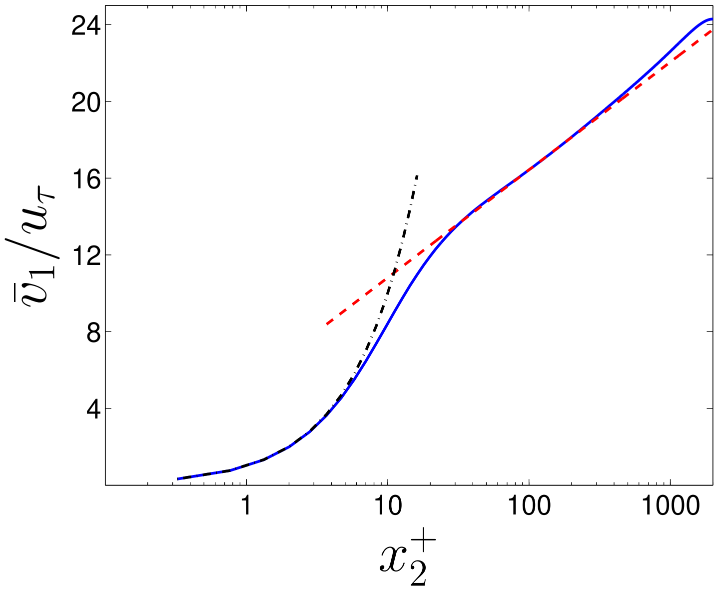
$$\frac{\bar{v}_1}{u_\tau} = \frac{u_\tau x_2}{\nu} \quad \text{or} \quad \bar{v}_1^+ = x_2^+$$

▶The log-law (turbulent region)

Velocity scale: u_τ; ▶length scale: $\ell \propto x_2 \quad \Rightarrow \quad \ell = \kappa x_2$

$$\frac{\partial \bar{v}_1}{\partial x_2} = \frac{u_\tau}{\kappa x_2} \quad \Rightarrow \quad \frac{\partial \bar{v}_1/u_\tau}{\partial x_2} = \frac{1}{\kappa x_2} \quad \Rightarrow \quad \frac{\partial \bar{v}_1/u_\tau}{\partial (x_2 u_\tau/\nu)} = \frac{1}{\kappa (x_2 u_\tau/\nu)}$$

$$\Rightarrow \quad \frac{\partial \bar{v}_1^+}{\partial x_2^+} = \frac{1}{\kappa x_2^+} \quad \text{integrate:} \quad \Rightarrow \quad \bar{v}_1^+ = \frac{1}{\kappa} \ln x_2^+ + B$$



Velocity profile in fully developed channel flow. $\bar{v}_1^+ = x_2^+$, $\bar{v}_1^+ = \dfrac{1}{\kappa}\ln x_2^+ + B$, $\kappa = 0.41$, $B = 5.2$.

▶In CFD, you may want to put the first cell ar $x_2^+ \equiv \dfrac{x_2 u_\tau}{\nu} \simeq 1$: how to find u_τ?

▶N.B. $\bar{v}_{1,centerline}/u_\tau = 24$ $\Rightarrow$ good estimate for u_τ ($\bar{v}_{1,centerline}/u_\tau$ increases weakly with Reynolds number)

▶Example: channel flow (or boundary layer), $x_2^+ \equiv \dfrac{x_2 u_\tau}{\nu} = 1$ gives

water: $x_2 = \nu x_2^+/u_\tau$ $=1 \cdot 10^{-6} \cdot 1/(1/24)$ $= 2.4 \cdot 10^{-5} m$ $= 2.4 \cdot 10^{-2} mm$

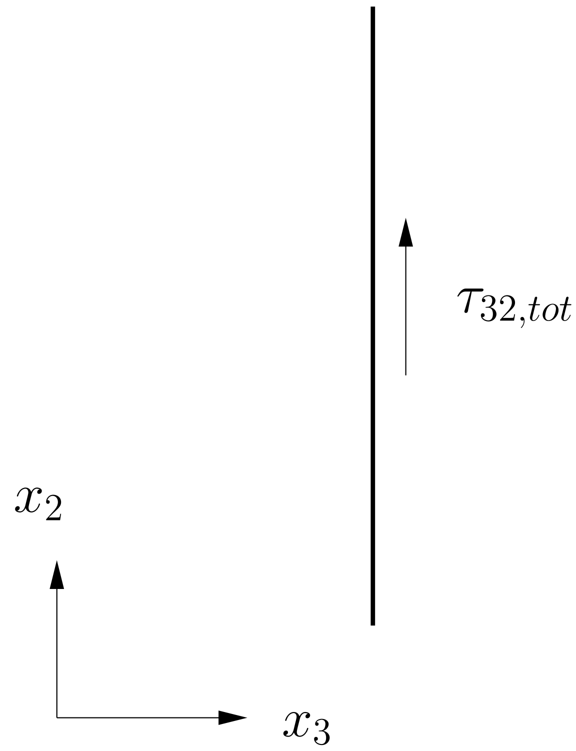
air: $x_2 = \nu x_2^+/u_\tau = 15 \cdot 1^{-6} \cdot 1/(1/24)$ $= 3.6 \cdot 10^{-4} m$ $= 3.6 \cdot 10^{-1} mm$

▶$0.2\delta/x_2$ (at $x_2^+ = 1$): estimate of ratio of largest to smallest turbulent length scales

▶estimate of ε? $u_\tau^3/(0.2\delta)$

¶See Section 6.3, Reynolds stresses in fully developed channel flow

- Symmetry plane or 2D: what value does $\tau_{32,tot} = \tau_{32} - \rho\overline{v_3'v_2'}$ take?

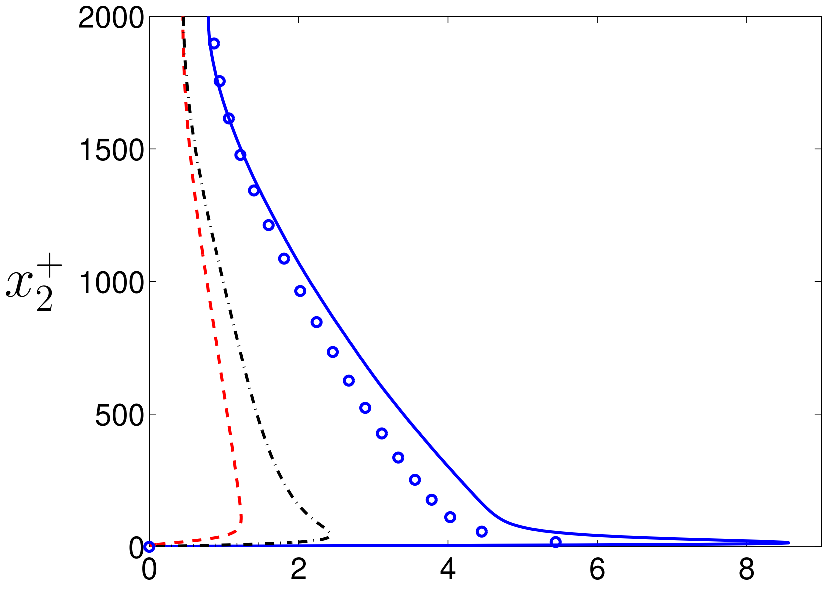


- $\bar{v}_3 = \partial/\partial x_3 = 0$

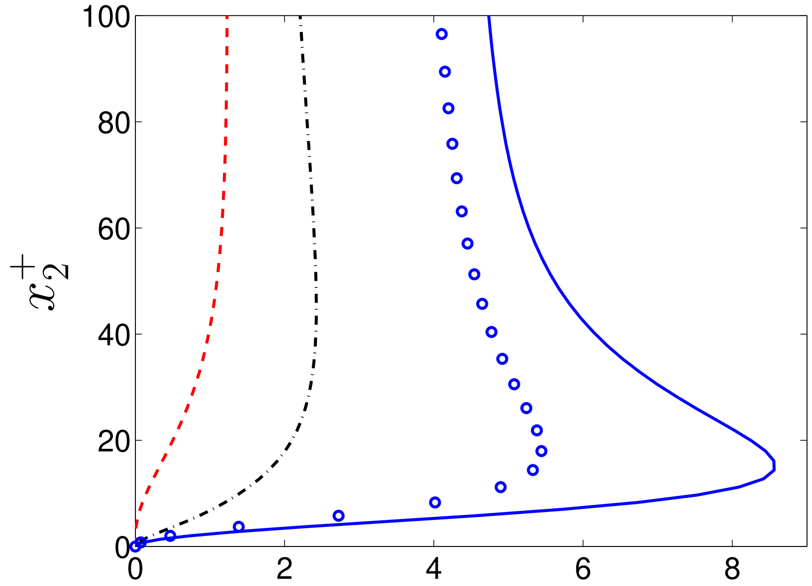
$$\tau_{32} = \mu\left(\frac{\partial \bar{v}_3}{\partial x_2} + \frac{\partial \bar{v}_2}{\partial x_3}\right) = 0, \quad \rho\overline{v_3'v_2'} = -\mu_t\left(\frac{\partial \bar{v}_3}{\partial x_2} + \frac{\partial \bar{v}_2}{\partial x_3}\right) = 0$$

