On-line Lecture 1

¶See Section 11.1.1, Flow equations

▶Boussinesq approximation: density variation only in gravitation (buoyancy) term

$$\frac{\partial \rho_0 \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho_0 \bar{v}_i \bar{v}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \mu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \rho_0 \frac{\partial \overline{v_i' v_j'}}{\partial x_j} - \rho_0 \beta(\bar{\theta} - \theta_0) g_i$$

$\bar{p}$ is hydrodynamic pressure: $\rho f_i \rightarrow (\rho - \rho_0) g_i$

If we let density depend on pressure and temperature, differentiation gives

$$d\rho = \left(\frac{\partial \rho}{\partial \theta}\right)_p d\theta + \left(\frac{\partial \rho}{\partial p}\right)_\theta dp$$

Incompressible flow: $\Rightarrow \partial \rho / \partial p = 0$

$$\beta = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial \theta}\right)_p \Rightarrow$$

$$d\rho = -\rho_0 \beta d\theta \Rightarrow \rho - \rho_0 = -\beta \rho_0 (\theta - \theta_0)$$

$$\rho_0 f_i = (\rho - \rho_0) g_i = -\rho_0 \beta(\bar{\theta} - \theta_0) g_i$$

¶See Section 11.1.2, Temperature equation

▶Temperature equation

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_i \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

where $\alpha = k/(\rho c_p)$.

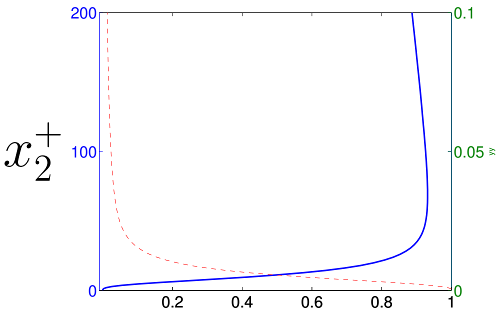
Introducing $\theta = \bar{\theta} + \theta'$ gives the mean temperature equation

$$\frac{\partial \bar{v}_i \bar{\theta}}{\partial x_i} = \alpha \frac{\partial^2 \bar{\theta}}{\partial x_i \partial x_i} - \frac{\partial \overline{v_i' \theta'}}{\partial x_i} \qquad (30.1)$$

▶Total (viscous plus turbulent) flux: momentum and temperature equation

$$-\frac{q_{2,tot}}{\rho c_p} = -\frac{q_{2,visc}}{\rho c_p} - \frac{q_{2,turb}}{\rho c_p} = \alpha\frac{\partial\bar{\theta}}{\partial x_2} - \overline{v_2'\theta'}, \quad \alpha = \frac{k}{\rho c_p}$$

$$\tau_{tot} = \tau_{visc} + \tau_{turb} = \mu\frac{\partial\bar{v}_1}{\partial x_2} - \rho\overline{v_1'v_2'}$$



Reynolds shear stress.

—: $-\rho\overline{v_1'v_2'}/\tau_w$

- - -: $\mu(\partial\bar{v}_1/\partial x_2)/\tau_w$.

¶See Section 11.2, The exact $\overline{v_i'v_j'}$ equation

- Set up the momentum equation for the instantaneous velocity $v_i = \bar{v}_i + v_i' \rightarrow$ Eq. (A)
- Time average $\rightarrow$ equation for $\bar{v}_i$, Eq. (B)
- Subtract Eq. (B) from Eq. (A) $\rightarrow$ equation for v_i', Eq. (C)
- Do the same procedure for $v_j \rightarrow$ equation for v_j', Eq. (D)
- Multiply Eq. (C) with v_j' and Eq. (D) with v_i', time average and add them together $\rightarrow$ equation for $\overline{v_i'v_j'}$

In Section 9 these steps are given in some detail.
The final $\overline{v_i'v_j'}$-equation (Reynolds Stress equation) reads (see Eq. 9.12)

▶ $\overline{v_i'v_j'}$-equation

$$\underbrace{\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)}}_{\Pi_{ij}}$$
$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_i'v_j'v_k'} + \frac{\overline{p'v_j'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu\frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k}}_{D_{ij,\nu}} \tag{30.2}$$
$$\underbrace{-g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}}_{G_{ij}} - \underbrace{2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

- Unknown terms

Π_{ij} Pressure-strain

$D_{ij,t}$ Turbulent diffusion

ε_{ij} Dissipation

¶See Section 11.3, The exact $\overline{v_i'\theta'}$ equation

▶ $\overline{v_i'\theta'}$ equation

$$\frac{\partial \theta'}{\partial t}+\frac{\partial}{\partial x_k}(v_k'\bar{\theta}+\bar{v}_k\theta'+v_k'\theta')=\alpha\frac{\partial^2\theta'}{\partial x_k\partial x_k}-\frac{\partial \overline{v_k'\theta'}}{\partial x_k} \tag{30.3}$$

$$\frac{\partial v_i'}{\partial t}+\frac{\partial}{\partial x_k}(v_k'\bar{v}_i+\bar{v}_k v_i'+v_k'v_i')=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 v_i'}{\partial x_k\partial x_k}-\frac{\partial \overline{v_i'v_j'}}{\partial x_k}-g_i\beta\theta' \tag{30.4}$$

Multiply Eq. 30.3 with v_i' and multiply Eq. 30.4 with θ', add them together and time average

$$\begin{aligned}&\overline{v_i'\frac{\partial}{\partial x_k}(v_k'\bar{\theta}+\bar{v}_k\theta'+v_k'\theta')+\theta'\frac{\partial}{\partial x_k}(\bar{v}_i v_k'+\bar{v}_k v_i'+v_i'v_k')}\\&=-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}+\alpha\overline{v_i'\frac{\partial^2\theta'}{\partial x_k\partial x_k}}+\nu\overline{\theta'\frac{\partial^2 v_i'}{\partial x_k\partial x_k}}-g_i\beta\overline{\theta'\theta'}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x_k}\bar{v}_k\overline{v_i'\theta'}=&\underbrace{-\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k}-\overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}}\underbrace{-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x_i}}}_{\Pi_{i\theta}}\underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}\\&\underbrace{+(\nu+\alpha)\frac{\partial^2\overline{v_i'\theta'}}{\partial x_k\partial x_k}}_{D_{i\theta,\nu}}\underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial\theta'}{\partial x_k}}}_{\varepsilon_{i\theta}}\underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}}\end{aligned}$$

- Unknown terms

$\Pi_{i\theta}$ Scramble

$D_{i\theta,t}$ Turbulent diffusion

$\varepsilon_{i\theta}$ Dissipation

¶The original derivation of the k equation is shown in Section 8.2.

►In 11.4, The k equation, we derive the k equation as follows

►Take the trace of the $\overline{v_i'v_j'}$ equation and divide by two

$$\bar{v}_k\frac{\partial\overline{v_i'v_j'}}{\partial x_k} = -\overline{v_i'v_k'}\frac{\partial\bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial\bar{v}_i}{\partial x_k} + \overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j}+\frac{\partial v_j'}{\partial x_i}\right)} - \frac{\partial}{\partial x_k}\left[\overline{v_i'v_j'v_k'} + \frac{\overline{p'v_j'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{jk}\right]$$
$$+\nu\frac{\partial^2\overline{v_i'v_j'}}{\partial x_k\partial x_k} - g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}$$

$$\bar{v}_k\frac{\partial\overline{v_i'v_i'}}{\partial x_k} = -\overline{v_i'v_k'}\frac{\partial\bar{v}_i}{\partial x_k} - \overline{v_i'v_k'}\frac{\partial\bar{v}_i}{\partial x_k} + \overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_i}+\frac{\partial v_i'}{\partial x_i}\right)} - \frac{\partial}{\partial x_k}\left[\overline{v_i'v_i'v_k'} + \frac{\overline{p'v_i'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{ik}\right]$$
$$+\nu\frac{\partial^2\overline{v_i'v_i'}}{\partial x_k\partial x_k} - g_i\beta\overline{v_i'\theta'} - g_i\beta\overline{v_i'\theta'} - 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_i'}{\partial x_k}}$$

$$\underbrace{\bar{v}_j\frac{\partial k}{\partial x_j}}_{C^k} = -\underbrace{\overline{v_j'v_k'}\frac{\partial\bar{v}_j}{\partial x_k}}_{P^k}\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_k'\left(\frac{1}{2}v_i'v_i'+\frac{p'}{\rho}\right)}\right]}_{D_t^k} + \underbrace{\nu\frac{\partial^2 k}{\partial x_k\partial x_k}}_{D_\nu^k}\underbrace{-g_i\beta\overline{v_i'\theta'}}_{G^k} - \underbrace{\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_i'}{\partial x_k}}}_{\varepsilon} \qquad (30.5)$$

- Unknown terms

$\overline{v_i'v_j'}$ Reynolds stress in P^k

D_t^k Turbulent diffusion

ε Dissipation

¶See Section 11.6, The Boussinesq assumption

►The Boussinesq assumption: a model for $\overline{v'_i v'_j}$

The diffusion term of time-averaged Navier-Stokes

$$\frac{\partial}{\partial x_j}\left\{\nu\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)-\overline{v'_i v'_j}\right\} \Rightarrow \frac{\partial}{\partial x_j}\left\{(\nu+\nu_t)\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)\right\}$$

$$\overline{v'_i v'_j} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)$$

►When this equation is contrated, the LHS is not zero ($\overline{v'_i v'_i}$) whereas the RHS is zero due to continuity ($v\bar{v}/\partial x_i = 0$)

►Add $(2/3)\delta_{ij}k$ on the RHS:

$$\overline{v'_i v'_j} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)+\boxed{\tfrac{2}{3}\delta_{ij}k}$$

►The turbulent viscosity: $\nu_t \propto v'\ell = k^{1/2}\frac{k^{3/2}}{\varepsilon} = c_\mu \frac{k^2}{\varepsilon}$

On-line Lecture 2

¶See Section 11.7.1, Production terms

- First let's repeat the definition of the strain-rate and vorticity tensors, see Eq. 1.11

$$\frac{\partial \bar{v}_i}{\partial x_j} = \bar{s}_{ij} + \bar{\Omega}_{ij}, \quad \bar{s}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right), \quad \bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_j}{\partial x_i} - \frac{\partial \bar{v}_j}{\partial x_i}\right)$$

- Recall that the product $\bar{s}_{ij}\bar{\Omega}_{ij} = 0$ (product of symmetric and anti-symmetric tensor, see Section 1.5)

▶Production term in k equation needs to be modeled.

$$\begin{aligned}
P^k = -\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j} &= \nu_t \left[\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k\right]\frac{\partial \bar{v}_i}{\partial x_j} \\
&= \nu_t \left[\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\right]\frac{\partial \bar{v}_i}{\partial x_j} - \frac{2}{3}\nu_t \delta_{ij} k \frac{\partial \bar{v}_i}{\partial x_j} \\
&= \nu_t \left[\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\right]\frac{\partial \bar{v}_i}{\partial x_j} - \cancelto{0}{\frac{2}{3}\nu_t \frac{\partial \bar{v}_i}{\partial x_i}} \\
&= 2\nu_t \bar{s}_{ij}(\bar{s}_{ij} + \bar{\Omega}_{ij}) = 2\nu_t \bar{s}_{ij}\bar{s}_{ij}
\end{aligned}$$

▶Diffusion term in k eq, Eq. 30.5, must be modelled.

▶The exact k equation:

$$\bar{v}_j \frac{\partial k}{\partial x_j} = -\overline{v'_j v'_k} \frac{\partial \bar{v}_j}{\partial x_k} - \frac{\partial}{\partial x_k} \left[\overline{v'_k \left(\frac{1}{2} v'_i v'_i + \frac{p'}{\rho} \right)} \right] + \nu \frac{\partial^2 k}{\partial x_k \partial x_k} - g_i \beta \overline{v'_i \theta'} - \nu \overline{\frac{\partial v'_i}{\partial x_k} \frac{\partial v'_i}{\partial x_k}}$$

▶The constitutive model for heat conduction, Fourier's law, (see Section 2.2)

$$q_i = -k \frac{\partial \bar{\theta}}{\partial x_i}.$$

Flux of k:

$$d^k_{j,t} = \frac{1}{2} \overline{v'_j v'_i v'_i} = -\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_j}$$

$$\Rightarrow -\frac{1}{2} \frac{\partial \overline{v'_j v'_i v'_i}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right)$$

▶The heat flux is an unknown both in the mean temperature equation, Eq. 30.1, and in the exact k equation above. Taking guidance from Fourier's law . It is modeled as

$$\overline{v'_i \theta'} = -\alpha_t \frac{\partial \bar{\theta}}{\partial x_i}, \quad \alpha_t = \frac{\nu_t}{\sigma_t}$$

¶See Section 11.8, The $k - \varepsilon$ model

▶Modeled k equation

$$\bar{v}_j \frac{\partial k}{\partial x_j} = 2\nu_t \bar{s}_{ij}\bar{s}_{ij} + \frac{\partial}{\partial x_j}\left\{\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right\} + g_i\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i} - \varepsilon$$

▶Exact k equation

$$\underbrace{\bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = -\underbrace{\overline{v'_j v'_k}\frac{\partial \bar{v}_j}{\partial x_k}}_{P^k} \underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v'_k\left(\frac{1}{2}v'_i v'_i + \frac{p'}{\rho}\right)}\right]}_{D_t^k} + \underbrace{\nu\frac{\partial^2 k}{\partial x_k \partial x_k}}_{D_\nu^k} \underbrace{-g_i\beta\overline{v'_i t\theta'}}_{G^k} - \underbrace{\nu\overline{\frac{\partial v'_i}{\partial x_k}\frac{\partial v'_i}{\partial x_k}}}_{\varepsilon}$$

