Tensors

(February 5, 2011)

The convection-diffusion equation for temperature reads

$$\frac{\partial}{\partial x}(\rho U T) + \frac{\partial}{\partial y}(\rho V T) + \frac{\partial}{\partial z}(\rho W T) = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right)$$

Using tensor notation it can be written as

$$\boxed{\frac{\partial}{\partial x_j}(\rho U_j T) = \frac{\partial}{\partial x_j}\left(\Gamma \frac{\partial T}{\partial x_j}\right)}$$

The Navier-Stokes equation reads (incompr. and $\mu = const.$)

$$\frac{\partial}{\partial x}(UU) + \frac{\partial}{\partial y}(VU) + \frac{\partial}{\partial z}(WU) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\frac{\partial}{\partial x}(UV) + \frac{\partial}{\partial y}(VV) + \frac{\partial}{\partial z}(WV) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

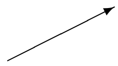
$$\frac{\partial}{\partial x}(UW) + \frac{\partial}{\partial y}(VW) + \frac{\partial}{\partial z}(WW) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

Using tensor notation it can be written as

$$\boxed{\frac{\partial}{\partial x_j}(U_j U_i) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} \qquad (62)}$$

a: A tensor of zeroth rank (scalar)

a_i: A tensor of first rank (vector)



$$a_i = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

a_{ij}: A tensor of second rank (tensor)

A common tensor in fluid mechanics (and solid mechanics) is the stress tensor σ_{ij}

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

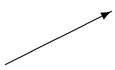
It is symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$. For fully, developed flow in a 2D channel (flow between infinite plates) σ_{ij} has the form:

$$\sigma_{12} = \sigma_{21} = \mu \frac{dU_1}{dx_2} (= \mu \frac{dU}{dy})$$

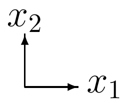
and the other components are zero. As indicated above, the coordinate directions (x_1, x_2, x_3) correspond to (x, y, z), and the velocity vector (U_1, U_2, U_3) corresponds to (U, V, W).

What is a tensor?

A tensor is a *physical* quantity. Consequently it is independent of which coordinate system. The tensor of rank one (vector) b_i below

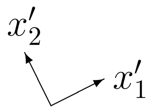


is physically the same whether expressed in the coordinate system (x_1, x_2)



where $b_i = (2, 1, 0)^T$

or in the coordinate system (x'_1, x'_2)



where $b_{i'} = (\sqrt{5}, 0, 0)^T$. The tensor is the same even if its *components* are different.

The stress tensor σ_{ij} is a physical quantity which expresses the stress experienced by the fluid (or the solid); this stress is the same irrespective of coordinate system.

Examples of equations using tensor notation

- Newton's second law

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F}$$

which on component form reads

$$\begin{aligned} m\frac{d^2x_1}{dt^2} &= F_1 \\ m\frac{d^2x_2}{dt^2} &= F_2 \\ m\frac{d^2x_3}{dt^2} &= F_3. \end{aligned} \tag{63}$$

On tensor notation:

$$m\frac{d^2x_i}{dt^2} = F_i$$

When an index appears once in each term (a free index) it indicates that the whole equation should be applied in each coordinate direction, cf. Eq. 63.

- Divergence $\nabla \cdot U = 0$

The equation above reads

$$\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} = 0 \;\Leftrightarrow\; \sum_{i=1}^{3} \frac{\partial U_i}{\partial x_i} = 0 \tag{64}$$

In tensor notation the following rule is introduced: if an index appears twice (a dummy index) within a term, we should apply summation over this index. Normally the summation is taken from 1 to 3 (the three coordinate directions). If our coordinate system is 2D, the summation goes, of course, only from 1 to 2.

Equation 64 is thus written as

$$\frac{\partial U_i}{\partial x_i} = 0. \tag{65}$$

Alternative notations for a derivative are $U_{i,i}$ or $\partial_i U_i$, so that Eq. 64 can be written as

$$U_{i,i} = 0 \;\text{ or }\; \partial_i U_i = 0$$

Note that, since the dummy index implies a summation over each term, it can be interchanged against any index, i.e.

$$\frac{\partial U_k}{\partial x_k} = 0.$$

is exactly the same equation as Eq. 65. Equation 62 can, for example, be written as

$$\frac{\partial}{\partial x_\ell}\left(U_\ell U_m\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x_m} + \nu \frac{\partial U_m^2}{\partial x_k \partial x_k}$$

where different dummy indices have been used (ℓ and k); this is perfectly correct, because the summation is carried out for each term separately. What is not allowed, however, it to choose the dummy index same as the free index, i.e. for the equation above we are not allowed to use m as a dummy index.

