Wake Conference June 8-9, 2011; Gotland University, Visby, Sweden

A Contribution to Wind Turbine Wake Dynamics Algorithms

Björn Montgomerie, Division of Fluid Dynamics, Dept. of Applied Mechanics
Chalmers University of Technology SE-412 96
Gothenburg, Sweden, May 2011

1. Introduction

Wind turbines are subject to aerodynamic forces and gravity. Both contribute to the external loading of the total structure. Only the aerodynamic aspects of external loading, on the blades of the rotor, are treated in this paper. More specifically the influence, of a wake characteristic, is addressed.

In wind turbine aerodynamics the loads on the blades are partly dictated by the makeup of the wake. This has dominantly been treated in models using the Blade Element Momentum (BEM) method, where the blades are represented by divisions of the blade into blade elements. An important aspect in the method is the reduction of the undisturbed natural wind at the rotor plane.

There is, however, one competing method, where concentrated vortices from the blades are modeled as active agents causing the reduction of the inflow velocity into the rotor plane. This method is also based on blade elements. This combination of methods can suitably be called the Blade Element Induction (BEI) method.

In the BEI model the word “induction” is used to describe the fact that, where a vortex exists in a fluid medium, there is a velocity field associated with it. If the strength of the vortex (circulation) and the details of its geometry are known the velocity field can be calculated. It is furthermore assumed that the velocity, caused by the vortices, can be superimposed upon the natural wind velocity field.

The technique used is to apply the Biot-Savart law (BS) to relate the vorticity to the velocity field. In practical application the induced velocity is calculated at any point in 3D space, e.g. on the rotor blade. But, it must be emphasized that the form of the participating vortices and the vortex filament core thickness must be correctly modeled, otherwise an error in the wake shape will reflect in an error in the blade load.

Vortices normally trail from the tips and the root of the blades. Since the blades rotate and the wind makes them drift downstream they obtain a spiral shape – one per blade. The geometry of the vortices can therefore be characterized in terms of four quantities. They are initial radial starting position on the blade, radius, pitch, and the ratio spiralCurvatureRadius/vortexCoreRadius. The latter ratio is usually ignored by most authors of papers on vortex wakes. Its influence is described in this report.

As mentioned above the velocity field can be calculated using the BS law only. There is, however, an exception. If a point of evaluation of the velocity, associated with the vortex, is chosen to be on the vortex itself, i.e. on the center of the vortex core, the BS velocity there, together with the free stream wind velocity, might be thought of as also being the transport velocity of the vortex in that point. That is not true. In fact the BS velocity contribution at that point is only about half of the true vortex core transport velocity in the case of a vortex ring. In an actual wind turbine vortex wake there is a corresponding effect. The cure of the deficiency, of using the BS method for induction alone, is at the core of the present investigation.

The report is thus focused on the additional correctional speed vector, which can be used to supplement the BS 3D velocity, such that a wake model mimics the true behavior of the curved vortex lines.

2. Numerical investigation of BS induction on a vortex ring

One case of vortex curvature influence is on the vortex ring, which will be used as a work bench in modeling a contribution to a new emerging 3D performance method. It is advantageous that vortex rings have been exhaustively studied by several authors. This gives formulas and a solid background from which to step up to the more difficult task of mastering the arbitrary vortex spiral. The basic formulas available for the velocity of the ring are simple. They do, however, depend on the so far unknown vortex core thickness.

Two-dimensional Vortex

The two-dimensional mathematical vortex provides a base for representing a physical vortex. However, as the distance to the center goes to zero the velocity goes to infinity. This is of course not the case in physical reality, so a modification to the original mathematical expression is called for. Several curve fit expressions from the literature on the topic, can be used for the purpose to mimic the physical behavior of the velocity near and in a vortex core. Below a simple sample from the literature was arbitrarily chosen. The simple formula (1), orange in the figure, is the mathematical 2D vortex velocity function.