- note that $\overline{v_3'v_3'} \neq 0$

Normal Reynolds stresses



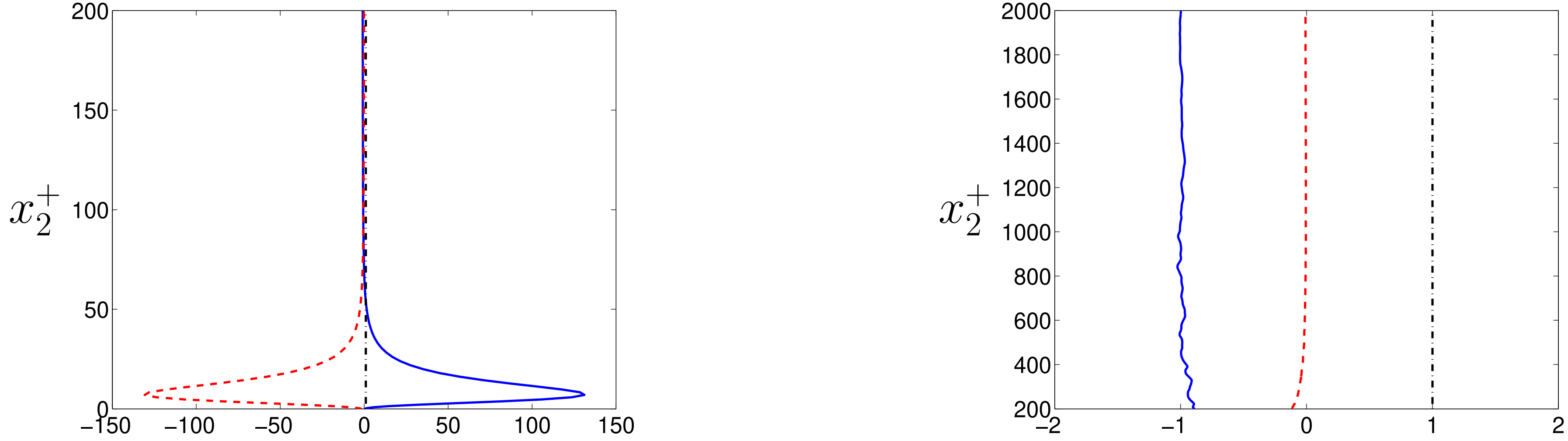
—: $\rho\overline{v_1'^2}/\tau_w$; - - -: $\rho\overline{v_2'^2}/\tau_w$; -·-·: $\rho\overline{v_3'^2}/\tau_w$



—: $-\rho\overline{v_1'v_2'}/\tau_w$; - - -: $\mu(\partial\bar{v}_1/\partial x_2)/\tau_w$.

▶Forces on a fluid element

$$0 = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial \bar{v}_1}{\partial x_2} - \rho \overline{v_1' v_2'}\right)$$

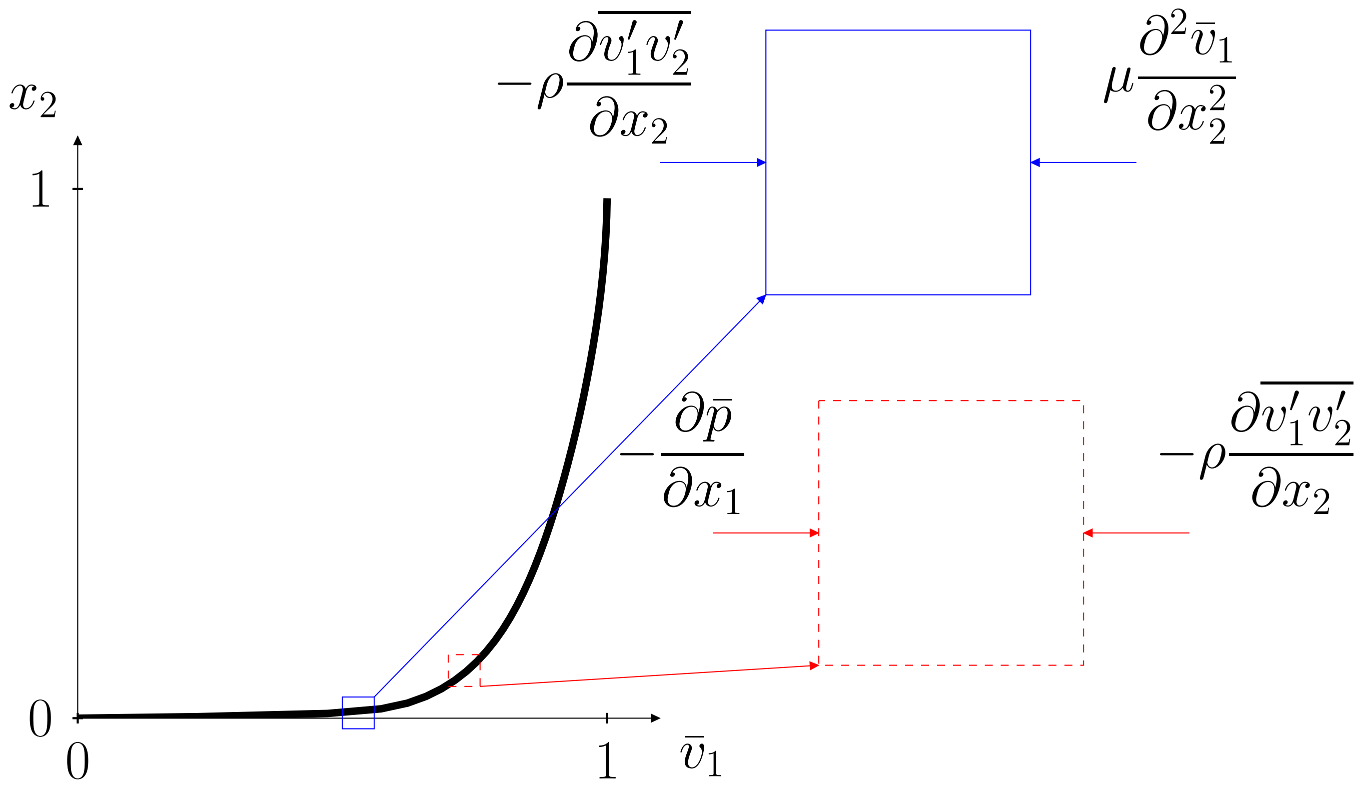


Near the wall

Far from the wall

Gradient of shear stresses. —: $-\rho(\partial\overline{v_1'v_2'}/\partial x_2)/\tau_w$; - - -: $\mu(\partial^2\bar{v}_1/\partial x_2^2)/\tau_w$; -·-·-: $-(\partial\bar{p}/\partial x_1)/\tau_w$.

▶Forces in a boundary layer. The dashed line and the solid line: $x_2^+ \simeq 400$ and $x_2^+ \simeq 20$, respectively



Lecture 8

¶See Section 8.1, Rules for time averaging

▶Time averaging

$$\bar{v} = \frac{1}{2T}\int_{-T}^{T} v dt, \quad \overline{v'} = 0 \tag{37.1}$$

▶What is the difference between $\overline{v_1' v_2'}$ and $\overline{v_1'}\ \overline{v_2'}$?