¶See Section 11.5, The ε equation

▶ε equation

$$C^\varepsilon = P^\varepsilon + D^\varepsilon + G^\varepsilon - \Psi^\varepsilon$$

Use the same source terms as in k equation and add turbulent time-scale ε/k to get the right dimensions:

$$P^\varepsilon + G^\varepsilon - \Psi^\varepsilon = \frac{\varepsilon}{k}\left(c_{\varepsilon 1}P^k + c_{\varepsilon 3}G^k - c_{\varepsilon 2}\varepsilon\right)$$

▶The final form modelled ε equation

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j} = c_{\varepsilon 1}\frac{\varepsilon}{k}P^k + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + c_{\varepsilon 3}\frac{\varepsilon}{k}G^k - c_{\varepsilon 2}\frac{\varepsilon}{k}\varepsilon$$

¶See Section 11.7.3, Dissipation term, ε_{ij}

$$\underbrace{\bar{v}_k\frac{\partial\overline{v'_i v'_j}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v'_i v'_k}\frac{\partial\bar{v}_j}{\partial x_k} - \overline{v'_j v'_k}\frac{\partial\bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)}}_{\Pi_{ij}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v'_i v'_j v'_k} + \frac{\overline{p' v'_j}}{\rho}\delta_{ik} + \frac{\overline{p' v'_i}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu\frac{\partial^2\overline{v'_i v'_j}}{\partial x_k\partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i\beta\overline{v'_j\theta'} - g_j\beta\overline{v'_i\theta'}}_{G_{ij}} - \underbrace{2\nu\overline{\frac{\partial v'_i}{\partial x_k}\frac{\partial v'_j}{\partial x_k}}}_{\varepsilon_{ij}}$$

►The dissipation term, ε_{ij}, in the $\overline{v'_i v'_j}$ equation (eq. 30.2), is modeled as follows:
Small-scale turbulence is isotropic

1. $\overline{v'^2_1} = \overline{v'^2_2} = \overline{v'^2_3}$.

2. All shear stresses are zero (if we flip one coordinate axis the sign will change: hence not isotropic) $\Rightarrow$

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} \tag{31.1}$$

¶See Section 11.7.2, Diffusion terms

$$\underbrace{\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)}}_{\Pi_{ij}}$$

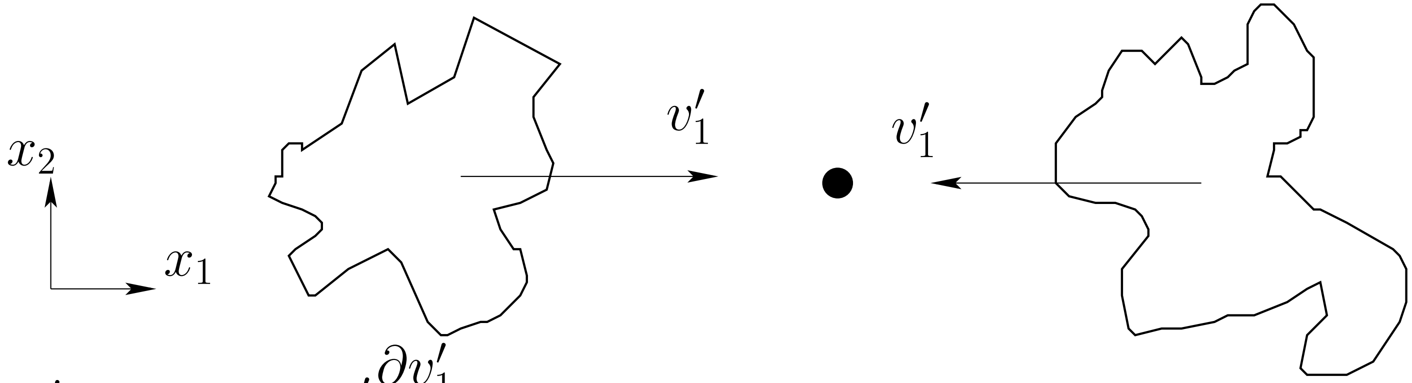
$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_i' v_j' v_k'} + \frac{\overline{p' v_j'}}{\rho}\delta_{ik} + \frac{\overline{p' v_i'}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'}}_{G_{ij}} - \underbrace{2\nu \overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

Flux of $\overline{v_i' v_j'}$:

$$D_{ij,t} = \overline{v_i' v_j' v_k'} = -\frac{\nu_t}{\sigma_k}\frac{\partial \overline{v_i' v_j'}}{\partial x_k}$$

$$\Rightarrow -\frac{\partial \overline{v_i' v_j' v_k'}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{\nu_t}{\sigma_k}\frac{\partial \overline{v_i' v_j'}}{\partial x_k}\right)$$



$\partial v_1'/\partial x_1 < 0$ and $p' > 0$ so that $p'\dfrac{\partial v_1'}{\partial x_1} < 0$

$$\frac{\partial v_2'}{\partial x_2} > 0, \quad \frac{\partial v_3'}{\partial x_3} > 0$$

If this happens then

$$\overline{v_1'^2} > \overline{v_2'^2}, \overline{v_1'^2} > \overline{v_3'^2}$$

$$\begin{aligned}\overline{p'\frac{\partial v_1'}{\partial x_1}} &\propto -\frac{\rho}{2t}\left[\left(\overline{v_1'^2}-\overline{v_2'^2}\right)+\left(\overline{v_1'^2}-\overline{v_3'^2}\right)\right]=-\frac{\rho}{t}\left[\overline{v_1'^2}-\frac{1}{2}\left(\overline{v_2'^2}+\overline{v_3'^2}\right)\right]\\ &= -\frac{\rho}{t}\left[\frac{3}{2}\overline{v_1'^2}-\frac{1}{2}\left(\overline{v_1'^2}+\overline{v_2'^2}+\overline{v_3'^2}\right)\right]=-\frac{\rho}{t}\left(\frac{3}{2}\overline{v_1'^2}-k\right)\end{aligned}$$

$$\Phi_{ij,1} \equiv \overline{p'\left(\frac{\partial v_i'}{\partial x_j}+\frac{\partial v_j'}{\partial x_i}\right)} = -c_1\rho\frac{\varepsilon}{k}\left(\overline{v_i'v_j'}-\frac{2}{3}\delta_{ij}k\right) \qquad (31.2)$$

see Eq. 11.57

¶See Eq. 11.2

▶Decaying grid turbulence

▶$\overline{v'_i v'_j}$ equation

$$\bar{v}_1 \frac{d\overline{v'_i v'_j}}{dx_1} = \overline{\frac{p'}{\rho}\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)} - \varepsilon_{ij} \tag{31.3}$$

An anisotropy stress tensor is defined as

$$a_{ij} = \frac{\overline{v'_i v'_j}}{k} - \frac{2}{3}\delta_{ij} \quad \Rightarrow \quad \overline{v'_i v'_j} = k a_{ij} + \frac{2k}{3}\delta_{ij} \tag{31.4}$$

In isotropic turbulence, $a_{ij} = 0$.

We insert Eq. 31.4 into Eq. 31.3 and use the models for the pressure strain term, $\phi_{ij,1}$ (Eq. 31.2) and dissipation, $\varepsilon_{ij} = (2/3)\delta_{ij}$ (Eq. 31.1) so that

$$\bar{v}_1 \left(\frac{d(k a_{ij})}{dx_1} + \frac{2}{3}\delta_{ij}\frac{dk}{dx_1}\right) = \underline{-c_1 \varepsilon a_{ij} - \frac{2}{3}\delta_{ij}\varepsilon}$$

$$\bar{v}_1\left(\frac{d(ka_{ij})}{dx_1}+\frac{2}{3}\delta_{ij}\frac{dk}{dx_1}\right)=\underline{-c_1\varepsilon a_{ij}-\frac{2}{3}\delta_{ij}\varepsilon}$$

$$\bar{v}_1\left(k\frac{da_{ij}}{dx_1}+a_{ij}\frac{dk}{dx_1}+\frac{2}{3}\delta_{ij}\frac{dk}{dx_1}\right)=\underline{-c_1\varepsilon a_{ij}-\frac{2}{3}\delta_{ij}\varepsilon}$$

►Using the k equation, $\bar{v}_1\frac{dk}{dx_1}=-\varepsilon$, and dividing by k give

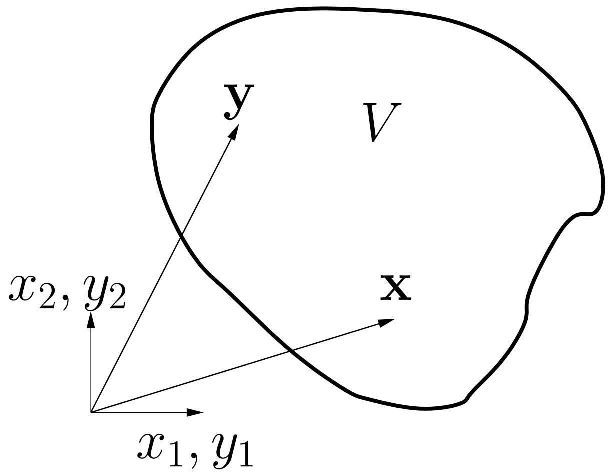
$$\bar{v}_1\frac{da_{ij}}{dx_1}=\underline{-c_1\frac{\varepsilon}{k}a_{ij}-\frac{2}{3}\delta_{ij}}\frac{\varepsilon}{k}+\frac{\varepsilon}{k}a_{ij}+\frac{2}{3}\delta_{ij}\frac{\varepsilon}{k}=\frac{\varepsilon}{k}a_{ij}(1-c_1)$$

$da_{ij}/dx<0$ (the turbulence becomes isotropic). Hence we find that c_1 must be larger than one.

On-line Lecture 3

¶See Section 11.7.5, Rapid pressure-strain term

▶Pressure strain: rapid part



1. Take the divergence of the incompressible Navier-Stokes equation assuming constant viscosity (see Eq. 6.6) i.e. $\frac{\partial}{\partial x_i}\left(v_j\frac{\partial v_i}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(-\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 v_i}{\partial x_j \partial x_j}\right)$ ... $\Rightarrow$ Equation **A**.
2. Take the divergence of the incompressible time-averaged Navier-Stokes equation assuming constant viscosity (see Eq. 6.10) i.e. $\frac{\partial}{\partial x_i}\left(\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(-\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{v_i' v_j'}}{\partial x_j}\right)$ ... $\Rightarrow$ Equation **B**.

Eq. A - Eq. B gives

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_j \partial x_j} = -\underbrace{2\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial v_j'}{\partial x_i}}_{\text{rapid term}} - \underbrace{\frac{\partial^2}{\partial x_i \partial x_j}\left(v_i' v_j' - \overline{v_i' v_j'}\right)}_{\text{slow term}} \tag{32.1}$$

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_j \partial x_j} = -\underbrace{2\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial v'_j}{\partial x_i}}_{\text{rapid term}} - \underbrace{\frac{\partial^2}{\partial x_i \partial x_j}\left(v'_i v'_j - \overline{v'_i v'_j}\right)}_{\text{slow term}} \tag{32.1}$$

$$\frac{\partial^2 \varphi}{\partial x_j \partial x_j} = f$$

►There exists an exact analytical solution given by Green's formula (derived from Gauss divergence law). The derivation is shown in an Appendix in the eBook.

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{f(\mathbf{y})dy_1 dy_2 dy_3}{|\mathbf{y}-\mathbf{x}|}$$

where $dy_1 dy_2 dy_3 = dV = d\mathbf{y}^3$. The integral is carried out for all points, $\mathbf{y}$, in volume V.

$$p'(\mathbf{x}) = \frac{\rho}{4\pi}\int_V \left[\underbrace{2\frac{\partial \bar{v}_i(\mathbf{y})}{\partial y_j}\frac{\partial v'_j(\mathbf{y})}{\partial y_i}}_{\text{rapid term}} + \underbrace{\frac{\partial^2}{\partial y_i \partial y_j}\left(v'_i(\mathbf{y})v'_j(\mathbf{y}) - \overline{v'_i(\mathbf{y})v'_j(\mathbf{y})}\right)}_{\text{slow term}}\right]\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|} \tag{32.2}$$

►Now make two assumptions in Eq. 32.2:

$$p'(\mathbf{x}) = \frac{\rho}{4\pi} \int_V \left[\underbrace{2 \frac{\partial \bar{v}_i(\mathbf{y})}{\partial y_j} \frac{\partial v'_j(\mathbf{y})}{\partial y_i}}_{\text{rapid term}} + \underbrace{\frac{\partial^2}{\partial y_i \partial y_j} \left(v'_i(\mathbf{y}) v'_j(\mathbf{y}) - \overline{v'_i(\mathbf{y}) v'_j(\mathbf{y})} \right)}_{\text{slow term}} \right] \frac{d\mathbf{y}^3}{|\mathbf{y} - \mathbf{x}|} \tag{32.2}$$

i) the turbulence is homogeneous which means that the last term in square brackets is zero.

ii) the variation of $\partial \bar{v}_i / \partial x_j$ in space is small because $\partial \bar{v}_i / \partial x_j$ varies much more slowly $\partial v'_j(\mathbf{y}) / \partial y_i$

Assumption *i)* ⇒ last term in the integral in Eq. 11.68 is zero, i.e.

$$\frac{\partial^2 \overline{v'_i v'_j}}{\partial y_i \partial y_j} = 0$$

Assumption *ii)* ⇒ mean velocity gradient moved outside the integral.