$$v = \frac{\Gamma}{2\pi r} \tag{1}$$

$$v = \frac{\Gamma r}{2\pi (r^2 + r_c^2)} \tag{2}$$

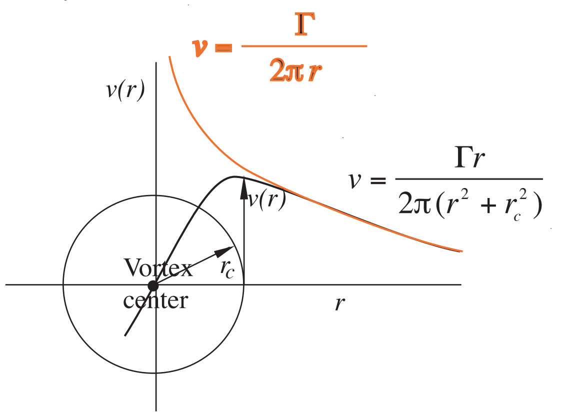
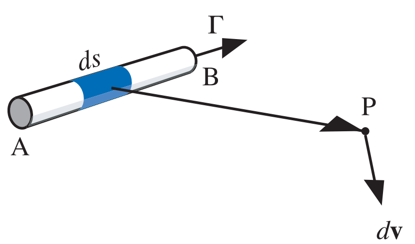


Figure 1

It is known from experiments that the velocity in the center of a vortex is zero. As can be seen the replacement of 1/*r,* for the slightly more complex expression (2), changes the curve to resemble known physical behavior. This arbitrariness will not become a deterrent to the further development of a future 3D program, including this method, because adjustments from measurement results will be used to set the future program "knobs".

Three-dimensional Vortex

In three dimensions the induced velocity **v** can be obtained from the integral in vector form below. An integral from A to B will give the total influence from the straight line segment depicted.



$$\mathbf{v} = \frac{1}{4\pi}\int \frac{\mathbf{\Gamma} \times \mathbf{p}}{|\mathbf{p}|^3} ds \tag{3}$$

Figure 2

The figure shows a straight line vortex element of finite length. For practical application it is advantageous to imagine the vortex segment and the point of induction evaluation (P) to define a plane. If this plane is identical with the plane of the present page the figure can be sketched as a 2D view seen in the following figure.

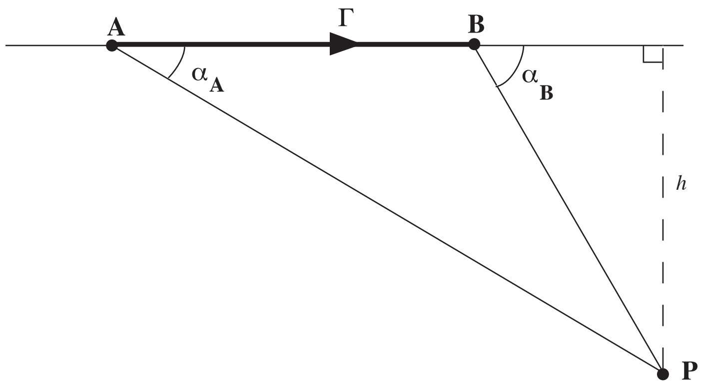


Figure 3

The induced velocity at point P is directed away from the reader, i.e. perpendicular to the plane of the paper and down through the paper. Had **Γ** been directed to the left the induced velocity would have been pointing up likewise.

The integral in the previous figure can easily be analytically evaluated using the angles at the vortex segment from A to B. The expression for the size of v at P then becomes

$$v = \frac{\Gamma}{4\pi h}\left(\cos\alpha_A - \cos\alpha_B\right) \qquad (4)$$

However, in order to implement the idea from the 2D case above this expression can be modified in a similar way, as follows. The expanded expression accommodates the finite core radius r_c.

$$v = \frac{\Gamma h}{4\pi\left(h^2 + r_c^2\right)}\left(\cos\alpha_A - \cos\alpha_B\right) \qquad (5)$$

This function for v is most useful for programming purposes. The vector 3D mechanism is then handled separately. Since the size of the velocity is known from Eq. (5) only a 3D unit vector is needed to represent the induced velocity in space. The technique consists of the following steps from basic vector algebra.

1. Create the cross product **u**= (**B** – **A**)x(**P** – **A**) or (**B**-**A**)x(**P**-**B**) … does not matter
2. Create the unit vector of **u** and call it **e**.
3. The induced velocity vector = v**e**.