▶Using 37.1 we get

$$\overline{v_1' v_2'} = \frac{1}{2T}\int_{-T}^{T} v_1' v_2' dt$$

whereas

$$\overline{v_1'}\ \overline{v_2'} = \left(\frac{1}{2T}\int_{-T}^{T} v_1' dt\right)\left(\frac{1}{2T}\int_{-T}^{T} v_1' dt\right)$$

which is zero

►What is the difference between $\overline{v_1'^2}$ and $\overline{v_1'}^2$? Using 37.1 we get

$$\overline{v_1'^2} = \frac{1}{2T}\int_{-T}^{T} v'^2 dt$$

whereas

$$\overline{v_1'}^2 = \left(\frac{1}{2T}\int_{-T}^{T} v' dt\right)^2$$

which is zero

►Show that $\overline{\bar{v}_1 v_1'^2} = \bar{v}_1\overline{v_1'^2}$. Using 37.1 we get

$$\overline{\bar{v}_1 v_1'^2} = \frac{1}{2T}\int_{-T}^{T} \bar{v}_1 v_1'^2 dt$$

and since $\bar{v}$ does not depend on t we can take it out of the integral as

$$\bar{v}_1\frac{1}{2T}\int_{-T}^{T} v_1'^2 dt = \bar{v}_1\overline{v_1'^2}$$

►Show that $\overline{\bar{v}}_1 = \bar{v}_1$. Using 37.1 we get

$$\overline{\bar{v}}_1 = \frac{1}{2T}\int_{-T}^{T} \bar{v}_1 dt$$

and since $\bar{v}$ does not depend on t we can take it out of the integral as

$$\bar{v}_1 \frac{1}{2T}\int_{-T}^{T} dt = \bar{v}_1 \frac{1}{2T} 2T = \bar{v}_1$$

►Show that $\overline{\bar{v}_1 v_1'} = 0$. Using 37.1 we get

$$\overline{\bar{v}_1 v_1'} = \frac{1}{2T}\int_{-T}^{T} \bar{v}_1 v_1' dt = \bar{v}_1 \frac{1}{2T}\int_{-T}^{T} v_1' dt = \bar{v}_1 \overline{v_1'} = 0$$

¶See Section 11.6, The Boussinesq assumption

▶The RANS equations read (see Eq. 36.1)

$$\frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{v}_i}{\partial x_j} - \overline{v_i' v_j'}\right)$$

▶The last term, the Reynolds stress, is unknown. ▶It must be modeled

▶This is called the closure problem ▶We need a turbulence model

▶Write the diffusion term above without assuming constant viscosity

$$\frac{\partial}{\partial x_j}\left\{\nu\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \overline{v_i' v_j'}\right\} \tag{37.2}$$

▶We replace $\overline{v_i' v_j'}$ by a turbulent viscosity, ν_t:

$$\frac{\partial}{\partial x_j}\left\{(\nu + \nu_t)\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\right\} \tag{37.3}$$

▶Identification of Eqs. 37.2 and 37.3 gives

$$-\overline{v_i' v_j'} = \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) \tag{37.4}$$

$$-\overline{v_i'v_j'} = \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \tag{37.4}$$

This equation is not valid upon contraction.

$$\overline{v_i'v_i'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_i} + \frac{\partial \bar{v}_i}{\partial x_i} \right)$$

►Left-side ($= \overline{v_i'v_i'}$) and right side ($= 0$) are different!

We modify Eq. 37.4 as

$$\overline{v_i'v_j'} = -\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) + \frac{1}{3}\delta_{ij}\overline{v_k'v_k'} = -2\nu_t \bar{s}_{ij} + \frac{2}{3}\delta_{ij}k \tag{37.5}$$

►Contracted left side: $\overline{v_i'v_i'} = 2k$ ►contracted right side: $2\nu_t \bar{s}_{ii} + \frac{2}{3}\delta_{ii}k = 0 + \frac{2}{3} \cdot 3k = 2k$

►ν: different for different fluids (air, water, …)

►ν_t: depends on the flow, i.e. $\nu_t = \nu_t(x_i)$

▶Now we need to model the turbulent viscosity in the Boussinesq aaumption (Eq. 37.5).

▶Recall the dimension of ν: ▶m^2/s

▶ν_t estimated as a turbulent velocity fluctuation times a turbulent length scale

$$\nu_t \propto \mathcal{U}\mathcal{L}$$

▶The velocity scale: $k^{1/2}$ ▶Dissipation (energy transfer eddy-to-eddy): $\varepsilon \propto \mathcal{U}^3/\mathcal{L}$

$\Rightarrow \quad \mathcal{L} \propto k^{3/2}/\varepsilon$ ▶We get

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}, \quad C_\mu = 0.09. \tag{37.6}$$

¶See Section 8.2, The Exact k Equation

▶ $k=\overline{v_i'v_i'}/2$ appears in the expression for the turbulence viscosity.

▶ The first step is to derive the k equation. ▶ Take N-S for v_i', multiply by v_i' and time average

$$\overline{v_i'\frac{\partial}{\partial x_j}\left[v_iv_j-\bar{v}_i\bar{v}_j\right]}=\underbrace{-\frac{1}{\rho}\overline{v_i'\frac{\partial}{\partial x_i}\left[p-\bar{p}\right]}}_{\textbf{IV}}+\underbrace{\nu\overline{v_i'\frac{\partial^2}{\partial x_j\partial x_j}\left[v_i-\bar{v}_i\right]}}_{V}+\underbrace{\overline{\frac{\partial\overline{v_i'v_j'}}{\partial x_j}v_i'}}_{\textbf{VI}}$$

Using $v_j=\bar{v}_j+v_j'$, the left side can be rewritten as

$$\overline{v_i'\frac{\partial}{\partial x_j}\left[(\bar{v}_i+v_i')(\bar{v}_j+v_j')-\bar{v}_i\bar{v}_j\right]}=\overline{v_i'\frac{\partial}{\partial x_j}\left[\underbrace{\bar{v}_iv_j'}_{\textbf{I}}+\underbrace{v_i'\bar{v}_j}_{\textbf{II}}+\underbrace{v_i'v_j'}_{\textbf{III}}\right]}$$

▶ Term I is rewritten as

$$\overline{v_i'\frac{\partial}{\partial x_j}\left(\bar{v}_iv_j'\right)}=\overline{v_i'v_j'}\frac{\partial\bar{v}_i}{\partial x_j}+\underbrace{\overline{\bar{v}_iv_i'\frac{\partial v_j'}{\partial x_j}}}_{\text{cont.eq.}}{}^{\nearrow 0}=\overline{v_i'v_j'}\frac{\partial\bar{v}_i}{\partial x_j}$$

▶ Term II

$$\overline{v_i'\frac{\partial}{\partial x_j}\left(v_i'\bar{v}_j\right)}=\overline{v_i'v_i'}\frac{\partial\bar{v}_j}{\partial x_j}{}^{\nearrow 0}+\overline{v_i'\bar{v}_j\frac{\partial v_i'}{\partial x_j}}\underset{\text{Trick 2}}{=}\bar{v}_j\frac{\partial}{\partial x_j}\left(\frac{\overline{v_i'v_i'}}{2}\right)=\bar{v}_j\frac{\partial k}{\partial x_j}$$

▶ Term III

$$\overline{v_i'\frac{\partial}{\partial x_j}\left(v_i'v_j'\right)}=\overline{v_i'v_i'\frac{\partial v_j'}{\partial x_j}}{}^{\nearrow 0}+\overline{v_j'v_i'\frac{\partial v_i'}{\partial x_j}}\underset{\text{Trick 2}}{=}\overline{v_j'\frac{\partial}{\partial x_j}\left(\frac{v_i'v_i'}{2}\right)}\underset{\text{Trick 1}}{=}\frac{\partial}{\partial x_j}\left(\frac{\overline{v_j'v_i'v_i'}}{2}\right)-\overline{v_i'v_i'\frac{\partial v_j'}{\partial x_j}}{}^{\nearrow 0}$$

$$\underbrace{\overline{v'_i v'_j}\frac{\partial \bar{v}_i}{\partial x_j} + \bar{v}_j\frac{\partial k}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\overline{v'_j v'_i v'_i}}{2}\right)}_{\textbf{I+II+III}} = \underbrace{-\frac{1}{\rho}\overline{v'_i\frac{\partial}{\partial x_i}\left[p-\bar{p}\right]}}_{\textbf{IV}} + \underbrace{\nu\overline{v'_i\frac{\partial^2}{\partial x_j \partial x_j}\left[v_i-\bar{v}_i\right]}}_{V} + \underbrace{\overline{\frac{\partial \overline{v'_i v'_j}}{\partial x_j}v'_i}}_{\textbf{VI}} \tag{37.7}$$