$$p'(\mathbf{x}) = \frac{\rho}{2\pi}\frac{\partial \bar{v}_i(\mathbf{x})}{\partial x_j}\int_V \frac{\partial v'_j(\mathbf{y})}{\partial y_i}\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|} - \frac{\rho}{4\pi}\int_V \frac{\partial^2}{\partial y_i \partial y_j}\left(v'_i(\mathbf{y})v'_j(\mathbf{y})\right)\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|}$$

►Multiply Eq. 32.2 with $\partial v'_i/\partial x_j + \partial v'_j/\partial x_i$ and average:

$$\begin{aligned}
\overline{\frac{p'(\mathbf{x})}{\rho}\left(\frac{\partial v'_i(\mathbf{x})}{\partial x_j}+\frac{\partial v'_j(\mathbf{x})}{\partial x_i}\right)} &= \frac{\partial \bar{v}_k(\mathbf{x})}{\partial x_\ell}\underbrace{\frac{1}{2\pi}\int_V \overline{\left(\frac{\partial v'_i(\mathbf{x})}{\partial x_j}+\frac{\partial v'_j(\mathbf{x})}{\partial x_i}\right)\frac{\partial v'_\ell(\mathbf{y})}{\partial y_k}}\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|}}_{M_{ijk\ell}} \\
&+ \underbrace{\frac{1}{4\pi}\int_V \overline{\left(\frac{\partial v'_i(\mathbf{x})}{\partial x_j}+\frac{\partial v'_j(\mathbf{x})}{\partial x_i}\right)\frac{\partial^2}{\partial y_k \partial y_\ell}\left(v'_k(\mathbf{y})v'_\ell(\mathbf{y})\right)}\frac{d\mathbf{y}^3}{|\mathbf{y}-\mathbf{x}|}}_{A_{ij}}
\end{aligned} \tag{32.3}$$

Short form of Eq. 32.3:

$$\overline{\frac{p'}{\rho}\left(\frac{\partial v'_i}{\partial x_j}+\frac{\partial v'_j}{\partial x_i}\right)} = A_{ij} + M_{ijk\ell}\frac{\partial \bar{v}_k}{\partial x_\ell} = \Phi_{ij,1} + \Phi_{ij,2}$$

- First term=slow term, $\Phi_{ij,1}$,
- second term=rapid term, $\Phi_{ij,2}$ (index 2 denotes the rapid part).

$$\Phi_{ij,2} = -c_2\rho\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) \quad \text{IP model}$$

¶See Section 11.7.6, Wall model of the pressure-strain term

▶Wall models of pressure-strain:

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,1w} + \Phi_{ij,2w}$$

$$\Phi_{22,1w} = -2c_{1w}\frac{\varepsilon}{k}\overline{v_2'^2}f, \quad f \propto \frac{L_t}{|x_i - x_{i,wall}|} = \frac{k^{\frac{3}{2}}}{2.55|n_{i,w}(x_i - x_{i,w})|\varepsilon}, \quad 0 < f < 1$$

Traceless ⇒

$$\Phi_{11,1w} = \Phi_{33,1w} = c_{1w}\frac{\varepsilon}{k}\overline{v_2'^2}f$$

The wall model for the shear stress is set as

$$\Phi_{12,1w} = -\frac{3}{2}c_{1w}\frac{\varepsilon}{k}\overline{v_1'v_2'}f$$

The general form reads:

$$\Phi_{ij,1w} = c_{1w}\frac{\varepsilon}{k}\left(\overline{v_k'v_m'}n_{k,w}n_{m,w}\delta_{ij} - \frac{3}{2}\overline{v_k'v_i'}n_{k,w}n_{j,w} - \frac{3}{2}\overline{v_k'v_j'}n_{i,w}n_{k,w}\right)f$$

The analogous wall model for the rapid part reads

$$\Phi_{ij,2w} = c_{2w}\left(\Phi_{km,2}n_{k,w}n_{m,w}\delta_{ij} - \frac{3}{2}\Phi_{ki,2}n_{k,w}n_{j,w} - \frac{3}{2}\Phi_{kj,2}n_{i,w}n_{k,w}\right)f$$

¶See Section 11.9, The modeled $\overline{v_i'v_j'}$ equation with IP model

►We can finally formulate the **modelled** $\overline{v_i'v_j'}$ equation (with IP model), the Reynolds Stress Model

(RSM)

$$
\begin{aligned}
\frac{\partial \overline{v_i' v_j'}}{\partial t} + & \quad \text{(unsteady term)} \\
\bar{v}_k \frac{\partial \overline{v_i' v_j'}}{\partial x_k} = & \quad \text{(convection)} \\
-\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k} & \quad \text{(production)} \\
-c_1 \frac{\varepsilon}{k} \left(\overline{v_i' v_j'} - \frac{2}{3} \delta_{ij} k \right) & \quad \text{(slow part)} \\
-c_2 \left(P_{ij} - \frac{2}{3} \delta_{ij} P^k \right) & \quad \text{(rapid part, IP model)} \\
+c_{1w} \rho \frac{\varepsilon}{k} \left[\, \overline{v_k' v_m'} n_k n_m \delta_{ij} - \frac{3}{2} \overline{v_i' v_k'} n_k n_j - \frac{3}{2} \overline{v_j' v_k'} n_k n_i \, \right] f & \quad \text{(wall, slow part)} \\
+c_{2w} \left[\, \Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2} \Phi_{ik,2} n_k n_j - \frac{3}{2} \Phi_{jk,2} n_k n_i \, \right] f & \quad \text{(wall, rapid part, IP model))} \\
+\nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial x_k \partial x_k} & \quad \text{(viscous diffusion)} \\
+\frac{\partial}{\partial x_k} \left[\frac{\nu_t}{\sigma_k} \frac{\partial \overline{v_i' v_j'}}{\partial x_m} \right] & \quad \text{(turbulent diffusion)} \\
-g_i \beta \overline{v_j' \theta'} - g_j \beta \overline{v_i' \theta'} & \quad \text{(buoyancy production)} \\
-\frac{2}{3} \varepsilon \delta_{ij} & \quad \text{(dissipation)}
\end{aligned}
\tag{32.4}
$$

¶See Section 11.10, Algebraic Reynolds Stress Model (ASM)

▶The Algebraic Reynolds Stress Model (ASM) is a simplified Reynolds Stress Model (RSM)

$$
\begin{aligned}
\mathrm{RSM}: C_{ij} - D_{ij} &= P_{ij} + \Phi_{ij} - \varepsilon_{ij} \\
k-\varepsilon: \; C^k - D^k &= P^k - \varepsilon
\end{aligned}
$$

Assumption in ASM:

$$
C_{ij} - D_{ij} = \left(C^k - D^k\right)\frac{\overline{v_i'v_j'}}{k}
$$

$$
\Rightarrow P_{ij} + \Phi_{ij} - \varepsilon_{ij} = \frac{\overline{v_i'v_j'}}{k}\left(P^k - \varepsilon\right)
$$

This gives

$$
\overline{v_i'v_j'} = \frac{2}{3}\delta_{ij}k + \frac{k}{\varepsilon}\frac{(1-c_2)\left(P_{ij} - \frac{2}{3}\delta_{ij}P^k\right) + \Phi_{ij,1w} + \Phi_{ij,2w}}{c_1 + P^k/\varepsilon - 1}
$$

¶See Section 11.13, Boundary layer flow

►We will consider source terms in the modelled $\overline{v_i' v_j'}$ equation for boundary layer flow.
We have $\bar{v}_2 \simeq 0$, $\partial \bar{v}_1/\partial x_2 \gg \partial \bar{v}_1/\partial x_1$. The production term reads:

$$P_{ij} = -\overline{v_i' v_k'} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j' v_k'} \frac{\partial \bar{v}_i}{\partial x_k}$$

In this special case we get (all velocity gradients except $\partial \bar{v}_1/\partial x_2$ are negligible):

$$\begin{aligned}
P_{11} &= -2\overline{v_1' v_2'} \frac{\partial \bar{v}_1}{\partial x_2} \\
P_{12} &= -\overline{v_2'^2} \frac{\partial \bar{v}_1}{\partial x_2} \\
P_{22} &= 0, \quad P_{33} = 0
\end{aligned}$$

►No production term in $\overline{v_2'^2}$ and $\overline{v_3'^2}$! How are they produced? Answer: the pressure strain (Robin Hood)

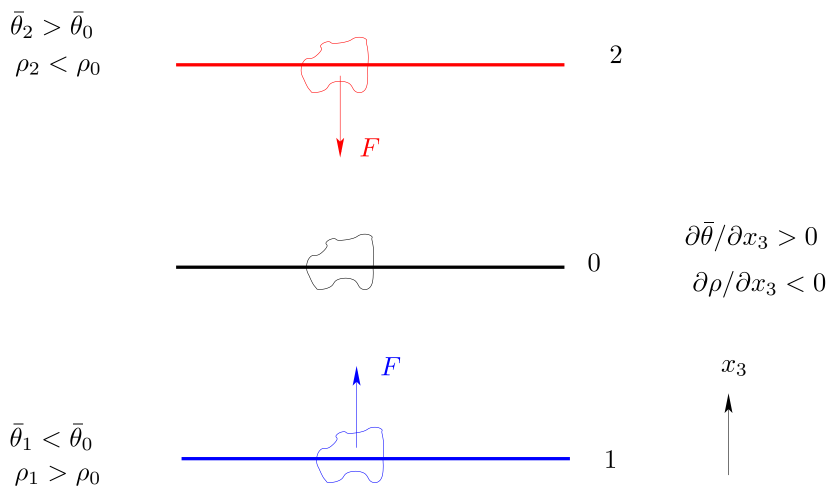
$$\begin{aligned}
\Phi_{22,1} &= c_1 \frac{\varepsilon}{k} \left(\frac{2}{3} k - \overline{v_2'^2} \right) > 0, \quad \frac{2}{3} k = (\overline{v_1'^2} + \overline{v_2'^2} + \overline{v_3'^2})/3 \\
\Phi_{22,2} &= c_2 \frac{1}{3} P_{11} = -c_2 \frac{2}{3} \overline{v_1' v_2'} \frac{\partial \bar{v}_1}{\partial x_2} > 0
\end{aligned}$$

$\varepsilon_{12} = 0$: No sink term in $\overline{v_1' v_2'}$ eq? Answer: the pressure strain terms $\Phi_{12,1}$ and $\Phi_{12,2}$ act as sink terms.

¶See Section 12.1, Stable and unstable stratification

▶Assume there is a non-constant temperature field and that the natural convection is important (no forced convection). We have then two different flow conditions, stable or unstable conditions.

▶We start with stable stratification for which $\partial\bar{\theta}/\partial x_3 > 0$.



$$G_{ij} = -g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}, \quad g_i = (0, 0, -g) \qquad \Rightarrow \overline{v_3'^2} \text{ eq.:} \quad G_{33} = 2g\beta\overline{v_3'\theta'}$$

which is the source term in the $\overline{v_3'^2}$ eq due to buoyancy.

Now we need $\overline{v_3'\theta'}$.

The main source term in this equation is (see Eq 30.3)

$$P_{3\theta} = -\overline{v_3'v_k'}\frac{\partial\bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial\bar{v}_3}{\partial x_k} = -\overline{v_3'v_k'}\frac{\partial\bar{\theta}}{\partial x_k} - \underbrace{\overline{v_k'\theta'}\frac{\partial\bar{v}_3}{\partial x_k}}_{0} = -\overline{v_3'^2}\frac{\partial\bar{\theta}}{\partial x_3} < 0$$

So

- $P_{3\theta} < 0$
- $\Rightarrow \overline{v_3'\theta'} < 0$
- $\Rightarrow G_{33} = 2g\beta\overline{v_3'\theta'} < 0$
- which dampens $\overline{v_3'^2}$ (but not $\overline{v_1'^2}$, $\overline{v_2'^2}$) as it should.

▶Above, we assumed stable conditions, $\partial\theta/\partial x_3 > 0$, which gives reduced vertical fluctuations.

▶If we assume un-stable conditions, $\partial\theta/\partial x_3 < 0$, we can in the same manner show that we get increased vertical fluctuations.

►$k-\varepsilon$ model. The buoyancy term reads (Eq. 30.5)

$$G^k = 0.5G_{ii} = -g_i\beta\overline{v_i'\theta'}, \qquad \overline{v_i'\theta'} = -\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_i}$$

For $g_i = (0, 0, -g)$ it reads

$$G^k = g\beta\overline{v_3'\theta'}$$

which gives, using Section 11.6,

$$G^k = -g\beta\frac{\nu_t}{\sigma_\theta}\frac{\partial\bar{\theta}}{\partial x_3}$$

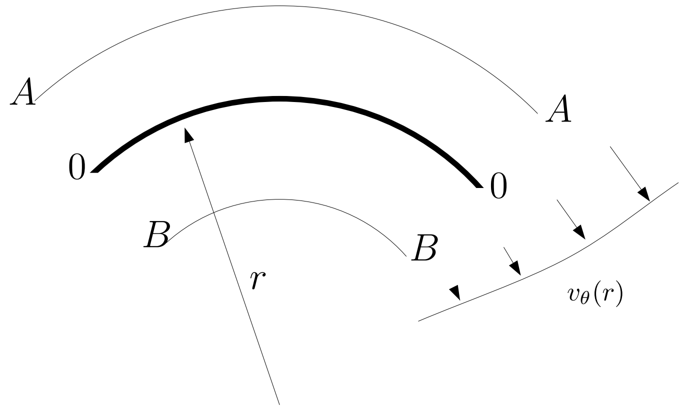
Hence $G^k < 0$ which dampens k (i.e. $\overline{v_1'^2}, \overline{v_2'^2}, \overline{v_3'^2}$).