3. Computer code "singleRingInduc"

The ideas described in the previous section lead to the development of a computer code for evaluation of the consequences of the model. The resulting application was called singleRingInduc. It calculates the induced velocity in the middle of a vortex ring and on the ring itself. This author initially thought that the latter velocity would be the transport velocity of the ring. So, it seems, is still the thinking of many other authors. The velocity from singleRingInduc is calculated from Eq. (5), where the formula is used for each inducing segment in a train of segments approximating the ring. Each segment gives a velocity contribution to the same (arbitrary) point on the ring. Such a point is characterized by preferably being at one of the connection points between two straight segments.

One important test, which the program passed during checking, is that the induction, calculated at the center of the ring, should converge rapidly toward 0.5 as the number of segments is increased if the core radius = 0, and the ring radius and Γ = 1.0. This is also seen directly in Eq. (3) since **Γ** and **p** are perpendicular. Consequently the cross product is the product of the vector sizes multiplied by sin(90deg)=1. The radius r is constant r = p and Γ is constant and can be taken outside of the integration. The remaining integral then just leads to the circumference of the circle =2πr, whence the result, for the velocity at the ring center, is 0.5 as the numerical program output below shows.

On the circle the velocity should be compared to the velocity as offered in the literature. The original formula due to Kelvin, see Ref. 1, has the following form

$$ringSpeed = \frac{\Gamma}{4\pi R}\left[\ln\left(\frac{8R}{r}\right) - \frac{1}{4}\right] \qquad (6)$$

where r is the core radius of the assumed circular cross section of the vortex core. By comparison the Kelvin formula results in a velocity, which is about twice as large compared with the result from integration of the BS formula according to the singleRingInduc result.

From the experience with the singleRingInduc application of the BS integration is seen to be finite even as the inducing segment approaches the point of evaluation on a node. This can also be concluded if the implications from the effect of Eq. (5) are pondered. This stable behavior will be necessary when the method is to be applied to a vortex spiral. A view of what happens, when the integration steps approach the point of evaluation, is enlightening in this context.

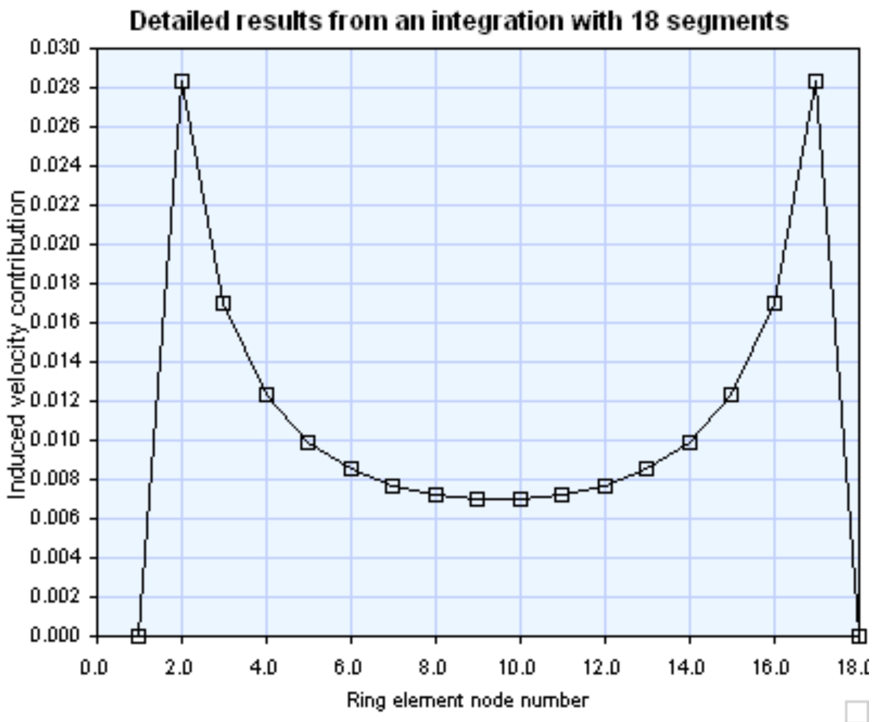


Figure 4

Each data point in Fig. 4 represents the induced velocity contribution from the segment, whose number is seen on the x axis. The striking way points 1 and 18 differ from all other points is explained by the fact that the point of evaluation is located on node point 1 of segment 1, which is the same as node point 19 of segment 18 (corresponds to segment n in Fig. 5). I.e. node 1 is located on the axes of both segment 1 and segment 18.
Besides the induction is calculated from Eq. (2). The induction contributions are therefore = 0, see Eq. (2), Fig. 1 and Fig. 5.