►First term on the right side (Term IV)

$$-\frac{1}{\rho}\overline{v'_i\frac{\partial p'}{\partial x_i}} \underset{\text{Trick 1}}{=} -\frac{1}{\rho}\frac{\partial \overline{p'v'_i}}{\partial x_i} + \frac{1}{\rho}\overline{p'\frac{\partial v'_i}{\partial x_i}}^{\,0}$$

►Second term on the right side (Term V), omit ν

$$\overline{v'_i\frac{\partial^2 v'_i}{\partial x_j \partial x_j}} \underset{\text{Trick 1}}{=} \overline{\frac{\partial}{\partial x_j}\left(\frac{\partial v'_i}{\partial x_j}v'_i\right)} - \overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}} \underset{\text{Trick 2}}{=} \overline{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial v'_i v'_i}{\partial x_j}\right)\right)} - \overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}} = \frac{\partial^2 k}{\partial x_j \partial x_j} - \overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}}$$

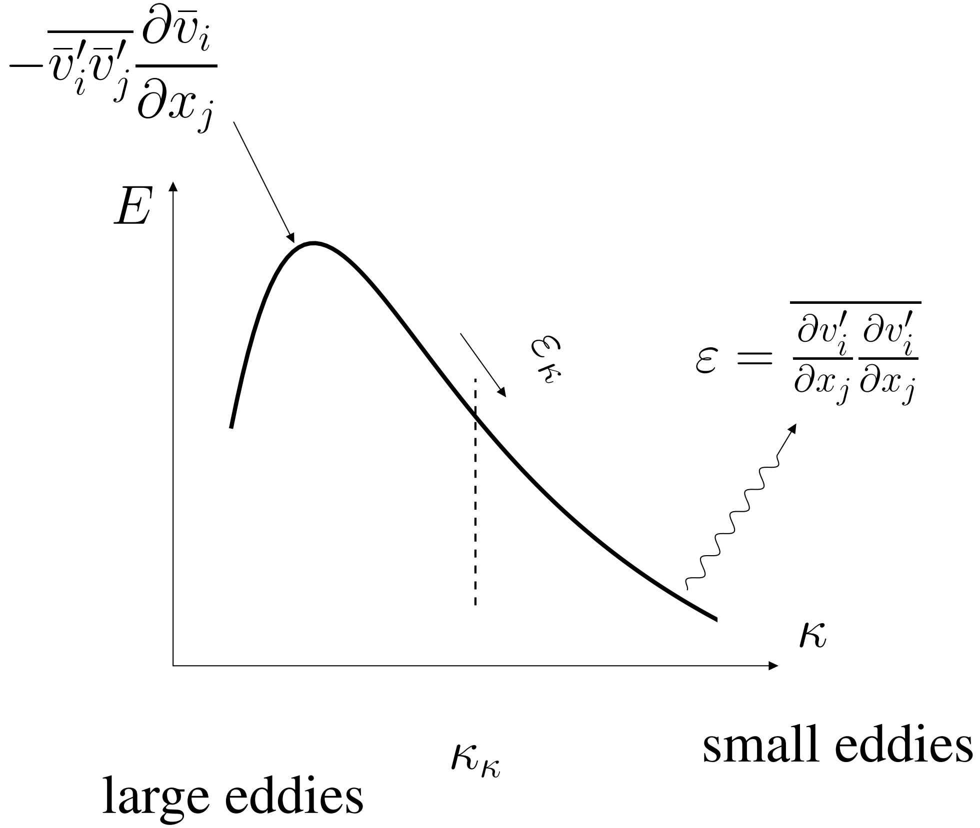
►Third term on the right side (Term VI)

$$\overline{\frac{\partial \overline{v'_i v'_j}}{\partial x_j}v'_i} = \frac{\partial \overline{v'_i v'_j}}{\partial x_j}\overline{v'_i} = 0$$

►Insert Terms IV, V and VI in Eq. 37.7

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{\textbf{I}} = \underbrace{-\overline{v'_i v'_j}\frac{\partial \bar{v}_i}{\partial x_j}}_{\textbf{II}} \underbrace{-\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v'_j p'} + \frac{1}{2}\overline{v'_j v'_i v'_i} - \nu\frac{\partial k}{\partial x_j}\right]}_{\textbf{III}} - \underbrace{\nu\overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}}}_{\textbf{IV}} \tag{37.8}$$

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{\textbf{I}} = \underbrace{-\overline{v'_i v'_j}\frac{\partial \bar{v}_i}{\partial x_j}}_{\textbf{II}} \underbrace{-\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v'_j p'} + \frac{1}{2}\overline{v'_j v'_i v'_i} - \nu\frac{\partial k}{\partial x_j}\right]}_{\textbf{III}} - \underbrace{\nu\overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}}}_{\textbf{IV}}$$



The terms have the following meaning.

I. **Convection**.

II. **Production**, P^k. The large turbulent scales extract energy from the mean flow. It is largest for small wavenumbers. It can be written as $P^k = -\overline{v'_i v'_j}\bar{S}_{ij}$. Hence only $\bar{S}_{ij}$ creates turbulence, not $\bar{\Omega}_{ij}$

III. The two first terms represent **turbulent diffusion** by pressure-velocity fluctuations, and velocity fluctuations, respectively. The last term is viscous diffusion.

IV. **Dissipation**, ε. It is largest for high wavenumbers

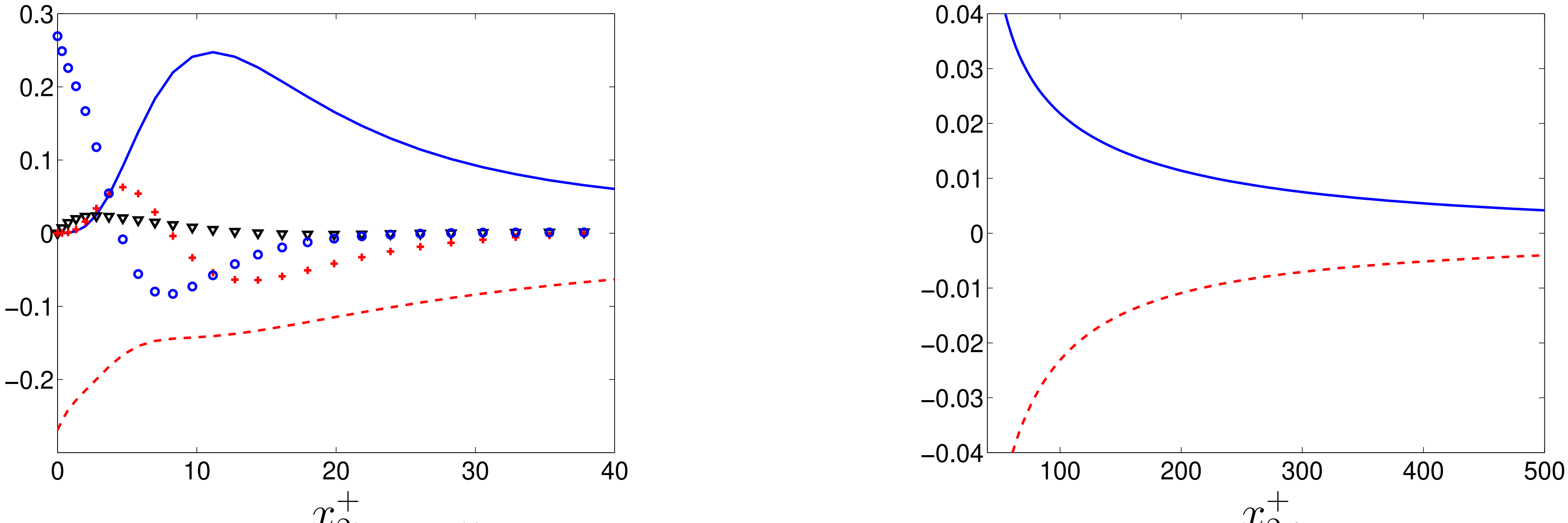
- The k equation in symbolic form:

$$C^k = P^k + D^k - \varepsilon$$

¶See Section 8.3, The Exact k Equation: 2D Boundary Layers

▶In 2D boundary-layer flow, $\partial/\partial x_2 \gg \partial/\partial x_1$, $\bar{v}_2 \ll \bar{v}_1$, we get