►Note that the $k-\varepsilon$ model incorrecly dampens all normal stress, not only the vertical one

On-line Lecture 4

¶See Section 12.2, Curvature effects

▶Streamline curvature affects the turbulence.



Flow aligned with the θ axis. $\partial v_\theta/\partial r > 0$

▶We assume inviscid flow ($\mu = 0$) and express the Navier-Stokes eq. in polar coordinates:

$$v_r \text{ eq. with } \mu = 0: \quad \frac{\rho v_\theta^2}{r} - \frac{\partial p}{\partial r} = 0 \tag{33.1}$$

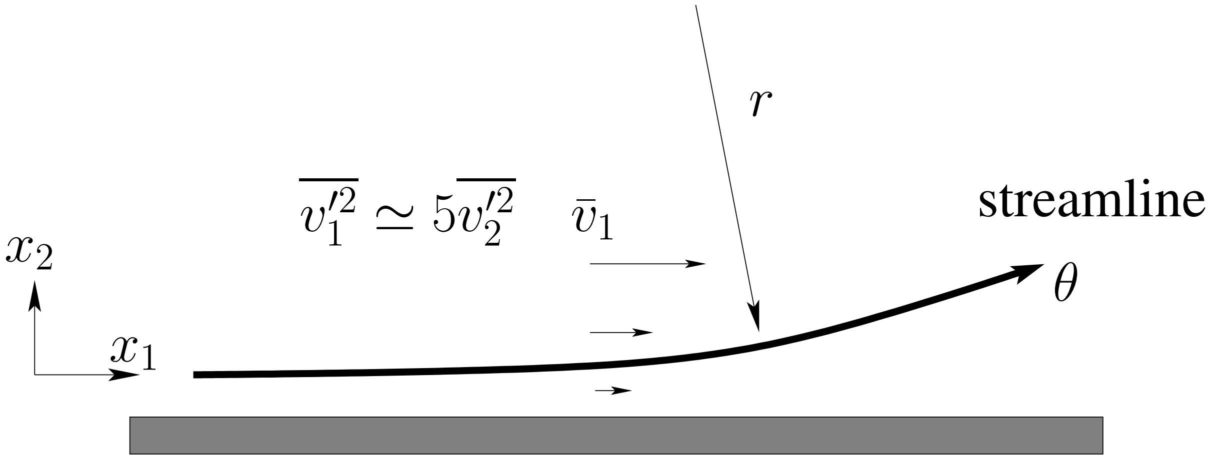
▶$(v_\theta)_A > (v_\theta)_0$, which from Eq. 33.1 gives $(\partial p/\partial r)_A > (\partial p/\partial r)_0$.

▶The streamline curvature stabilized (decreases) the turbulence

▶Change sign of velocity gradient $\partial v_\theta/\partial r < 0$: now streamline curvature will increase the turbulence

▶Now we will find out how well the effect of streamline curvature is modeled by RSM (and ASM).

▶We choose a boundary layer flow as below



A boundary layer flow that gradually departs from the wall. $\dfrac{\partial \bar{v}_2}{\partial x_1} > 0,\ \dfrac{\partial \bar{v}_1}{\partial x_2} > 0$

▶For this flow, the production terms read (boxed terms appear because $\dfrac{\partial \bar{v}_2}{\partial x_1}$ is not negligible)

$$
\begin{aligned}
RSM,\ \overline{v_1'^2} - eq.: \quad P_{11} &= -2\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2} \\
RSM,\ \overline{v_1'v_2'} - eq.: \quad P_{12} &= \boxed{-\overline{v_1'^2}\frac{\partial \bar{v}_2}{\partial x_1}} - \overline{v_2'^2}\frac{\partial \bar{v}_1}{\partial x_2} \\
RSM,\ \overline{v_2'^2} - eq.: \quad P_{22} &= \boxed{-2\overline{v_1'v_2'}\frac{\partial \bar{v}_2}{\partial x_1}} \\
k - \varepsilon \quad P^k &= \nu_t \left\{ \left(\frac{\partial \bar{v}_1}{\partial x_2} \right) + \boxed{\left(\frac{\partial \bar{v}_2}{\partial x_1} \right)} \right\}^2
\end{aligned}
$$

$$
\begin{aligned}
RSM,\ \overline{v_1'^2}-eq.: \quad P_{11} &= -2\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2}\\
RSM,\ \overline{v_1'v_2'}-eq.: \quad P_{12} &= \boxed{-\overline{v_1'^2}\frac{\partial \bar{v}_2}{\partial x_1}} - \overline{v_2'^2}\frac{\partial \bar{v}_1}{\partial x_2}\\
RSM,\ \overline{v_2'^2}-eq.: \quad P_{22} &= \boxed{-2\overline{v_1'v_2'}\frac{\partial \bar{v}_2}{\partial x_1}}\\
k-\varepsilon \quad P^k &= \nu_t\left\{\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)+\boxed{\left(\frac{\partial \bar{v}_2}{\partial x_1}\right)}\right\}^2
\end{aligned}
$$

▶ $\frac{\partial \bar{v}_1}{\partial x_2} > 0,\ \frac{\partial \bar{v}_2}{\partial x_1} > 0$, ▶ $-\overline{v_1'^2}\frac{\partial \bar{v}_2}{\partial x_1}$ increases $|P_{12}|$ (recall that $\overline{v_1'^2} \gg \overline{v_2'^2}$) , ▶ $\Rightarrow |\overline{v_1'v_2'}|$ increases

▶ $\Rightarrow \overline{v_1'^2}$ and $\overline{v_2'^2}$ increase , ▶ $\Rightarrow |P_{12}|$ increases even more (a positive feedback loop)

▶Hence: the turbulence increases (as it should). ▶We have a de-stabilizing streamline curvature.

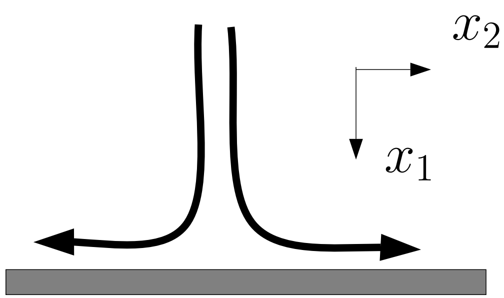
▶Change the sign of $\frac{\partial \bar{v}_1}{\partial x_2}$ gives decreased turbulence (stabilizing streamline curvature)

▶ $k-\varepsilon$ model: it does react to streamline curvaturve but much less ▶Why?

Contrary to RSM, the two velocity gradients are multiplied by the same coefficient

¶See Section 12.3, Stagnation flow

▶Stagnation 2D flow



The flow pattern for stagnation flow.

▶Near the plate, strong deceleration, i.e. large $\dfrac{\partial \bar{v}_1}{\partial x_1}$. ▶Continuity equation ⇒ large $\dfrac{\partial \bar{v}_2}{\partial x_2}$

▶The velocity gradient $\dfrac{\partial \bar{v}_1}{\partial x_2}$ and $\dfrac{\partial \bar{v}_2}{\partial x_1}$ are in this flow negligible.

$$RSM/ASM \;:\; 0.5\,(P_{11}+P_{22}) = -\overline{v_1'^2}\frac{\partial \bar{v}_1}{\partial x_1} - \overline{v_2'^2}\frac{\partial \bar{v}_2}{\partial x_2} = -\frac{\partial \bar{v}_1}{\partial x_1}(\overline{v_1'^2} - \overline{v_2'^2})$$

$$k-\varepsilon \;:\; P^k = 2\nu_t\left\{\left(\frac{\partial \bar{v}_1}{\partial x_1}\right)^2 + \left(\frac{\partial \bar{v}_2}{\partial x_2}\right)^2\right\}$$

For RSM/ASM, $\overline{v_1'^2} - \overline{v_2'^2}$ nearly cancels whereas for $k-\varepsilon$ they don't since the sum of squared velocity gradients is used.

¶See Section 13, Realizability

▶Realizability

$$\begin{aligned}\overline{v_i'^2} &\geq 0 \text{ for all } i \\ \frac{\left|\overline{v_i' v_j'}\right|}{\left(\overline{v_i'^2}\,\overline{v_j'^2}\right)^{1/2}} &\leq 1 \text{ no summation over } i \text{ and } j,\ i \neq j\end{aligned}$$

$$\overline{v_1'^2} = \frac{2}{3}k - 2\nu_t \frac{\partial \bar{v}_1}{\partial x_1} = \frac{2}{3}k - 2\nu_t \bar{s}_{11}$$

$\bar{s}_{11}$ largest in the principal coordinate directions. Hence, let's find the eigenvalues of $\bar{s}_{ij}$

$$|\bar{s}_{ij} - \delta_{ij}\lambda| = 0$$

which gives in 2D

$$\begin{vmatrix} \bar{s}_{11} - \lambda & \bar{s}_{12} \\ \bar{s}_{21} & \bar{s}_{22} - \lambda \end{vmatrix} = 0$$

The resulting equation characteristic equation is

$$\begin{aligned}\lambda^2 &- I_1^{2D}\lambda + I_2^{2D} = 0 \\ I_1^{2D} &= \bar{s}_{ii} = 0 \quad \textbf{continuity equation} \\ I_2^{2D} &= \frac{1}{2}(\bar{s}_{ii}\bar{s}_{jj} - \bar{s}_{ij}\bar{s}_{ij}) = \det(C_{ij}) = -\bar{s}_{ij}\bar{s}_{ij}/2\end{aligned}$$

$$\lambda^2 + I_2^{2D} = 0$$

$$\lambda_{1,2} = \pm\left(-I_2^{2D}\right)^{1/2} \;=\; \pm\left(\frac{\bar{s}_{ij}\bar{s}_{ij}}{2}\right)^{1/2}$$

$$\overline{v_1'^2} \;=\; \frac{2}{3}k - 2\nu_t \bar{s}_{11}$$

$$\left(\overline{v_1'^2}\right)_{\lambda_1} \;=\; \frac{2}{3}k - 2\nu_t\lambda_1 = \frac{2}{3}k - 2\nu_t\left(\frac{\bar{s}_{ij}\bar{s}_{ij}}{2}\right)^{1/2} > 0$$

$$\Rightarrow \nu_t \leq \frac{k}{3|\lambda_1|} = \frac{k}{3}\left(\frac{2}{\bar{s}_{ij}\bar{s}_{ij}}\right)^{1/2}$$

In 3D

$$|\lambda_k| = k\left(\frac{2\bar{s}_{ij}\bar{s}_{ij}}{3}\right)^{1/2}$$

¶See Section 14, Non-linear Eddy-viscosity Models

▶It is non-linear in velocity gradients.

▶The advantage is better normal stresses and a certain ability to handle streamline curvature.

$$
\begin{aligned}
a_{ij} &\equiv \frac{\overline{v_i'v_j'}}{k} - \frac{2}{3}\delta_{ij}\\
a_{ij} &= \boxed{-2c_\mu \tau \bar{s}_{ij}} + c_1\tau^2\left(\bar{s}_{ik}\bar{s}_{kj} - \frac{1}{3}\bar{s}_{mk}\bar{s}_{mk}\delta_{ij}\right) + c_2\tau^2\left(\bar{\Omega}_{ik}\bar{s}_{kj} - \bar{s}_{ik}\bar{\Omega}_{kj}\right)\\
&+ c_3\tau^2\left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{mk}\bar{\Omega}_{mk}\delta_{ij}\right) + c_4\tau^3\left(\bar{s}_{ik}\bar{s}_{km}\bar{\Omega}_{mj} - \bar{\Omega}_{im}\bar{s}_{mk}\bar{s}_{kj}\right)\\
&+ c_5\tau^3\left(\bar{\Omega}_{im}\bar{\Omega}_{mm}\bar{s}_{mj} + \bar{s}_{im}\bar{\Omega}_{mm}\bar{\Omega}_{mj} - \frac{2}{3}\bar{\Omega}_{mn}\bar{\Omega}_{nm}\bar{s}_{mm}\delta_{ij}\right) + c_6\tau^3\bar{s}_{km}\bar{s}_{km}\bar{s}_{ij} + c_7\tau^3\bar{\Omega}_{km}\bar{\Omega}_{km}\bar{s}_{ij}\\
\bar{s}_{ij} &= \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right), \quad \bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial \bar{v}_j}{\partial x_i}\right), \quad \tau = \frac{k}{\varepsilon} \text{ or } \tau = \frac{1}{c_\mu \omega}
\end{aligned}
\tag{33.2}
$$

▶symmetric, ▶trace-less ▶, only linear $\bar{s}_{ij}$ and quadratic $\bar{s}_{ik}\bar{s}_{kj}$ terms (no cubic terms, $\bar{s}_{ik}\bar{s}_{km}\bar{s}_{mj}$).