- The left-hand side of Navier-Stokes $U_\ell \partial U_m / \partial x_\ell$

For simplicity, let's assume 2D. The left-hand side of the equation above includes both a free index (m) and a dummy index (ℓ). Let's first write out the summation on component form so that

$$U_1 \frac{\partial U_m}{\partial x_1} + U_2 \frac{\partial U_m}{\partial x_2}.$$

The free index indicates that the equation should be written in each coordinate direction (x_1 and x_2 in this case, since we have assumed 2D flow), cf. Eq. 63, i.e.

$$\begin{aligned}&U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2}\\ &U_1 \frac{\partial U_2}{\partial x_1} + U_2 \frac{\partial U_2}{\partial x_2}\end{aligned}$$

Contraction

If two free indices are set equal, they are turned into dummy indices, and the rank of the tensor is decreased by two. This is called *contraction*.If the tensor equation

$$a_{ij} = b_j c d_i - f_{ij}$$

is contracted, the result is

$$a_{ii} = b_i c d_i - f_{ii}.$$

For a tensor of rank two, a_{ij}, contraction is simply summation of the diagonal elements, i.e. $a_{11} + a_{22} + a_{33}$.

Two Tensor Rules

• The summation rule

A summation over a dummy index corresponds to a scalar product or a divergence; it should not appear more than twice. The following expressions are not valid:

$$a_{kkk} = 0,\ a_{iik} b_{ij} = d_{kj},\ a_i b_i c_i = d$$

• Free Index

In an expression the free index (indices) must be the same in all terms The following expressions are not valid:

$$a_{ikk} = b_j,\ c_i a_i b_j = d_k,\ a_{ij} d_{jk} = c_{im}$$

Special Tensors

• Kroenecker's δ (identity tensor)

It is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Contraction of δ_{ij} yields

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

Another example of contraction can now be given. We have the expression for the turbulent stress tensor based on the Boussinesq hypothesis (see Section 2.2 in LD)

$$\rho\overline{u_i u_j} = -\mu_t \left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) + \frac{2}{3}\delta_{ij}\rho k. \tag{66}$$

Contraction gives

$$\rho\overline{u_i u_i} = -2\mu_t \frac{\partial \bar{U}_i}{\partial x_i} + \frac{2}{3}\delta_{ii}\rho k = -2\mu_t \frac{\partial \bar{U}_i}{\partial x_i} + 2\rho k.$$

For incompressible flow the first term on the right-hand side is zero (due to continuity) so that

$$\overline{u_i u_i} = 2k,$$

which actually is the definition of k. Thus Eq. 66 is valid upon contraction; this should always be the case. As can be seen, contraction of Eq. 66 corresponds simply to the sum of the diagonal components (elements 11, 22 & 33).

- Levi-Civita's ε_{ijk} (permutation tensor)

It is defined as

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } (i, j, k) \text{ are cyclic permutations of } (1, 2, 3) \\ 0 & \text{if at least two indices are equal} \\ -1 & \text{otherwise} \end{cases} \tag{67}$$

Examples:

$$\varepsilon_{123} = 1,\ \varepsilon_{132} = -1,\ \varepsilon_{113} = 0$$
$$\varepsilon_{312} = 1,\ \varepsilon_{321} = -1,\ \varepsilon_{233} = 0$$

Symmetric and antisymmetric tensors

A tensor a_{ij} is symmetric if $a_{ij} = a_{ji}$.

A tensor b_{ij} is antisymmetric if $b_{ij} = -b_{ji}$. It follows that for an antisymmetric tensor all diagonal components must be zero (for example, $b_{11} = -b_{11} \Rightarrow b_{11} = 0$).

The (inner) product of a symmetric and antisymmetric tensor is always zero. This can be shown as follows:

$$a_{ij}b_{ij} = a_{ji}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij},$$

where we first used the fact that $a_{ij} = a_{ji}$ (symmetric), then that $b_{ij} = -b_{ji}$ (antisymmetric), and finally we interchanged the indices i and j, since they are dummy indices. Thus the product must be zero.