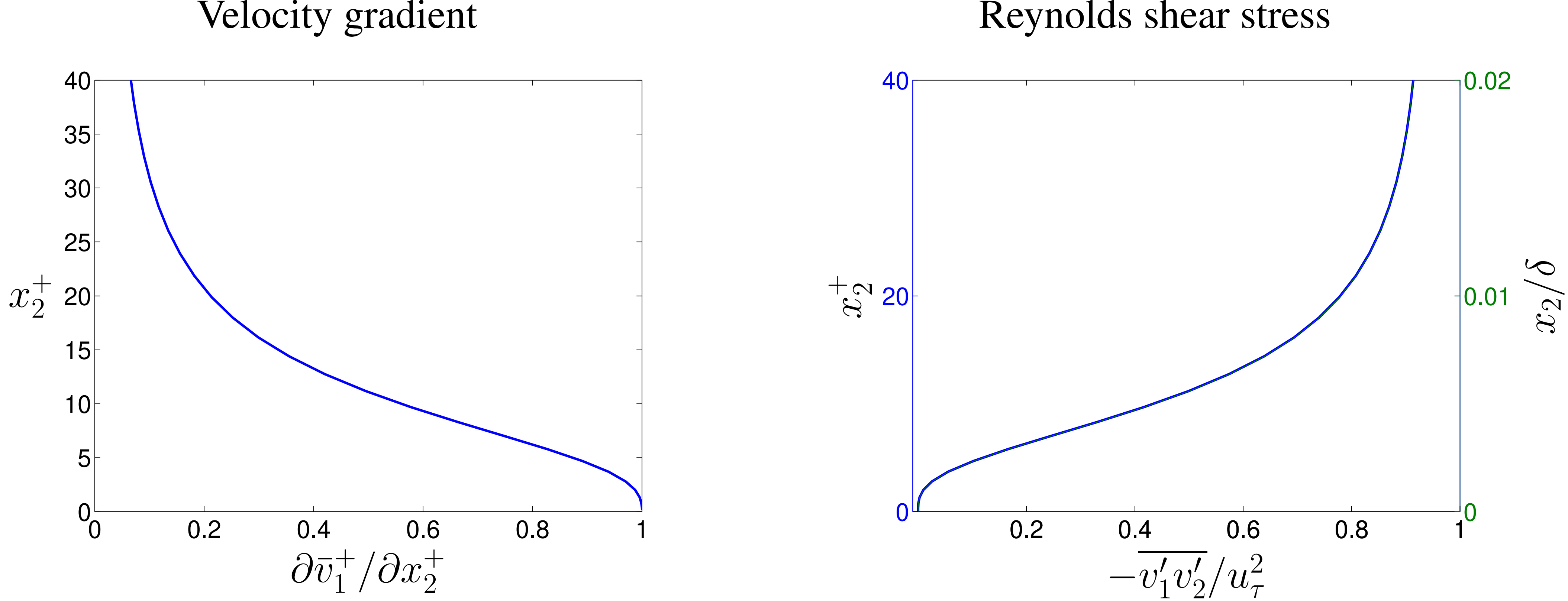
$$\frac{\partial \bar{v}_1 k}{\partial x_1} + \frac{\partial \bar{v}_2 k}{\partial x_2} = -\overline{v_1' v_2'}\frac{\partial \bar{v}_1}{\partial x_2} - \frac{\partial}{\partial x_2}\left[\frac{1}{\rho}\overline{p' v_2'} + \frac{1}{2}\overline{v_2' v_i' v_i'} - \nu \frac{\partial k}{\partial x_2}\right] - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$



Zoom near the wall

Outer region

—: P^k; - - -: $-\varepsilon$; ∇ : $-\partial\overline{v'p'}/\partial x_2$; +: $-\partial\overline{v_2' v_i' v_i'/2}/\partial x_2$; ○: $\nu \partial^2 k/\partial x_2^2$.



The production term $-\overline{v_1' v_2'} \dfrac{\partial \bar{v}_1}{\partial x_2}$

Lecture 9

¶See Section 8.6, The transport equation for $\bar{v}_i\bar{v}_i/2$

▶The main source term in k eq is P^k. ▶Hence, k gets energy via P^k. ▶From where?

▶Answer: from $K = \bar{v}_i\bar{v}_i/2$. ▶Let's derive the transport eq. for K.

Multiply the RANS equations by $\bar{v}_i$ so that

$$\underbrace{\bar{v}_i\left(\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j}\right)}_{\text{I}} = \bar{v}_i\left(\underbrace{-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}}_{\text{II}} + \nu\underbrace{\frac{\partial^2 \bar{v}_i}{\partial x_j\partial x_j}}_{\text{III}} - \underbrace{\frac{\partial \overline{v_i'v_j'}}{\partial x_j}}_{\text{IV}}\right)$$

Term I:

$$\bar{v}_i\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j} = \bar{v}_i\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j} + \bar{v}_i\bar{v}_i\cancelto{0}{\frac{\partial \bar{v}_j}{\partial x_j}} \underset{\text{Trick 2}}{=} \frac{1}{2}\bar{v}_j\frac{\partial \bar{v}_i\bar{v}_i}{\partial x_j} = \frac{\partial \bar{v}_j K}{\partial x_j}$$

Term II:

$$-\bar{v}_i\frac{\partial \bar{p}}{\partial x_i} \quad \text{main source term in, for example, channel flow} \quad \left(-\bar{v}_1\frac{\partial \bar{p}}{\partial x_1} > 0\right)$$

Term III:

$$\nu\bar{v}_i\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{v}_i}{\partial x_j}\right) \underset{\text{Trick 1}}{=} \nu\frac{\partial}{\partial x_j}\left(\underbrace{\bar{v}_i\frac{\partial \bar{v}_i}{\partial x_j}}_{\partial/\partial_j(\bar{v}_i\bar{v}_i/2)}\right) - \nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j} \underset{\text{Trick 2}}{=} \nu\frac{\partial^2 K}{\partial x_j\partial x_j} - \nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j}.$$

Term IV:

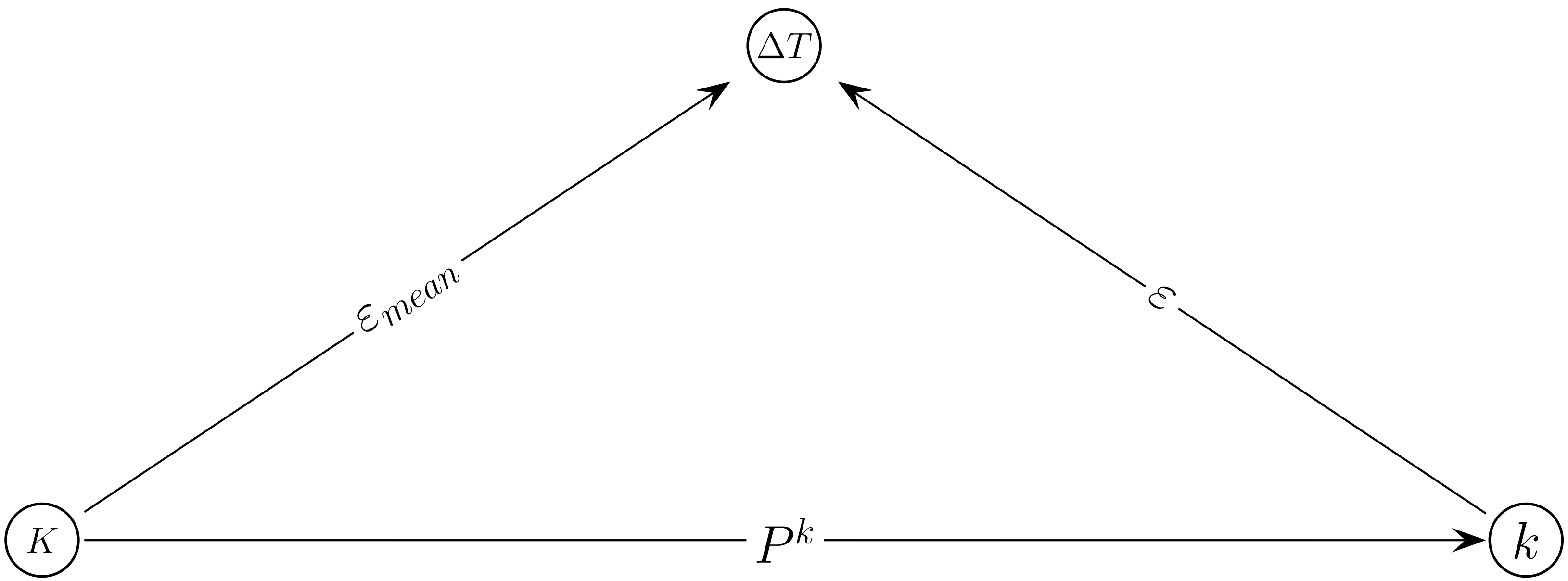
$$-\bar{v}_i \frac{\partial \overline{v_i' v_j'}}{\partial x_j} \underset{\text{Trick 1}}{=} -\frac{\partial \bar{v}_i \overline{v_i' v_j'}}{\partial x_j} + \overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j}.$$