The integrated velocities for various values of the core radius are plotted in Fig. 6. In future applications, to spiral shaped vortex filaments, the term d**V**_Kelvin will be a given function, from the present study, which is to be used as follows, where the corrected speed (**V**corr):

Vcorr = **V**BS + d**V**_Kelvin (7)

Different from the ring study is that Eq. (7) should be interpreted as a vector equation, because **V**BS and d**V**_Kelvin will generally have different direction in the context of an arbitrary vortex wake.

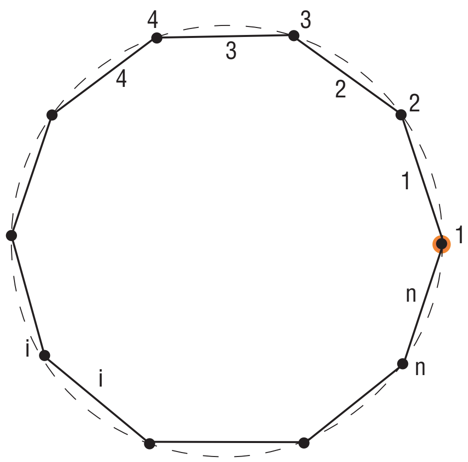


Figure 5

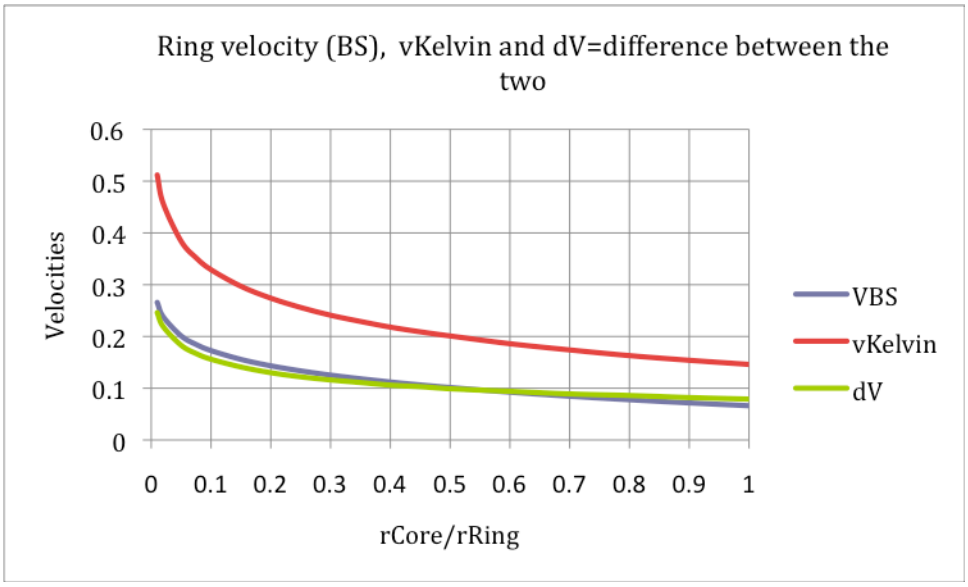


Figure 6

As it occurs the curvature correction term (dV_Kelvin) happens to be almost identical to the velocity curve as calculated by the singleRingInduc program. The significance of the two curves being near equal is that the Biot-Savart integration gives only one half of the transportation speed contribution in the case of the vortex ring. The other half is concluded to come from some *curvature* effect.

Curve fits to the dV_Kelvin curve can be created for the continued development of the method. The ring velocity, as calculated, is then to be scaled by the ratio Γ/R. As a nice consequence no curve fitting is needed to accommodate also Γ and R, which would have complicated the curve fitting chore considerably.