▶Why no cubic terms? Caley-Hamilon theorem which is based on the characteristic equation in 3D

$$\lambda^3 + I_1^{3D}\lambda^2 - I_2^{3D}\lambda + I_3^{3D} = 0$$

▶ Let's verify that the three first terms are indeed symmetric and traceless

▶ Term 1

$$-2c_\mu \tau \bar{s}_{ij} \quad \text{symmetric: } \bar{s}_{ij} = \bar{s}_{ji}, \quad \text{traceless: } \bar{s}_{ii} = 0$$

▶ Term 2

$$\left(\bar{s}_{ik}\bar{s}_{kj} - \frac{1}{3}\bar{s}_{mk}\bar{s}_{mk}\delta_{ij} \right) \quad \text{symmetric} \ : \ \bar{s}_{ik}\bar{s}_{kj} = \bar{s}_{jk}\bar{s}_{ki} = \bar{s}_{kj}\bar{s}_{ik} \quad \delta_{ij} = \delta_{ji}$$

$$\text{traceless} \ : \ \bar{s}_{ik}\bar{s}_{ki} - \frac{1}{3}\bar{s}_{mk}\bar{s}_{mk}\delta_{ii} = \bar{s}_{ik}\bar{s}_{ki} - \bar{s}_{mk}\bar{s}_{mk} = \bar{s}_{ik}\bar{s}_{ki} - \bar{s}_{ik}\bar{s}_{ik} = 0$$

▶ Term 3

$$\left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{mk}\bar{\Omega}_{mk}\delta_{ij} \right) \quad \text{symmetric} \ : \ \bar{\Omega}_{ik}\bar{\Omega}_{jk} = \bar{\Omega}_{jk}\bar{\Omega}_{ik} \quad \delta_{ij} = \delta_{ji}$$

$$\text{traceless} \ : \ \bar{\Omega}_{ik}\bar{\Omega}_{ik} - \frac{1}{3}\bar{\Omega}_{mk}\bar{\Omega}_{mk}\delta_{ii} = \bar{\Omega}_{ik}\bar{\Omega}_{ik} - \bar{\Omega}_{mk}\bar{\Omega}_{mk} = 0$$

▶With constants c_1, c_2, ... Eq. 33.2 read for a boundary layer flow

$$\overline{v_1'^2} = \frac{2}{3}k + \frac{0.82}{12}k\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2$$
$$\overline{v_2'^2} = \frac{2}{3}k - \frac{0.5}{12}k\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2$$
$$\overline{v_3'^2} = \frac{2}{3}k - \frac{0.16}{12}k\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2$$

▶We find that the nornal stresses are indeed not equal (coontrary to the standard linear $k - \varepsilon$ model)

On-line Lecture 5

¶See Section 15, The V2F Model

- Four equations are solved, k, ε (or ω), $\overline{v_2'^2}$ and f.
- f is proportional to the pressure-strain term in the eq. for the wall-normal fluctuation ($\overline{v_1'^2}$, $\overline{v_2'^2}$ or $\overline{v_3'^2}$)
- Strength: better in modeling damping of the turbulence near walls, e.g. stagnation flow

▶The exact $\overline{v_2'^2}$ equation (see Eq. 30.2) – modeling the turbulent diffusion – reads for a boundary layer

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v} \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu+\mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] \underbrace{-2\overline{v_2'\partial p'/\partial x_2}}_{\Phi_{22}} - \rho\varepsilon_{22}$$

▶Re-formulate and introduce a model for the dissipation, $\varepsilon_{22} = \frac{\overline{v_2'^2}}{k}\varepsilon$, and the pressure-strain, Φ_{22},

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v}_2 \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu+\mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] + \underbrace{\Phi_{22} - \rho\varepsilon_{22} + \rho\frac{\overline{v_2'^2}}{k}\varepsilon}_{fk} - \rho\frac{\overline{v_2'^2}}{k}\varepsilon$$

▶An equation is formulated for f, where Φ_{22} is taken from the RSM (see Eq. 32.4)

$$L^2\frac{\partial^2 f}{\partial x_2^2} - f = -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right), \quad T \propto \frac{k}{\varepsilon}, \quad L \propto \frac{k^{3/2}}{\varepsilon}$$

$$L^2\frac{\partial^2 f}{\partial x_2^2} - f = -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right), \quad T \propto \frac{k}{\varepsilon}, \quad L \propto \frac{k^{3/2}}{\varepsilon}$$

▶Let's try to understand the f equation.

- Far from the wall $\frac{\partial^2 f}{\partial x_2^2} \simeq 0 \Rightarrow \; -f \to -\frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right)$ i.e. $f = \frac{\Phi_{22}}{k} - \frac{1}{T}\left(\frac{\overline{v_2'^2}}{k} - \frac{2}{3}\right)$
- Insert this f into the $\overline{v_2'^2}$ equation on the previous slide gives

$$\frac{\partial \rho \bar{v}_1 \overline{v_2'^2}}{\partial x_1} + \frac{\partial \rho \bar{v} \overline{v_2'^2}}{\partial x_2} = \frac{\partial}{\partial x_2}\left[(\mu + \mu_t)\frac{\partial \overline{v_2'^2}}{\partial x_2}\right] + \Phi_{22} - \rho\varepsilon_{22}$$

▶We see that the $\overline{v_2'^2}$ eq. in the V2F model reverts back to the origonal $\overline{v_2'^2}$ equation. ▶Good.

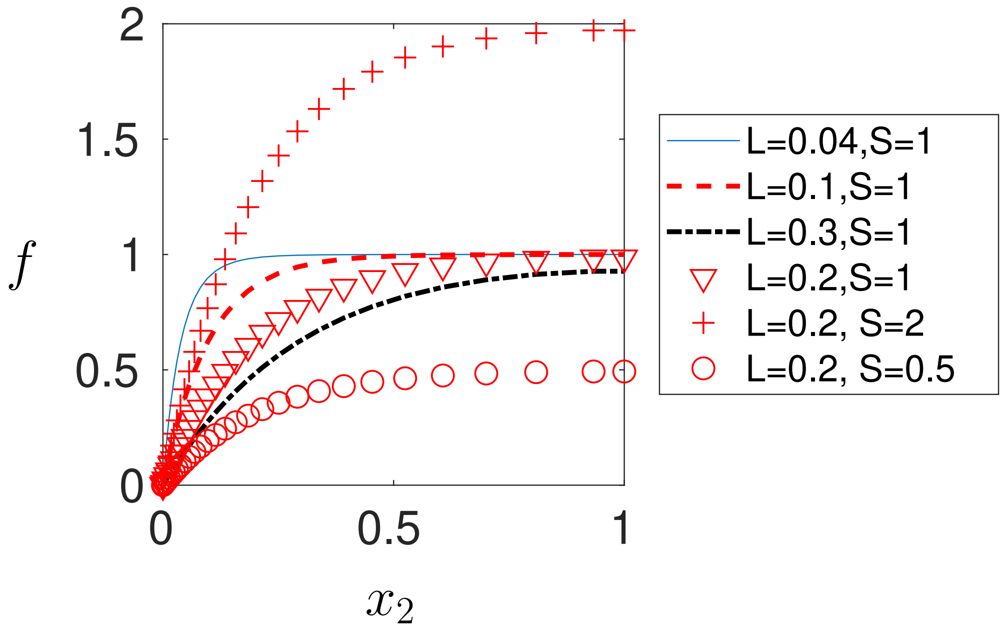
▶Next, let's see how the f equation behaves near the wall.

▶We formulate a simplified f in one dimension

$$L^2\frac{\partial^2 f}{\partial x_2^2} - f = -S \tag{34.1}$$

▶Now solve it (with a 1D finite volume code) for different L and S

$$L^2 \frac{\partial^2 f}{\partial x_2^2} - f = -S \tag{34.1}$$



Solution of Eq. 34.1 for different L and S

► V2F model. Wall boundary conditions

Near the wall, the $\overline{v_2'^2}$ equation reads (viscos, dissipation and f source term)

$$0 = \nu \frac{\partial^2 \overline{v_2'^2}}{\partial x_2^2} + fk - \frac{\overline{v_2'^2}}{k}\varepsilon$$

Replace k using $\varepsilon = 2\nu k / x_2^2$ gives

$$0 = \frac{\partial^2 \overline{v_2'^2}}{\partial x_2^2} + \frac{f \varepsilon x_2^2}{2\nu^2} - \frac{2\overline{v_2'^2}}{x_2^2}$$

► Assume $f \simeq$ const and $\varepsilon \simeq$ const as $x_2 \to 0$.

$$\overline{v_2'^2} = A x_2^2 + \frac{B}{x_2} - \varepsilon f \frac{x_2^4}{20\nu^2}$$

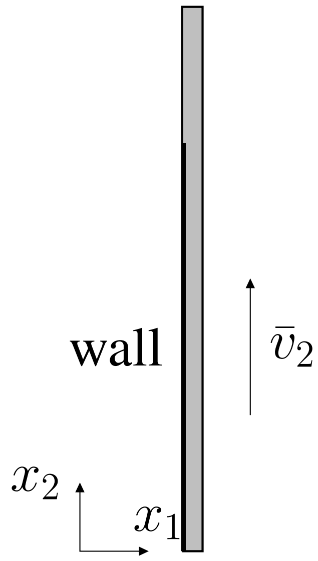
► Since $\overline{v_2'^2} = \mathcal{O}(x_2^4)$ as $x_2 \to 0$, $A = B = 0$, we get the b.c.

$$f = -\frac{20\nu^2}{\varepsilon} \frac{\overline{v_2'^2}}{x_2^4}$$

▶Above, we have presented the V2F model in 1D. In 3D, it reads

$$
\begin{aligned}
\frac{\partial \rho \bar{v}_j v^2}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[(\mu+\mu_t)\frac{\partial v^2}{\partial x_j}\right] + \rho f k - \rho\frac{v^2}{k}\varepsilon \\
L^2\frac{\partial^2 f}{\partial x_j \partial x_j} - f &= -\frac{\Phi_{22}}{\rho k} - \frac{1}{T}\left(\frac{v^2}{k} - \frac{2}{3}\right) \\
\frac{\Phi_{22}}{\rho k} &= \frac{C_1}{T}\left(\frac{2}{3} - \frac{v^2}{k}\right) + C_2\frac{P^k}{k}
\end{aligned}
$$

▶How does the V2F model behave near a vertical wall?



Boundary later along a vertical wall

$$L^2 \frac{\partial^2 f}{\partial x_j \partial x_j} - f = -\frac{\Phi_{22}}{\rho k} - \frac{1}{T}\left(\frac{v^2}{k} - \frac{2}{3}\right), \quad \frac{\Phi_{22}}{\rho k} = \frac{C_1}{T}\left(\frac{2}{3} - \frac{v^2}{k}\right) + C_2 \frac{P^k}{k}$$

- $\overline{v_1'^2} < \overline{v_2'^2}, \quad \overline{v_1'^2} < \overline{v_3'^2}$ ▶The key-term is P^k.
- For the horizontal plate, P^k is dominated by $\dfrac{\partial \bar{v}_1}{\partial x_2}$ ▶$v^2 = \overline{v_2'^2}$
- For the vertical plate, P^k is dominated by $\dfrac{\partial \bar{v}_2}{\partial x_1}$
- Hence, in this case (the vertical plate), v^2 corresponds to $\overline{v_1'^2}$
- P^k in the expression of Φ_{22} explains why v^2 is equal to $\overline{v_2'^2}$, $\overline{v_1'^2}$ or $\overline{v_3'^2}$ depending on orientation of the nearest wall (the largest velocity gradient).

¶See Section 16, The SST Model

▶The SST (Shear Stress Transport) model

1. Combination of a $k-\omega$ model (in the inner boundary layer) and $k-\varepsilon$ model (in the outer region of the boundary layer as well as outside of it)
   (a) $k-\omega$ is good for near-wall turbulence (well-defined b.c., no additional near-wall terms)
   (b) $k-\omega$ has a problem with far-field boundary conditions; $k-\varepsilon$ can handle these b.c.
2. A limitation of the shear stress in adverse pressure gradient regions

▶$\omega = \varepsilon/(\beta^* k) = \varepsilon/(c_\mu k)$. Use this to obtain an eq. for ω

$$\frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{\varepsilon}{\beta^* k}\right) = \frac{1}{\beta^* k}\frac{d\varepsilon}{dt} - \frac{\varepsilon}{\beta^* k^2}\frac{dk}{dt} = \frac{1}{\beta^* k}\frac{d\varepsilon}{dt} - \frac{\omega}{k}\frac{dk}{dt}$$

▶Production term

$$P_\omega = \frac{1}{\beta^* k}P_\varepsilon - \frac{\omega}{k}P^k = \frac{1}{\beta^* k}C_{\varepsilon 1}\frac{\varepsilon}{k}P^k - \frac{\omega}{k}P^k = (C_{\varepsilon 1} - 1)\frac{\omega}{k}P^k$$

▶Destruction term

$$\Psi_\omega = \frac{1}{\beta^* k}\Psi_\varepsilon - \frac{\omega}{k}\Psi_k = \frac{1}{\beta^* k}C_{\varepsilon 2}\frac{\varepsilon^2}{k} - \frac{\omega}{k}\varepsilon = (C_{\varepsilon 2} - 1)\beta^* \omega^2$$

$$\frac{d\omega}{dt} = \frac{1}{\beta^* k}\frac{d\varepsilon}{dt} - \frac{\omega}{k}\frac{dk}{dt}$$

▶Viscous diffusion term

$$\begin{aligned} D_\omega^\nu &= \frac{\nu}{\beta^* k}\frac{\partial^2 \varepsilon}{\partial x_j^2} - \frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2} = \frac{\nu}{k}\frac{\partial^2 \omega k}{\partial x_j^2} - \frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2} \\ &= \frac{\nu}{k}\left[\frac{\partial}{\partial x_j}\left(\omega\frac{\partial k}{\partial x_j} + k\frac{\partial \omega}{\partial x_j}\right)\right] - \nu\frac{\omega}{k}\frac{\partial^2 k}{\partial x_j^2} = \frac{2\nu}{k}\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \omega}{\partial x_j}\right) \end{aligned}$$

▶The ω eq. (which really is an ε eq. when the $k-\varepsilon$ constants are used) reads

$$\begin{aligned} \frac{\partial}{\partial x_j}(\bar{v}_j\omega) &= \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \alpha\frac{\omega}{k}P^k - \beta\omega^2 + \frac{2}{k}\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i} \\ \alpha &= C_{\varepsilon 1} - 1 = 0.44, \beta = (C_{\varepsilon 2} - 1)\beta^* = 0.0828 \end{aligned}$$

▶Inner region: $k-\omega$ coefficients; outer region: $k-\varepsilon$ coefficients. Blending function reads

$$F_1 = \tanh(\xi^4), \quad \xi \propto \frac{L_t}{x_n} = \frac{k^{1/2}}{\omega x_n}$$

▶$F_1 = 1$ in the near-wall region and $F_1 = 0$ in the outer region. The β-coefficient, e.g., reads

$$\beta_{SST} = F_1\beta_{k-\omega} + (1 - F_1)\beta_{k-\varepsilon}$$

▶SST model. Limitation of shear stress in adverse pressure gradient flow (APG).