▶ $K = \frac{1}{2}\bar{v}_i\bar{v}_i$ equation

$$\frac{\partial \bar{v}_j K}{\partial x_j} = \underbrace{\overline{v'_i v'_j}\frac{\partial \bar{v}_i}{\partial x_j}}_{-P^k,\ \text{sink}} \underbrace{-\frac{\bar{v}_i}{\rho}\frac{\partial \bar{p}}{\partial x_i}}_{\text{source}} - \frac{\partial}{\partial x_j}\left(\bar{v}_i\overline{v'_i v'_j} - \nu\frac{\partial K}{\partial x_j}\right) \underbrace{-\nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j}}_{\varepsilon_{mean},\ \text{sink}}$$

▶ $k = \frac{1}{2}\overline{v'_i v'_i}$ equation (see Eq. 37.8)

$$\frac{\partial \bar{v}_j k}{\partial x_j} = \underbrace{-\overline{v'_i v'_j}\frac{\partial \bar{v}_i}{\partial x_j}}_{P^k,\ \text{source}} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v'_j p'} + \frac{1}{2}\overline{v'_j v'_i v'_i} - \nu\frac{\partial k}{\partial x_j}\right] \underbrace{-\nu\overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}}}_{\varepsilon,\ \text{sink}}$$



Transfer of energy between mean kinetic energy (K), turbulent kinetic energy (k) and internal energy (denoted as an increase in temperature, ΔT). $K = \frac{1}{2}\bar{v}_i\bar{v}_i$ and $k = \frac{1}{2}\overline{v'_i v'_i}$.

¶See Section 11.7.1, Production terms

▶The exact k eq. reads (see Eq. 37.8)

$$\frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} \tag{38.1}$$

▶Production term needs to be modelled.

$$\begin{aligned} P^k &= -\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j} = \left[\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k\right]\frac{\partial \bar{v}_i}{\partial x_j} = 2\nu_t \bar{s}_{ij}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{2}{3}\delta_{ij}k\frac{\partial \bar{v}_i}{\partial x_j} \\ &= 2\nu_t \bar{s}_{ij}(\bar{s}_{ij} + \bar{\Omega}_{ij}) - k\frac{\partial \bar{v}_i}{\partial x_i} = 2\nu_t \bar{s}_{ij}\bar{s}_{ij} \end{aligned}$$

▶Also the diffusion term needs to be modeled. Example: heat flux is modelled as

$$\overline{v_i'\theta'} = -\frac{\nu_t}{\sigma_t}\frac{\partial \bar{\theta}}{\partial x_i}$$ ▶$q_i = k\frac{\partial \bar{\theta}}{\partial x_i}$

▶The diffusion term in k eq, Eq. 38.1, is modelled as

$$\frac{1}{2}\overline{v_j' v_i' v_i'} = \overline{v_j' k'} = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j} \Rightarrow -\frac{1}{2}\frac{\partial \overline{v_j' v_i' v_i'}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)$$

¶See Section 11.8, The $k-\varepsilon$ model

▶Modelled k equation

$$\frac{\partial k}{\partial t} + \bar{v}_j\frac{\partial k}{\partial x_j} = 2\nu_t \bar{s}_{ij}\bar{s}_{ij} + \frac{\partial}{\partial x_j}\left\{\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right\} - \varepsilon \tag{38.2}$$

¶See Section 11.5, The ε equation

►$\nu_t = C_\mu \frac{k^2}{\varepsilon}$ $\Rightarrow$ We need an equation for ε: ►Look at k equation in symbolic form:

$$C^k = P^k + D^k - \varepsilon$$

►ε equation in symbolic form:

$$C^\varepsilon = P^\varepsilon + D^\varepsilon - \Psi^\varepsilon$$

►Use the source terms as in k eq, and add turbulent time-scale ε/k to get correct dimensions:

$$P^\varepsilon - \Psi^\varepsilon = \frac{\varepsilon}{k}(c_{\varepsilon 1} P^k - c_{\varepsilon 2}\varepsilon)$$

►The final form of the modelled ε equation

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}(c_{\varepsilon 1} P^k - c_{\varepsilon 2}\varepsilon) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \tag{38.3}$$

▶Summary of the $k - \varepsilon$ model.

- The Reynolds stress tensor, $\overline{v_i' v_j'}$, needs to be modeled, see Eq. 36.1
- We use the Boussinsq assumption, see Eq. 37.5, to replace the unknown $\overline{v_i' v_j'}$ with the turbulent viscosity, ν_t (a new unknown).
- We make a model for $\nu_t = C_\mu \dfrac{k^2}{\varepsilon}$, see Eq. 37.6, which includes k and ε
- We formulate modeled equations for k (Eq. 38.2) and ε (Eq. 38.3)
- Now we have closed Eq. 36.1. The equations we need to solve are
  - The time-averaged continuity equation (Eq. 36.1)
  - Three time-averaged Navier-Stokes equations (Eq. 36.1)
  - Two equations for k (Eq. 38.2) and ε (Eq. 38.3)
  - The equation for turbulent viscosity, $\nu_t = C_\mu k^2/\varepsilon$ (Eq. 37.6)

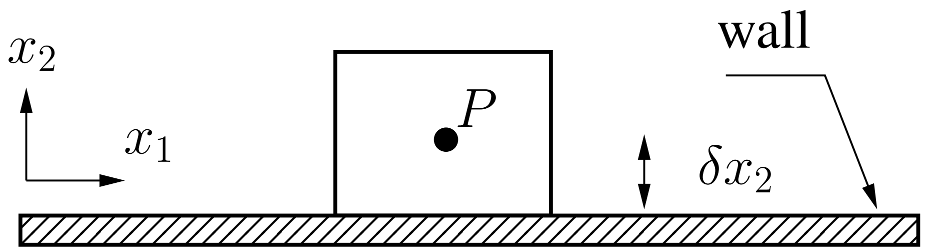
Lecture 10

¶See Section 11.14, Wall boundary conditions

▶Two options for treating the wall boundary conditions.

- Coarse mesh near the walls. Assume that the logarithmic law applies. This is called wall functions
- Fine mesh. Modify the turbulence models to account for the viscous effects.
  This is called Low-Reynolds number models

¶See Section 11.14.1, Wall Functions



Wall-adjacent cell.

►We don't resolve the boundary layer. We assume that the velocity obeys rhe log-law at this location

►The log-law reads $\dfrac{\bar{v}_1}{u_\tau} = \dfrac{1}{\kappa} \ln\left(\dfrac{u_\tau x_2}{\nu}\right) + B$

It is re-written as

$$\frac{\bar{v}_1}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{E u_\tau x_2}{\nu}\right), \quad E = 9.0, \quad B = \frac{1}{\kappa} \ln E$$