Below is one example of the output from the singleRingInduc program. In fact there is an expression due to Moore, which includes the means to express the ring speed when the vortex core has an elliptic cross section. It was in fact this formula that was implemented in the code. This explains why the input variables `a` and `b` emerge in this context. They are the elliptic half axes in Moore's formula, see Ref. 4. In this whole study `a` was set equal to `b`. The consequence is that the Moore formula yields exactly the same result as the Kelvin formula.

```
INPUT:
 Ring radius =  1.000
 Core radius =  0.030
 Core axis elliptic a =  0.030
 Core axis elliptic b =  0.030
 Circulation =  1.000
 Point of velocity evaluation on node point. (kind=3 recommended)
 Required ring velocity accuracy level (%) = 1.000

OUTPUT:
 Ring velocity according to Moore = 0.425
 Number of segments required to satisfy the required accuracy =  33
 --------------------------------
   Nr of seg.  Vcenter    Vring
 --------------------------------
        3     0.8240     0.0530
        5     0.5774     0.0983
        7     0.5359     0.1254
       10     0.5166     0.1535
       14     0.5081     0.1790
       18     0.5047     0.1965
       27     0.5018     0.2168
       36     0.5008     0.2214
       72     0.4999     0.2218
      120     0.4997     0.2218
      180     0.4996     0.2218
      270     0.4996     0.2218
      360     0.4996     0.2218
 --------------------------------
 Velocity term to add =  0.203
```

Explanation of the singleRingInduc input:

In line number 2, at the input, the core radius is set equal to 0.03, while the ring radius is 1.0 throughout the calculations. The elliptic quantities a and b are part of Eq. Moore's formula. Thus only the Moore formula makes use of `a` and `b`. The program has three possibilities, called kind, for selection of point of speed evaluation. The kinds are:

1. On circle near mid-point of segment
2. On mid-point of segment
3. On node point (the preferred choice)

The program also outputs the minimum number of ring elements to be used.
This minimum meets a criterion, which is constructed in the following manner. – The last Vring value, seen in the output above, is considered the "best", because it is based on the highest number of segments. This should approximate the circle in the best way. But, it can be observed that several Vring values, before the last line with 360 segments, have already reached the best value (using four decimals). After having made this observation it is tempting to speculate about the accuracy versus number of chosen segments. But, it is necessary first to define accuracy. The following example can pass as a definition for this particular purpose in this particular context.

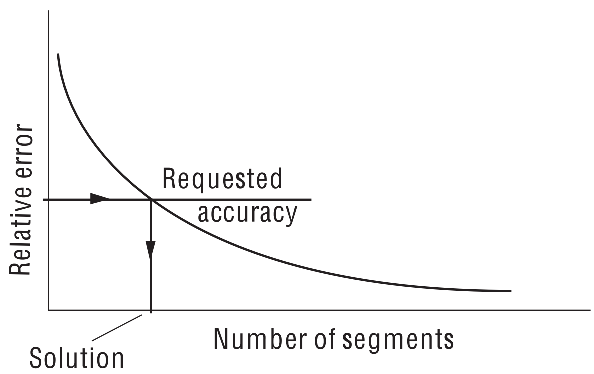


Figure 7

Error(Nr segments) = Vring(Nr segments) - Vring(360 segments)

Relative error = Error(Nr segments)/ Vring(360 segments)

In the table of the output above a column of relative error values could have been added. Then a diagram, like the one sketched in Fig. 7, could be plotted and used as the two arrows indicate. The requested *accuracy*, seen in the diagram, is equivalent with requested *error*. The solution will generally be a floating point number. But, the nearest higher integer number was used as an output from the process. This idea is implemented in the code of singleRingInduc. Therefore, the "minimum number of segments", necessary to satisfy an arbitrary number for relative error (relative accuracy), is generated and outputted from singleRingInduc.

Every such analysis was thus carried out for every chosen set of input. The program was run systematically varying the core thickness only. Ring radius, circulation, selector type (kind) for choice of evaluation point and accuracy/error request were held constant. Collecting the data from several such runs gave rise to the following table.