▶The $k-\omega$ gives too high shear stress. The JK model $-\overline{v_1'v_2'} = a_1 k$ ($a_1 = c_\mu^{1/2}$) gives good results.

▶Two formulas for ν_t. ▶$\Omega = \partial \bar{v}_1/\partial x_2$. ▶Formulate JK model with Boussinesq.

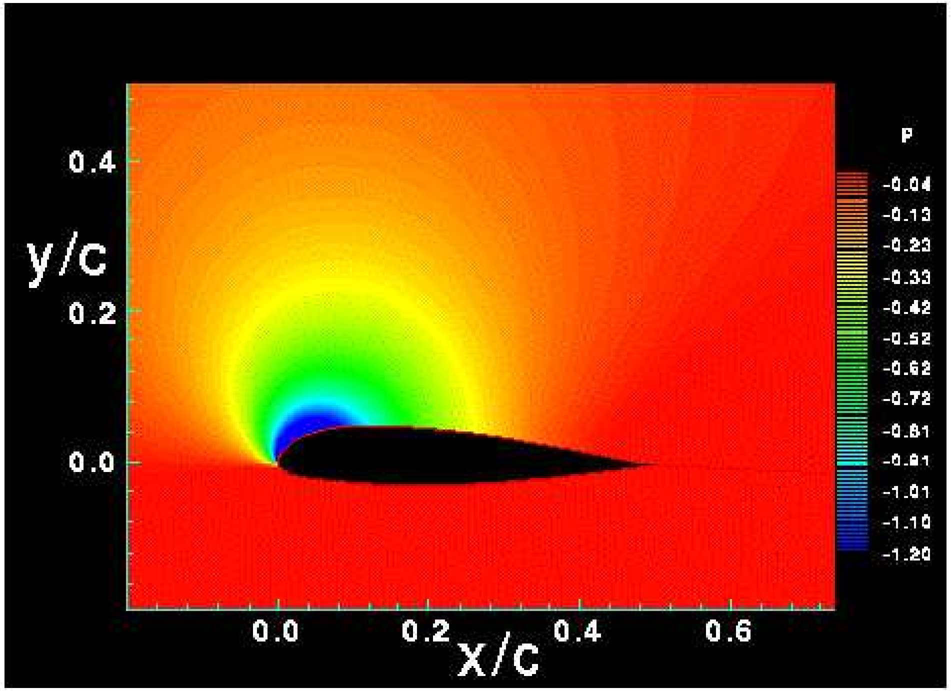
$$\left.\begin{array}{ll} \text{JK Model:} & \nu_t = \dfrac{-\overline{v_1'v_2'}}{\Omega} = \dfrac{a_1 k}{\Omega} \\ k-\omega \text{ model:} & \nu_t = \dfrac{k}{\omega} = \dfrac{a_1 k}{a_1 \omega} \end{array}\right\} \nu_t = \frac{a_1 k}{\underline{\max(a_1\omega, F_2\Omega)}}$$

F_2 is one near walls and zero elsewhere

▶The purpose of the underlined term above is:

- the second part, $F_2\Omega$ (the Johnson-King model), should be used in APG flow (where $P^k > \varepsilon$)
- the first part, $a_1\omega$ (the usual Boussinesq model), should be in the remaining part of the flow domain
- F_2 makes sure that the Johnson-King model is used only near the wall

▶ Adverse pressure gradient flow (APG).



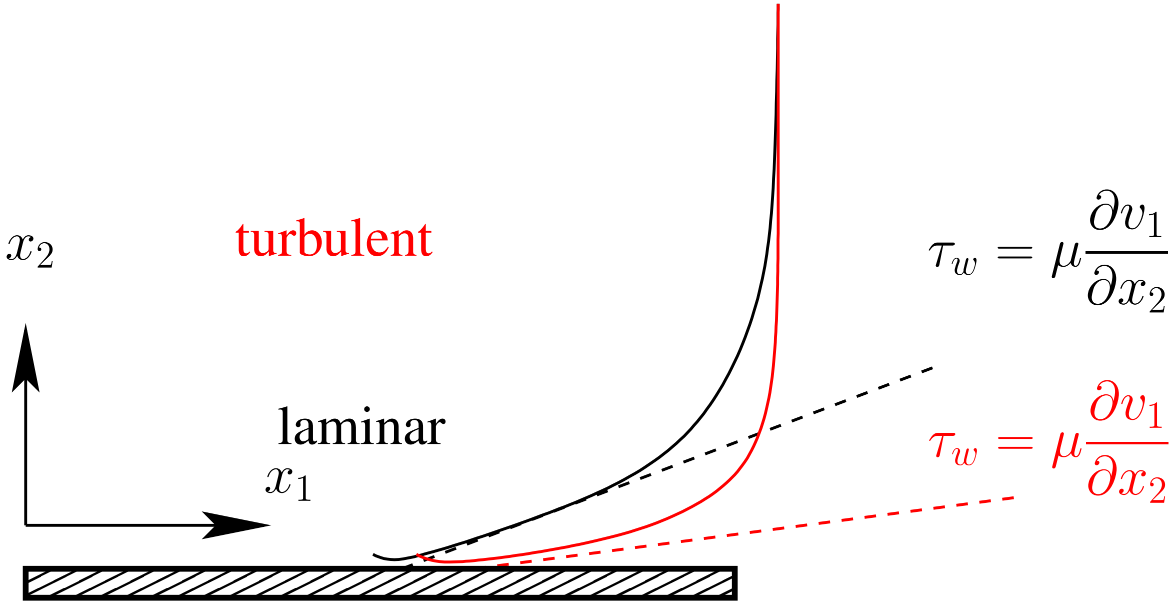
Flow around an airfoil. Angle of attack, $\alpha = 13^o$. Pressure contours.

On-line Lecture 6

¶See Section 5, Turbulence

▶ $v_i = \bar{v}_i + v'_i$, is irregular and consists of eddies of different size

▶ increases diffusivity

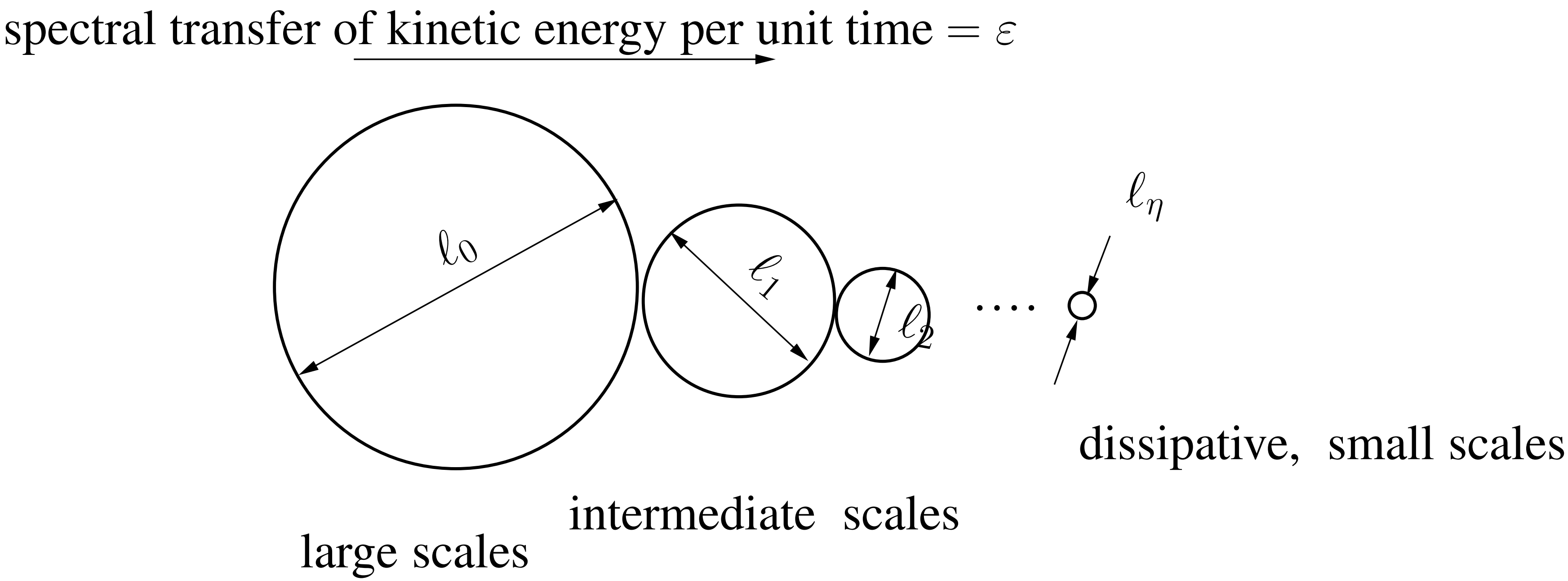


Difference between a laminar and turbulent boundary later

▶ occurs at large Reynolds numbers. Pipes: $Re_D = \frac{VD}{\nu} \simeq 2300$; boundary layers: $Re_x = \frac{Vx}{\nu} \simeq 500\,000$.

▶ is three-dimensional

▶ is dissipative. Kinetic energy, $v'_i v'_i/2$, in the small (dissipative) eddies are transformed into thermal energy (increases temperature).



▶Dissipation $\varepsilon = \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$ ▶All dissipation energy (say 90%) takes place at the small scales.

▶We want to characterize the dissipation of kinetic energy at small scales in two relevant quantities: ε, ν

$$\begin{array}{cccc} v_\eta & = & \nu^a & \varepsilon^b \\ [m/s] & = & [m^2/s] & [m^2/s^3] \end{array}$$

$$\begin{array}{rrcl} [m] & 1 & = & 2a + 2b \\ [s] & -1 & = & -a - 3b \end{array}$$

▶This gives the Kolmogorov scales, $a = b = 1/4$

$$v_\eta = (\nu\varepsilon)^{1/4}, \ \ell_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \ \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

▶At the next slide, we will look at energy spectra. It is based on Fourier series.

▶Any periodic function, f, can be expressed as a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(\kappa_n x) + b_n \sin(\kappa_n x)), \quad f = v', \quad \kappa_n = \frac{n\pi}{L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\kappa_n x)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\kappa_n x)dx$$

▶Parseval's formula states that the kinetic energy can be computed as

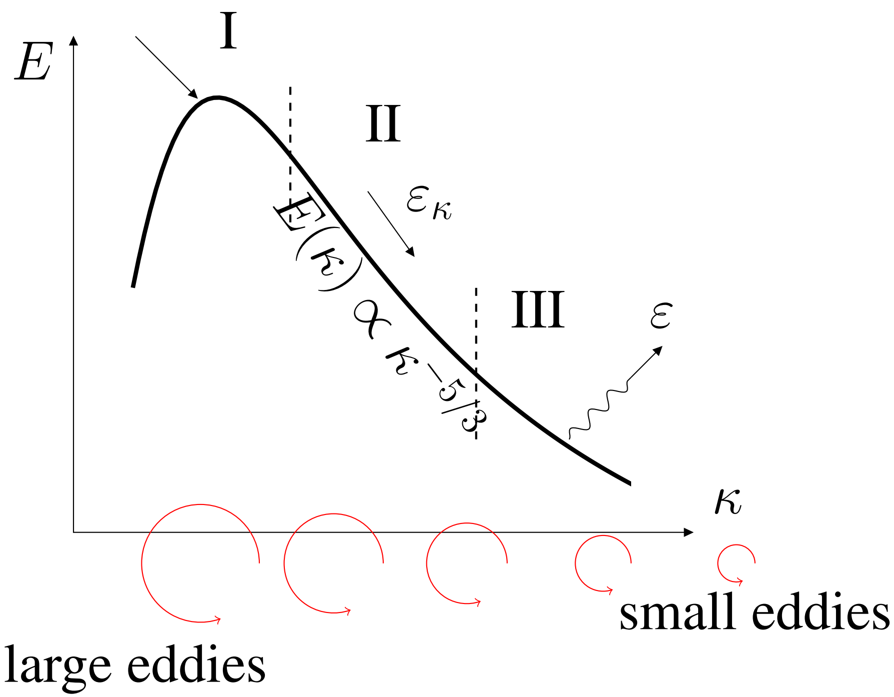
$$\int_{-L}^{L} v'^2(x)dx = \frac{L}{2}a_0^2 + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{35.1}$$

▶Hence, you can compute the kinetic energy by:

- integrating in Fourier (wavenumber) space (right-hand side)
- or integrating in physical space over all fluctuations (left-hand side)

▶Spectrum for turbulent kinetic energy, k

$$-\langle \bar{v}_i' \bar{v}_j' \rangle \frac{\partial \langle \bar{v}_i \rangle}{\partial x_j}$$