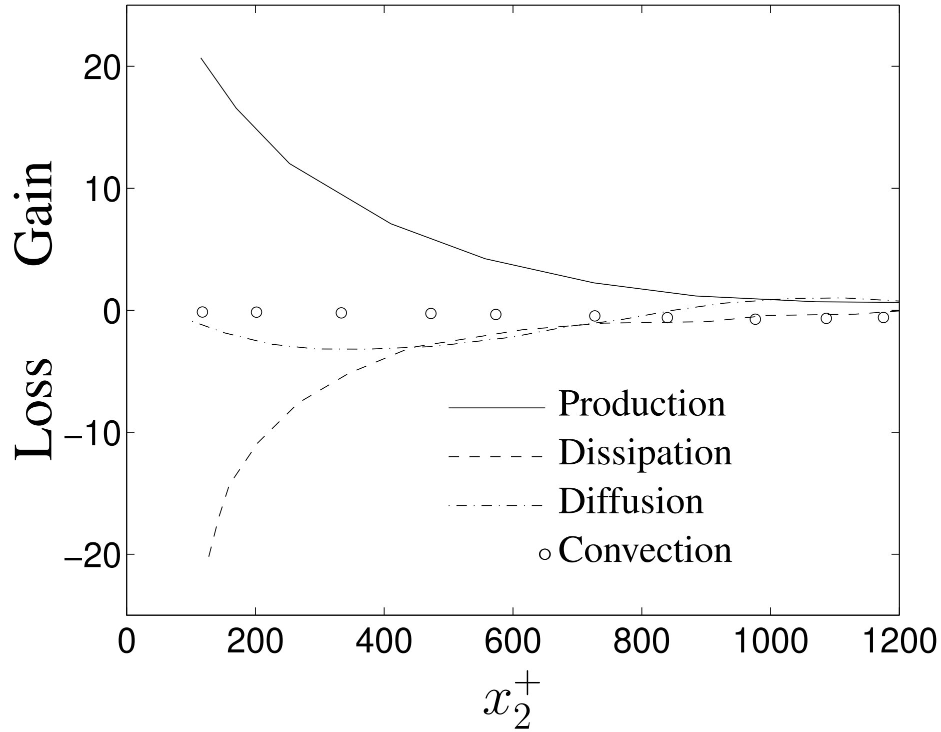
Friction velocity is computed as

$$u_\tau = \frac{\kappa \bar{v}_{1,P}}{\ln(E u_\tau \delta x_2/\nu)}$$

►subscript P denotes the wall-adjacent cell ►It is obtained by iteration. ►Finally

$$\tau_w = \rho u_\tau^2 \quad \text{is used a force wall boundary condition.} \tag{39.1}$$

►B.c. for k.



Boundary along a flat plate. Energy balance in k equation.

►In the log-region, production and dissipation in the k eq. are large. The k eq. reads

$$0 = P^k - \rho\varepsilon = -\rho\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2} - \rho\varepsilon \quad \Rightarrow \quad 0 = -\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2} - \varepsilon \tag{39.2}$$

In the log-region

$$\frac{\tau_w}{\rho} = -\overline{v_1'v_2'} = \nu_t\frac{\partial \bar{v}_1}{\partial x_2} \quad \Rightarrow \frac{\partial \bar{v}_1}{\partial x_2} = \frac{-\overline{v_1'v_2'}}{\nu_t} \tag{39.3}$$

Inserting Eq. 39.3 into Eq. 39.2 gives

$$0 = \frac{\overline{v_1'v_2'}^2}{\nu_t} - \varepsilon = \frac{u_\tau^4}{\nu_t} - \varepsilon \quad \text{with} \quad \nu_t = C_\mu k^2/\varepsilon \quad \Rightarrow \quad k_P = C_\mu^{-1/2}u_\tau^2 \tag{39.4}$$

▶B.c. for ε.

▶Velocity gradient in the log-region: when deriving the log-law we assumed

$\partial \bar{v}_1/\partial x_2 \simeq u_\tau/(\kappa x_2)$

▶Shear stress in log-region $-\overline{v_1'v_2'} \simeq u_\tau^2$

▶The production term reads $P^k = -\overline{v_1'v_2'}\dfrac{\partial \bar{v}_1}{\partial x_2} = \dfrac{u_\tau^3}{\kappa \delta x_2}$

Eq. 39.2 gives

$$\varepsilon_P = P^k = \frac{u_\tau^3}{\kappa \delta x_2}$$

▶Note that both the k and ε b.c. are applied at the wall-adjacent cells in the interior domain.

▶Hence, formally they are not b.c.

▶The b.c. for the wall-parallel velocity is expressed as a wall shear stress

¶See Section 11.14.2, Low-Re Number Turbulence Models

►In Low-Re number models we resolve the boundary layer, i.e. we use a refined grid near the wall.

►First (wall-adjacent) at $x_2^+ < 1$ ►B.c. $v_i = 0$.

►The turbulence near the wall is not fully turbulent: ►the viscous effect is large.
$\Rightarrow$ We must modify the turbulence model.

►Analyze the turbulence near the wall. ►Taylor expansion of v_i' near the wall (also valid for $\bar{v}_i$) gives

$$\begin{aligned} v_1' &= a_0 + a_1 x_2 + a_2 x_2^2 + \ldots \\ v_2' &= b_0 + b_1 x_2 + b_2 x_2^2 + \ldots \\ v_3' &= c_0 + c_1 x_2 + c_2 x_2^2 + \ldots \end{aligned} \tag{39.5}$$

►At the wall: $v_1' = v_2' = v_3' = 0$ which gives $a_0 = b_0 = c_0$.
►Furthermore $\partial v_1'/\partial x_1 = \partial v_3'/\partial x_3 = 0$: ►continuity eq. gives $\partial v_2'/\partial x_2 = 0$ $\Rightarrow$ $b_1 = 0$.

Equation 39.5 now reads

$$\begin{aligned} v_1' &= a_1 x_2 + a_2 x_2^2 + \ldots \\ v_2' &= b_2 x_2^2 + \ldots \\ v_3' &= c_1 x_2 + c_2 x_2^2 + \ldots \end{aligned} \tag{39.6}$$

$$
\begin{aligned}
v_1' &= a_1 x_2 + a_2 x_2^2 + \dots \\
v_2' &= \phantom{a_1 x_2 +{}} b_2 x_2^2 + \dots \\
v_3' &= c_1 x_2 + c_2 x_2^2 + \dots
\end{aligned}
\tag{39.6}
$$

►Using Eq. 39.6 we can write

$$
\begin{aligned}
\overline{v_1'^2} &= \overline{a_1^2} x_2^2 + \dots && = \mathcal{O}(x_2^2) \\
\overline{v_2'^2} &= \overline{b_2^2} x_2^4 + \dots && = \mathcal{O}(x_2^4) \\
\overline{v_3'^2} &= \overline{c_1^2} x_2^2 + \dots && = \mathcal{O}(x_2^2) \\
\overline{v_1' v_2'} &= \overline{a_1 b_2} x_2^3 + \dots && = \mathcal{O}(x_2^3) \\
k &= \frac{1}{2}(\overline{a_1^2 + c_1^2}) x_2^2 + \dots && = \mathcal{O}(x_2^2) \\
\partial \bar{v}_1 / \partial x_2 &= \overline{a_1} + \dots && = \mathcal{O}(x_2^0) \\
\partial v_1' / \partial x_2 &= a_1 + \dots && = \mathcal{O}(x_2^0) \\
\partial v_2' / \partial x_2 &= 2 b_2 x_2 + \dots && = \mathcal{O}(x_2^1) \\
\partial v_3' / \partial x_2 &= c_1 + \dots && = \mathcal{O}(x_2^0) \\
\varepsilon &\propto \overline{\frac{\partial v_1'}{\partial x_2}\frac{\partial v_1'}{\partial x_2}} + \overline{\frac{\partial v_2'}{\partial x_2}\frac{\partial v_2'}{\partial x_2}} + \overline{\frac{\partial v_3'}{\partial x_2}\frac{\partial v_3'}{\partial x_2}} = \mathcal{O}(x_2^0) + \mathcal{O}(x_2^2) + \mathcal{O}(x_2^0) && = \mathcal{O}(x_2^0) \\
\nu_t &= C_\mu \frac{k^2}{\varepsilon} \propto \frac{\mathcal{O}(x_2^4)}{\mathcal{O}(x_2^0)} && = \mathcal{O}(x_2^4)
\end{aligned}
\tag{39.7}
$$

¶See Section 11.14.3, Low-Re $k-\varepsilon$ Models

▶Now let's compare the exact and the modeled k equation near the wall

▶The exact k equation (see Eq. 37.8)

$$\rho\bar{v}_1\frac{\partial k}{\partial x_1}+\rho\bar{v}_2\frac{\partial k}{\partial x_2}=\underbrace{-\rho\overline{v_1'v_2'}\frac{\partial\bar{v}_1}{\partial x_2}}_{\mathcal{O}(x_2^3)}-\frac{\partial\overline{p'v_2'}}{\partial x_2}\underbrace{-\frac{\partial}{\partial x_2}\left(\frac{1}{2}\rho\overline{v_2'v_i'v_i'}\right)}_{\mathcal{O}(x_2^3)}+\underbrace{\mu\frac{\partial^2 k}{\partial x_2^2}}_{\mathcal{O}(x_2^0)}-\underbrace{\mu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\mathcal{O}(x_2^0)}$$