Table 1

Ring radius = 1 Circul = 1 kind = 3 req accuracy = 0.01

coreRadius	VBS	vMoore	dV_Moore	Min nr segm
0.01	0.266	0.512	0.246	67
0.02	0.238	0.457	0.219	42
0.05	0.201	0.384	0.183	33
0.075	0.185	0.352	0.167	27
0.1	0.173	0.329	0.156	18
0.15	0.156	0.297	0.141	18
0.2	0.143	0.274	0.13	17
0.25	0.134	0.256	0.122	15
0.3	0.125	0.241	0.116	13
0.35	0.118	0.229	0.111	10
0.4	0.112	0.218	0.106	10
0.45	0.106	0.209	0.103	10
0.5	0.101	0.201	0.099	10
0.6	0.092	0.186	0.094	10
0.7	0.085	0.174	0.089	10
0.8	0.078	0.163	0.086	11
0.9	0.072	0.154	0.082	12
1	0.066	0.146	0.079	13

Using the left and the right columns the following plot was generated.

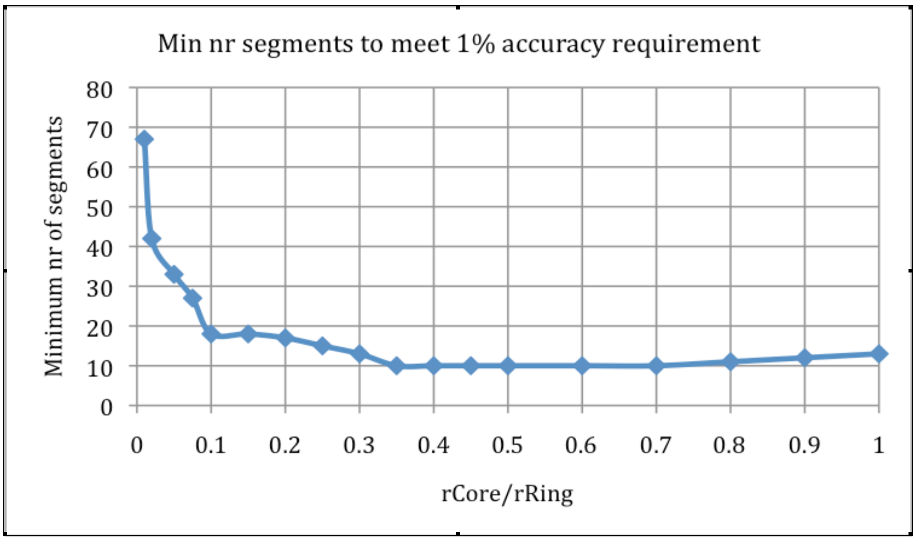


Figure 8

That information can be used in an aeroelastic wind turbine code when the time step is to be inputted. This is because the time step has a relation to the azimuth angle of the rotor blade. The requirements seem to be rather lenient on number of necessary segments, despite the rather harsh requirement chosen (1% relative error). Behind a wind turbine the minimum core of the tip vortex is probably around 4%. This would require about 36 segments, which corresponds to 10° of azimuthal travel. This is probably needed considering other requirements such as resolution to accommodate gusting and structural vibrations.

The described exercise could be repeated for say 2% error etc in order to form a database for minimum number of segments.

4. Application to an arbitrary 3D vortex

The numerical exercises, described above, show that the curvature effect can be obtained as the difference between the Kelvin ring velocity and a value predicted from a BS integration. Furthermore, the curvature effect is a function of curvature only. Curvature in this context is to be interpreted to be r_c/R, or the inverted value. It so happens that the vectors of the BS velocity contribution and the Kelvin velocity have the same direction. In the more general case, such as a wind turbine wake vortex, the curvature velocity contribution will not have the same direction as that obtained from a BS integration. The velocity direction, resulting from the BS integration, is an inherent outcome of the method and no particular additional statement need be considered for this contribution. The curvature contribution, however, must be given by a dedicated rule. Taking the support from the vortex ring situation the following thoughts present themselves.

Suppose that the local curvature velocity contribution, on the ring, has a component, which is directed outward or inward. Then the ring radius would grow or shrink. It does not. Therefore it can be concluded that the curvature induction direction is normal to the plane of the ring. Extending the thought to a small arc of the ring it must be that this small arc also creates its curvature induced velocity perpendicular to the plane of the arc. Considering this as a fact it is not far fetched to assume that the local curvature of any curved vortex will induce a velocity on itself, which is perpendicular to its plane.