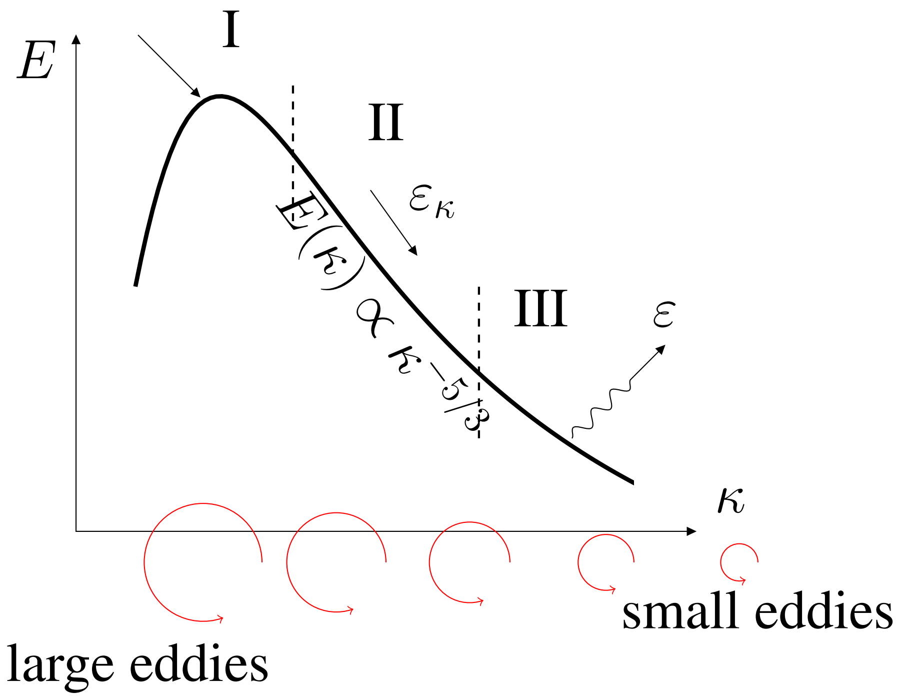


▶$E(\kappa_n) \propto a_n^2 + b_n^2$, see the Fourier series on the previous slide ▶

$$k = \int_0^\infty E(\kappa) d\kappa = \sum_0^\infty E(\kappa_n) \Delta \kappa_n \tag{35.2}$$

▶which corresponds to Parseval's formula

$$-\langle \bar{v}'_i \bar{v}'_j \rangle \frac{\partial \langle \bar{v}_i \rangle}{\partial x_j}$$



▶The turbulence spectrum is divided into three regions:

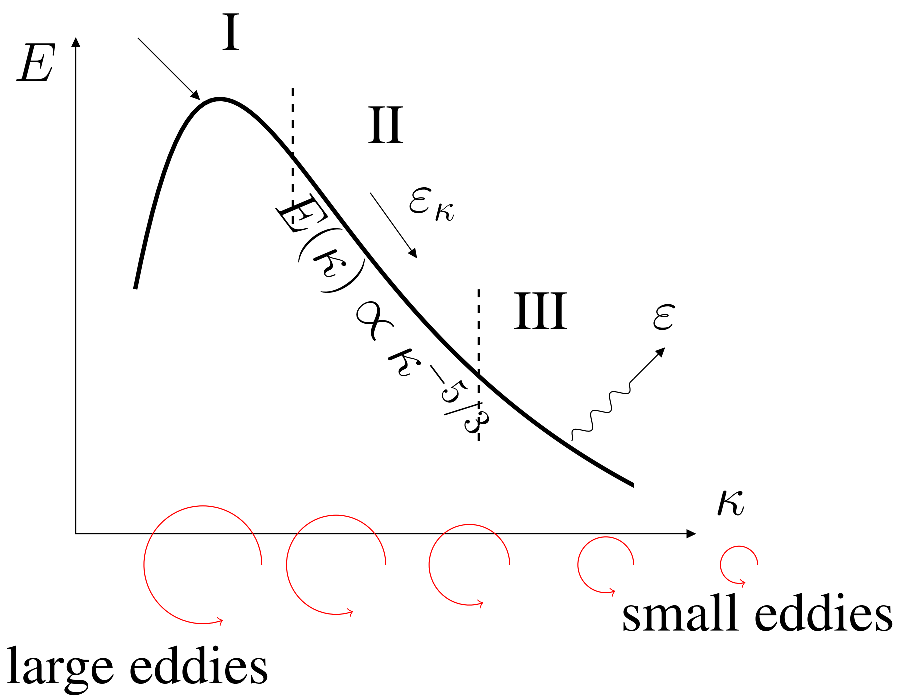
I. Large eddies carry most of the turb. kinetic energy. They extract energy from the mean flow, P^k.

II. Inertial subrange. Independent of both large eddies (mean flow) and viscosity. Isotropic eddies.

III. Dissipation range. Isotropic eddies ($\overline{v'_i v'_j} = c_1 \delta_{ij}$) described by the Kolmogorov scales.

▶Turb. kinetic energy in Region II

$$-\langle \bar{v}_i' \bar{v}_j' \rangle \frac{\partial \langle \bar{v}_i \rangle}{\partial x_j}$$



▶Turb. kinetic energy in Region II depends on: ▶ε and ▶eddy size $1/\kappa$ Recall: ▶$k = \int_0^\infty E(\kappa) d\kappa$

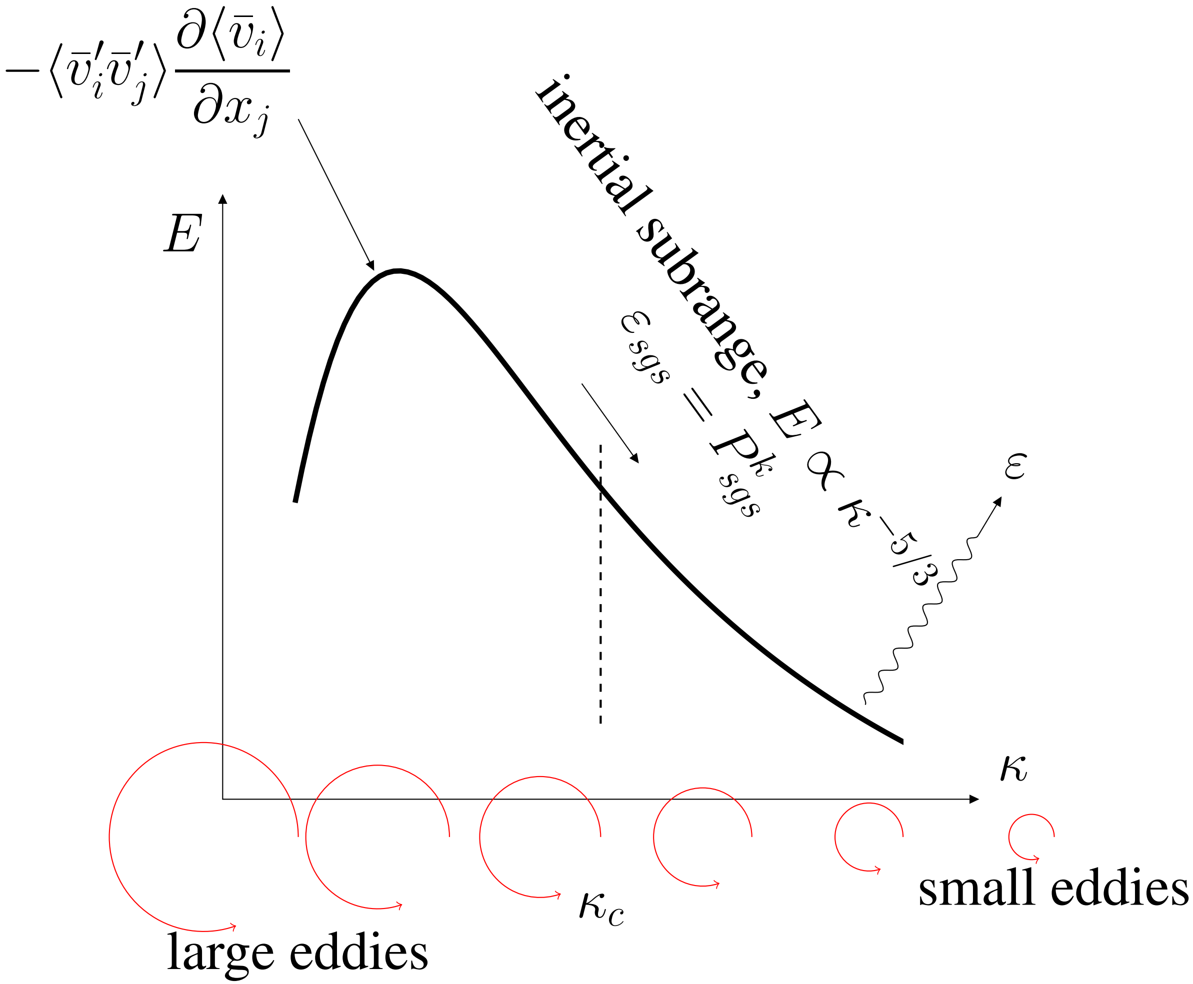
$$\begin{array}{cccc} E & = & \kappa^a & \varepsilon^b \\ [m^3/s^2] & = & [1/m] & [m^2/s^3] \end{array}$$

$$\begin{array}{cccc} [m] & 3 & = & -a+2b \\ [s] & -2 & = & -3b \end{array}$$

$b = 2/3, a = -5/3$ so that ▶$E(\kappa) = C_K \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}}$

▶This is called von Kármán spectrum or $-5/3$ law

▶Energy transfer from eddy-to-eddy



$$\varepsilon_\kappa \propto v_\kappa^2/(\ell_\kappa/v_\kappa) \propto \frac{v_\kappa^3}{\ell_\kappa} \propto \frac{v_0^3}{\ell_0}$$

▶Find relation between largest and smallest scales: $Re = v_0 \ell_0/\nu$, $v_\eta = (\nu\varepsilon)^{1/4}$, $\varepsilon = v_0^3/\ell_0$

$$\frac{v_0}{v_\eta} = (\nu\varepsilon)^{-1/4} v_0 = \left(\nu v_0^3/\ell_0\right)^{-1/4} v_0 = (v_0 \ell_0/\nu)^{1/4} = Re^{1/4}$$

$$\frac{\ell_0}{\ell_\eta} = \left(\frac{\nu^3}{\varepsilon}\right)^{-1/4} \ell_0 = \left(\frac{\nu^3 \ell_0}{v_0^3}\right)^{-1/4} \ell_0 = \left(\frac{\nu^3}{v_0^3 \ell_0^3}\right)^{-1/4} = Re^{3/4}$$

$$\frac{\tau_o}{\tau_\eta} = \left(\frac{\nu \ell_0}{v_0^3}\right)^{-1/2} \tau_0 = \left(\frac{v_0^3}{\nu \ell_0}\right)^{1/2} \frac{\ell_0}{v_0} = \left(\frac{v_0 \ell_0}{\nu}\right)^{1/2} = Re^{1/2}$$

▶We do a DNS (Direct Numerical Simulation) at a certain Reynolds number.

▶Now if we double the Re number, how much finer must the grid be?

$$\underbrace{2^{3/4}}_{x_1\,dir} \times \underbrace{2^{3/4}}_{x_2\,dir} \times \underbrace{2^{3/4}}_{x_3\,dir} \times \underbrace{2^{1/2}}_{time} = 2^{11/4} \simeq 7$$

▶Hence, doubling the Re number requires 7 times more computational effort

▶This explains why DNS (Direct Numerical Simulation) is too expensive at high Re numbers:

▶Why dissipation only at small scale/eddies?

- Let's show that $\varepsilon = \nu \overline{\frac{\partial v'_i}{\partial x_j}\frac{\partial v'_i}{\partial x_j}}$ gets larger the smaller the scales/eddies.
- The velocity gradient for an eddy can be estimated as

$$\left(\frac{\partial v}{\partial x}\right)_\kappa \propto \frac{v_\kappa}{\ell_\kappa} \propto \left(v_\kappa^2\right)^{1/2} \kappa$$

▶Energy spectrum: recall that k for wavenumber κ is $k \propto E\Delta\kappa$ (see Eq. 35.2). We get

$$E(\kappa) \propto k_\kappa/\kappa \propto v_\kappa^2/\kappa \propto \kappa^{-5/3} \quad \Rightarrow \quad v_\kappa^2 \propto \kappa^{-2/3}$$

We get

$$\left(\frac{\partial v}{\partial x}\right)_\kappa \propto \left(\kappa^{-2/3}\right)^{1/2} \kappa \propto \kappa^{-1/3}\kappa \propto \kappa^{2/3}$$

▶Hence ε increases as κ increases, i.e. ε gets larger for small eddies.