This can of course also be shown be writing out $a_{ij}b_{ij}$ on component form, i.e.

$$a_{ij}b_{ij} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + \ldots + a_{32}b_{32} + a_{33}b_{33} = 0$$

By inserting

$$\begin{aligned}
&a_{12} = a_{21}, \quad a_{13} = a_{31}, \quad a_{23} = a_{32} \\
&b_{11} = b_{22} = b_{33} = 0 \\
&b_{12} = -b_{21}, \quad b_{13} = -b_{31}, \quad b_{23} = -b_{32}
\end{aligned}$$

the relation above, i.e. $a_{ij}b_{ij} = 0$, is verified.

Vector Product

The vector cross product

$$\vec{c} = \vec{a} \times \vec{b}$$

is on tensor notation written

$$c_i = \varepsilon_{ijk} a_j b_k. \tag{68}$$

This is easily shown by writing it on component form. Using Sarrus' rule we get

$$\vec{c} = \begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)^T$$

We find that the first component of Eq. 68 is

$$\begin{aligned} c_1 &= \varepsilon_{1jk} a_j b_k = \\ &= \varepsilon_{111} a_1 b_1 + \varepsilon_{112} a_1 b_2 + \varepsilon_{113} a_1 b_3 \\ &+ \varepsilon_{121} a_2 b_1 + \varepsilon_{122} a_2 b_2 + \varepsilon_{123} a_2 b_3 \\ &+ \varepsilon_{131} a_3 b_1 + \varepsilon_{132} a_3 b_2 + \varepsilon_{133} a_3 b_3 \\ &= \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2. \end{aligned}$$

Recall that ε_{ijk} is zero if any two indices are equal (see Eq. 67, p. 65).

Derivative Operations

- The derivative of a vector $\vec{B}$:

Tensor notation	Vector notation
$\dfrac{\partial B_i}{\partial x_j}$ or $B_{i,j}$	$grad(\vec{B})$ or $\nabla \vec{B}$

The result is a tensor of rank two (rank of B_i plus one)

- The gradient of a scalar a:

Tensor notation	Vector notation
$\dfrac{\partial a}{\partial x_j}$ or $a_{,j}$	$grad(a)$ or ∇a

The result is a *vector*.

• The divergence of a vector $\vec{B}$:

Tensor notation	Vector notation
$\dfrac{\partial B_j}{\partial x_j}$ or $B_{j,j}$	$div(\vec{B})$ or $\nabla \cdot \vec{B}$

The result is a *scalar*.

• The curl of a vector $\vec{B}$:

Tensor notation	Vector notation
$\varepsilon_{ijk}\dfrac{\partial B_k}{\partial x_j}$ or $\varepsilon_{ijk} B_{k,j}$	$rot(\vec{B})$ or $\nabla \times \vec{B}$

The result is a *vector*.

• The Laplace operator on a scalar a:

Tensor notation	Vector notation
$\dfrac{\partial^2 a}{\partial x_j \partial x_j}$ or $a_{,jj}$	$\nabla \cdot (\nabla a) = \nabla^2 a$

The result is a *scalar*.

Integral Formulas

Stokes theorem

$$\oint_C \vec{B} \cdot d\vec{x} = \int_S (\nabla \times \vec{B}) \cdot d\vec{S},$$

where the surface S is bounded by the line C. On tensor notation:

$$\oint_C B_i dx_i = \int_S \varepsilon_{ijk} B_{k,j} dS_i \text{ or } \int_S \varepsilon_{ijk} \frac{\partial B_k}{\partial x_j} dS_i.$$

Gauss theorem

$$\int_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B} dV,$$

where the volume V is bounded by the surface S. On tensor notation:

$$\int_S B_i dS_i = \int_V B_{i,i} dV \text{ or } \int_V \frac{\partial B_i}{\partial x_i} dV$$

Multiplication of tensors

Two tensor can be multiplied in two ways: either the number of free indices is reduced by two (inner product), or it is unchanged (outer product). The product

$$a_{ijk} b_{k\ell} = c_{ij\ell}$$

represents an inner product; the rank of the product is the sum of the rank of the two tensors $(3 + 2 = 5)$ on the left-hand side minus two $(5 - 3 = 2)$. An outer product between the two tensors reads

$$a_{ijk} b_{m\ell} = d_{ijk\ell m}.$$

Now the rank of the resulting tensor $d_{ijk\ell m}$ (rank 5) is the sum of the rank of the two tensors $(3 + 2 = 5)$.