In circumstances where the vortex is modeled from a sequence of straight segments these segments touch each other in a special point, which can be called a “node”. An application of the above conclusion is now straightforward. The two segments, being equivalent with three adjacent nodes, provide a basis for calculating two important quantities. One is a normal to the plane of the segments (or the three nodes). Its unit vector is also needed and it gives the 3D direction of the curvature velocity contribution. The other is the radius of curvature (R). The curvature term is only a look-up number from an existing table, where the curvature (r_c/R or R/r_c) is the input. The number obtained is then multiplied by the ratio Γ/R. After calculating the BS contribution at the node the total velocity of the node is obtained from a summation of the vectors for wind speed, BS velocity and the curvature velocity.

It is possible, in fact even likely, that two consecutive segments will have the same direction, i.e. the three nodes are in line. If so the proposed mechanism, using curve fits, must be circumvented, since R would be = infinity.

5. Assumptions for a simulated wind turbine vortex wake

A summary of key assumptions, as follows, constitutes the whole idea behind the method proposed in this document. This should be the basis for a vortex method for wind turbines.

1. The Biot-Savart (BS) law of induction is necessary but not sufficient.

2. The almost double velocity result from the Kelvin formula indicates that there is a special curvature effect, which the BS induction, using straight segments, cannot reproduce.

3. The complete method can rest on adding the curvature effect to the BS result.

4. The curvature effect can be obtained from the difference between the Kelvin velocity and the BS velocity, when both are applied to a circular vortex ring. This difference has a general validity (although tied to the choice of form for the Kelvin-like formula).

5. In any integration of induced velocities, e.g. in application to a trailing vortex spiral behind a wind turbine, a representation of the vortex curve must be a train of straight segments. A node is the connection point between segments. A node moves with a velocity affected by the vector sum of the free stream, the BS and the curvature contributions.

6. A recommendation for a minimum number of azimuthal divisions has been obtained in the study of the vortex ring. This can also be used to support the analyst who is preparing the input for a wind turbine simulation run. The minimum recommended number of segments, for a ring, will normally match the azimuthal division requirements satisfying other needs. These needs are typically contained in the resolution requirements to handle gusts and vibrations.

6. Conclusions

1. An important element of for wind turbine wake modeling has been established. It is partly derived from observed behaviors of a vortex ring. A generalization to arbitrarily formed vortex filaments has tentative support.

2. Computer code “knobs” must constitute the road to successful replication of natural vortex wake behavior. The knobs, which are foreseen at this stage, are a development of the vortex core size and ellipticity variations on which the findings from the vortex ring study depend. This work requires a workable vortex wake computer code and thorough numerical exploration.

3. More relevant bibliographical studies should be carried out in order to support, expand or cancel the conclusions of this report. In this effort experts also should be consulted.

7. References

1. “Hydrodynamics” – Sir Horace Lamb
Sixth edition first published 1932
ISBN 0 – 486 – 60256 – 7 (Now available in paperback from Dover Publ. and amazon.com)

Note: Reference 1 (includes the Kelvin vortex ring speed) is now easy to come by, while the following might be more difficult to obtain. It seems to be the original paper in which Kelvin presented the vortex ring velocity:

Kelvin, L.: The translatory velocity of a circular vortex ring. Phil. Mag. **33**, 511–512 (1867).

One of many papers that make reference to Kelvin’s ring speed is the following thoughtful report.

2. “On the motion of thin vortex tubes” - Anthony Leonard
Theor. Comput. Fluid Dyn. (2010) 24:369–375
DOI 10.1007/s00162-009-0126-5
ORIGINAL ARTICLE
Received: 6 January 2009 / Accepted: 2 June 2009 / Published online: 9 July 2009
© Springer-Verlag 2009

3. Leonard, A. 1985. “Computing three-dimensional incompressible flows with vortex elements.
Ann. Rev. Fluid Mech. 17, 5

4. The velocity of a vortex ring with a thin core of elliptical cross section
D. W. Moore, Deptmt of Mathematics, Imperial College, London
Proc. R. Soc. London, A370, 407-417 (1980)

The vortex ring velocity formula from the Moore report is

$$ringSpeed = \frac{\Gamma}{4\pi R}\left[\ln\left(\frac{16R}{a+b}\right) - \frac{1}{4}\right]$$