The modeled k equation (see Eq. 38.2)

$$\rho\bar{v}_1\frac{\partial k}{\partial x_1}+\rho\bar{v}_2\frac{\partial k}{\partial x_2}=\underbrace{\mu_t\left(\frac{\partial\bar{v}_1}{\partial x_2}\right)^2}_{\mathcal{O}(x_2^4)}+\underbrace{\frac{\partial}{\partial x_2}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_2}\right)}_{\mathcal{O}(x_2^4)}+\underbrace{\mu\frac{\partial^2 k}{\partial x_2^2}}_{\mathcal{O}(x_2^0)}-\underbrace{\rho\varepsilon}_{\mathcal{O}(x_2^0)}$$

▶the exact and the modeled dissipation term behave in the same way

▶this is not true for the production term and the turbulent diffusion term

▶To make the modeled production term behave as $\mathcal{O}(x_2^3)$:

replace C_μ with $C_\mu f_\mu$ (damping function) where $f_\mu \propto \mathcal{O}(x_2^{-1})$ e.g. $f_\mu = 1/x_2^+$ or as in Assignment 2

▶$C_\mu \rightarrow C_\mu f_\mu$ also fixes the modeled turb. diffusion term

▶Now we look at the modeled ε eq. (see Eq. 38.3) ▶$C_\mu \rightarrow C_\mu f_\mu \quad \Rightarrow \quad \mu_t \propto \mathcal{O}(x_2^3)$

$$\underbrace{\rho\bar{v}_1\frac{\partial\varepsilon}{\partial x_1}}_{\mathcal{O}(x_2^1)}+\underbrace{\rho\bar{v}_2\frac{\partial\varepsilon}{\partial x_2}}_{\mathcal{O}(x_2^2)}=\underbrace{C_{\varepsilon 1}\frac{\varepsilon}{k}P^k}_{\mathcal{O}(x_2^1)}+\underbrace{\frac{\partial}{\partial x_2}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_2}\right)}_{\mathcal{O}(x_2^2)}+\underbrace{\mu\frac{\partial^2\varepsilon}{\partial x_2^2}}_{\mathcal{O}(x_2^0)}-\underbrace{C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}}_{\mathcal{O}(x_2^{-2})}$$

▶Modification of turbulent viscosity modifies both production and turbulent diffusion

▶Terms that are non-zero when $x_2 \rightarrow 0$: ▶the viscous diffusion term and the destruction term.

▶But they can't balance each other: ▶the first is $\propto \mathcal{O}(x_2^0)$ and the second $\propto \mathcal{O}(x_2^{-2})$.

▶We fix this by multiplying the destruction term by $f_2 \propto \mathcal{O}(x_2^2)$

▶Suitable form of f_2? ▶$f_2 = \min\left(x^{+2}, 1\right)$ ▶better: $f_2 = (1 - \exp(-x_2^+))^2$

$$\text{Taylor expansion gives } f_2 = \left(1 - \underbrace{(1 - x_2^+ + x_2^{+2}\ldots)}_{\exp(-x_2^+)}\right)^2 = \left(x_2^+ - x_2^{+2}\ldots\right)^2$$

$$= x_2^{+2} - 2x_2^{+3} + x_2^{+4}\ldots = \mathcal{O}(x_2^2)$$

¶See Section 11.14.5, Different ways of prescribing ε at or near the wall

▶Boundary condition for k (since $v_i' \to 0$ near the wall)

$$k = 0$$

The exact form of k equation for boundary layer flow

$$\bar{v}_1 \frac{\partial k}{\partial x_1} + \bar{v}_2 \frac{\partial k}{\partial x_2} = -\overline{v_1' v_2'} \frac{\partial \bar{v}_1}{\partial x_2} - \frac{\partial}{\partial x_2}\left[\frac{1}{\rho}\overline{p' v_2'} + \frac{1}{2}\overline{v_2' v_i' v_i'} - \nu \frac{\partial k}{\partial x_2}\right] - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶Boundary condition for No I for ε: ▶look at the k eq. near the wall.

▶The only non-vanishing terms are

$$0 = \mu \frac{\partial^2 k}{\partial x_2^2} - \rho\varepsilon \quad \Rightarrow \varepsilon_{wall} = \nu \frac{\partial^2 k}{\partial x_2^2} \tag{39.8}$$

▶Eq. 39.8 can be used as boundary conditions of ε. Not good. It includes a 2nd derivative of k

▶Let's try somthing else. ▶Exact form of the ε term near wall (see Eq. 39.7):

$$\varepsilon = \nu \left\{\overline{\left(\frac{\partial v_1'}{\partial x_2}\right)^2 + \left(\frac{\partial v_2'}{\partial x_2}\right)^2 + \left(\frac{\partial v_3'}{\partial x_2}\right)^2}\right\} = \nu \left\{\overline{\left(\frac{\partial v_1'}{\partial x_2}\right)^2 + \left(\frac{\partial v_3'}{\partial x_2}\right)^2}\right\}$$

where we have assumed: $\partial/\partial x_2 \gg \partial/\partial x_1 \simeq \partial/\partial x_3$ and $\partial v_1'/\partial x_2 \simeq \partial v_3'/\partial x_2 \gg \partial v_2'/\partial x_2$.

$$\varepsilon = \nu \left\{ \overline{\left(\frac{\partial v_1'}{\partial x_2}\right)^2 + \left(\frac{\partial v_3'}{\partial x_2}\right)^2} \right\}$$

►Taylor expansion gives (see Eq. 39.7)

$$\varepsilon = \nu \left(\overline{a_1^2} + \overline{c_1^2}\right) + \ldots \qquad (39.9)$$

The turbulent kinetic energy (see Eq. 39.7)

$$k = \frac{1}{2} \left(\overline{a_1^2} + \overline{c_1^2}\right) x_2^2 + \ldots \qquad (39.10)$$

so that

$$\left(\frac{\partial \sqrt{k}}{\partial x_2}\right)^2 = \frac{1}{2} \left(\overline{a_1^2} + \overline{c_1^2}\right) + \ldots \qquad (39.11)$$

Eqs. 39.9 and 39.11 gives

$$\varepsilon_{wall} = 2\nu \left(\frac{\partial \sqrt{k}}{\partial x_2}\right)^2 .$$

►This is b.c. No II for ε It also includes a derivative of k. Not so good ...

$$\varepsilon = \nu \left(\overline{a_1^2} + \overline{c_1^2} \right) + \dots \tag{39.9}$$

$$k = \frac{1}{2} \left(\overline{a_1^2} + \overline{c_1^2} \right) x_2^2 + \dots \tag{39.10}$$

►Often the following boundary condition is used

$$\varepsilon_{wall} = \left(\frac{2\nu k}{x_2^2} \right) \tag{39.12}$$

Comparing Eqs. 39.10 and 39.12 we see that Eq. 39.12 is satisfied.

►This is b.c. No III for ε

- **Summary of the low-Re number model.**

  - Fine mesh near the wall. The first cell center is located at $x_2^+ \lesssim 1$.
  - This means that standard wall b.c. can be used, i.e. $\bar{v}_1 = \bar{v}_2 = \bar{v}_3 = k = 0$.
  - There are three different (I-III) options for the wall b.c. for ε:
    usually $\varepsilon_P = 2\nu k_P/(x_2^2)$ is used for the wall-adjacent cells

- **Summary of wall-functions.**

  - Coarse mesh near the wall. The first cell center is located at $30 \lesssim x_2^+ \lesssim 400$.
    It is located in the log region.
  - Friction velocity, u_τ, computed from the log-law.
  - A shear stress b.c. is used for the wall-parallel velocity component: $\tau_w = \rho u_\tau^2$
  - In the log-region we know that the k equation can be simplified as $0 = P^k - \varepsilon$
    $\Rightarrow k_P = C_\mu^{-1/2} u_\tau^2$ (k_P is prescribed for the wall-adjacent cells)
  - We use the simplified k equation also for ε: $0 = P^k - \varepsilon$ gives

    $$0 = -\overline{v_1' v_2'}\frac{\partial \bar{v}_1}{\partial x_2} - \varepsilon \quad \Rightarrow 0 = -u_\tau^2 \frac{u_\tau}{\kappa x_2} - \varepsilon \quad \Rightarrow \varepsilon_P = \frac{u_\tau^3}{\kappa x_2}$$

    ε_P is prescribed for the wall-adjacent cells