On-line Lecture 7

¶See Section 18, Large Eddy Simulations

in RANS:

$$\langle\Phi\rangle = \frac{1}{2T}\int_{-T}^{T}\Phi(t)dt, \quad \Phi = \langle\Phi\rangle + \Phi', \quad \langle\Phi'\rangle = 0 \quad \Rightarrow \langle\Phi\rangle = \langle\langle\Phi\rangle\rangle$$

in LES:

$$\bar{\Phi}(x,t) = \frac{1}{\Delta x}\int_{x-0.5\Delta x}^{x+0.5\Delta x}\Phi(\xi,t)d\xi, \quad \Phi = \bar{\Phi} + \Phi'', \quad \overline{\Phi''} \neq 0 \quad \Rightarrow \bar{\bar{\Phi}} \neq \bar{\Phi}$$

Momentum equations in DNS:

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}(v_i v_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 v_i}{\partial x_j \partial x_j} \tag{36.1}$$

Momentum equations in LES:

$$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{v}_i \bar{v}_j) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}, \quad \tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j \tag{36.2}$$

Momentum equations in LES:

$$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{v}_i \bar{v}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}, \qquad \tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j \tag{36.3}$$

▶Filter pressure gradient in Eq. 36.1

$$\overline{\frac{\partial p}{\partial x_i}} = \frac{1}{V}\int_V \frac{\partial p}{\partial x_i} dV \overset{?}{=} \frac{\partial}{\partial x_i}\left(\frac{1}{V}\int_V p dV\right) = \frac{\partial \bar{p}}{\partial x_i}$$

$$\overline{\frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i}\left(\frac{1}{V}\int_V p dV\right) + \mathcal{O}\left((\Delta x)^2\right) = \frac{\partial \bar{p}}{\partial x_i} + \mathcal{O}\left((\Delta x)^2\right)$$

▶Filter non-linear term in Eq. 36.1

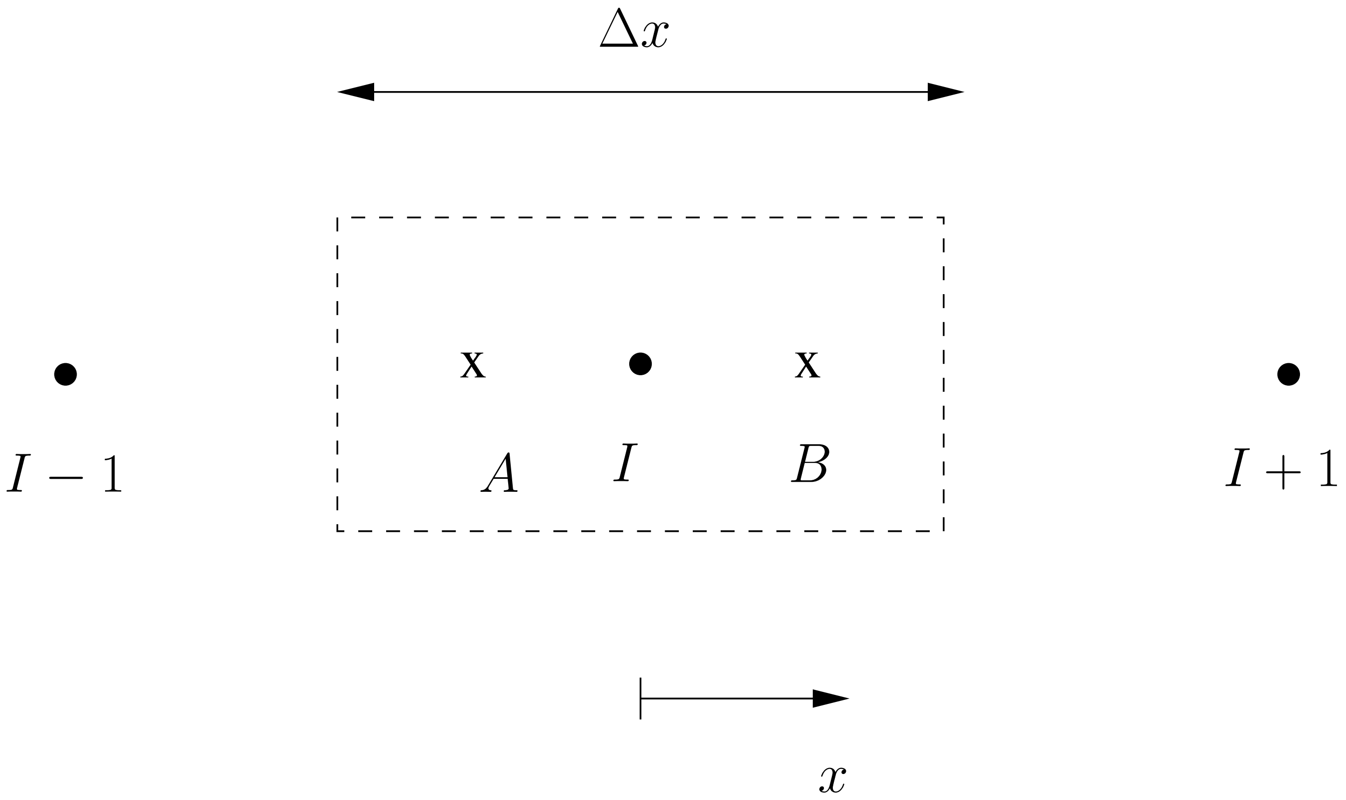
$$\overline{\frac{\partial v_i v_j}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(\overline{v_i v_j}\right) + \mathcal{O}\left((\Delta x)^2\right)$$

$$\text{Left side}: \quad \frac{\partial}{\partial x_j}\left(\overline{v_i v_j}\right) \underbrace{- \frac{\partial}{\partial x_j}\left(\overline{v_i v_j}\right) + \frac{\partial}{\partial x_j}\left(\bar{v}_i \bar{v}_j\right)} = \frac{\partial}{\partial x_j}\left(\bar{v}_i \bar{v}_j\right)$$

$$\text{Right side}: \quad \underbrace{-\frac{\partial}{\partial x_j}\left(\overline{v_i v_j}\right) + \frac{\partial}{\partial x_j}\left(\bar{v}_i \bar{v}_j\right)} = -\frac{\partial \tau_{ij}}{\partial x_j}$$

▶ $$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{v}_i \bar{v}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}, \qquad \tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

►Filtering twice (used for turbulence modeling)

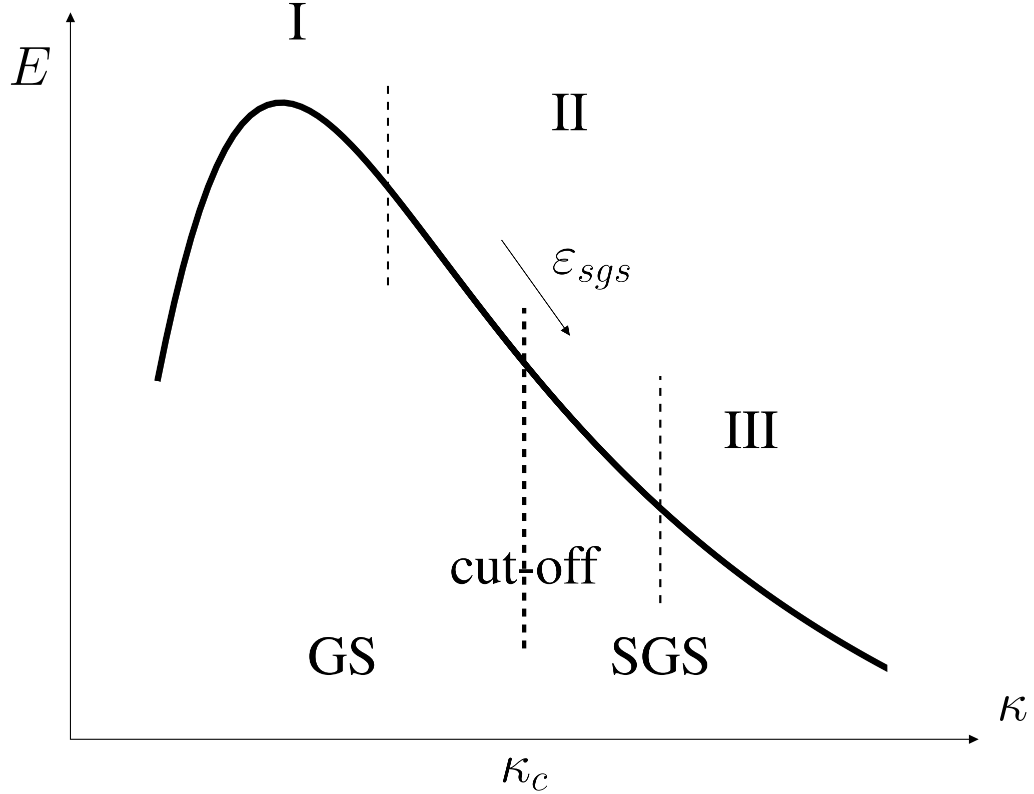


Control volume (dashed lines). $\bar{\bar{v}}_I \neq \bar{v}_I$

$$\bar{\bar{v}}_I = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \bar{v}(\xi)d\xi = \frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{0} \bar{v}(\xi)d\xi + \int_{0}^{\Delta x/2} \bar{v}(\xi)d\xi\right) =$$

$$= \frac{1}{\Delta x}\left(\frac{\Delta x}{2}\bar{v}_A + \frac{\Delta x}{2}\bar{v}_B\right) = \frac{1}{2}\left[\left(\frac{1}{4}\bar{v}_{I-1} + \frac{3}{4}\bar{v}_I\right) + \left(\frac{3}{4}\bar{v}_I + \frac{1}{4}\bar{v}_{I+1}\right)\right] = \frac{1}{8}\left(\bar{v}_{I-1} + 6\bar{v}_I + \bar{v}_{I+1}\right) \neq \bar{v}_I$$

¶See Section 18.3, Resolved & SGS scales (GS & SGS)

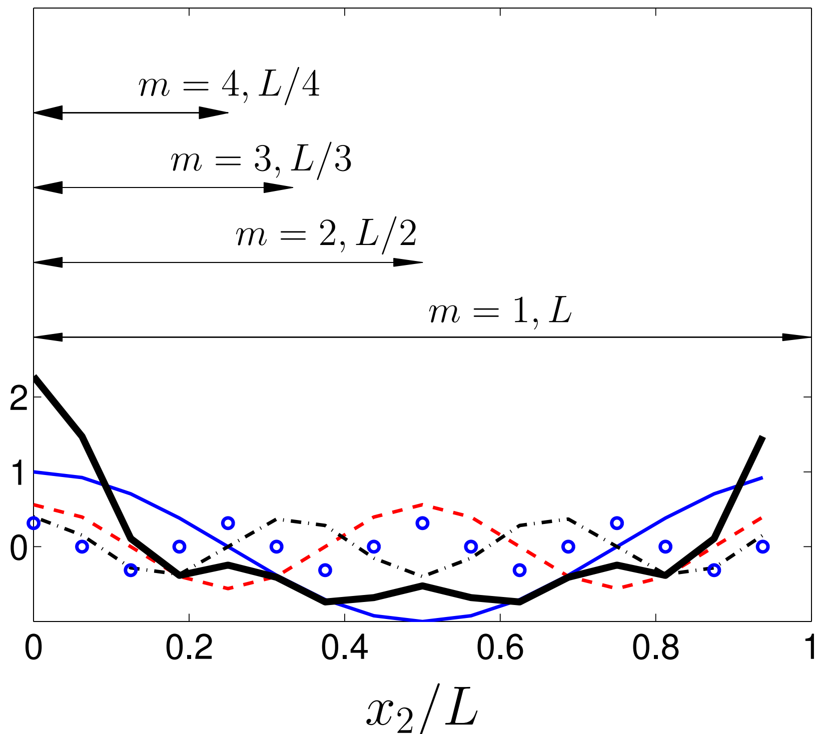


$\kappa \leq \kappa_c$: Grid (=resolved) Scales; $\kappa > \kappa_c$= Sub-Grid Scales

¶See Section 18.5, Highest resolved wavenumbers

▶A Fourier series (see Appendix H)

$$v_1'(x) = \sum_{n=-\infty}^{\infty} c_n \exp(\imath \kappa_n x_1)) \quad \text{only symmetric part, i.e. real}$$



v_2' vs. x_2/L. —: term 1 ($m=1$); - -: term 2 ($m=2$); -·-: term 3 ($m=3$); ∘: term 4 ($m=4$); ▬: v_2'

Matlab code is given in Section I.3.

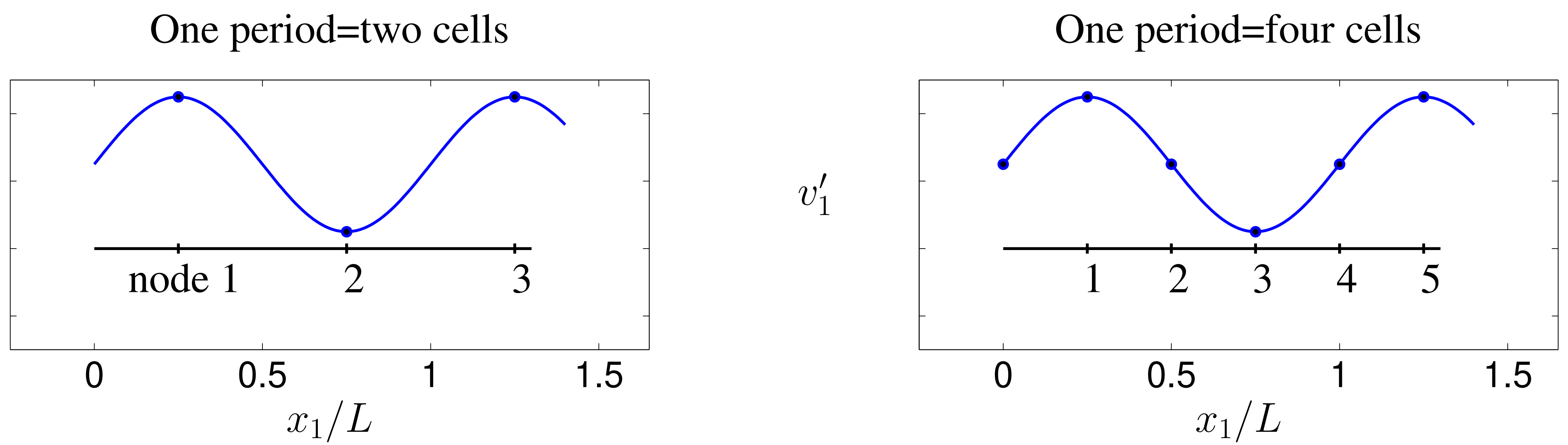
▶We construct v_2' as a sum of four Fourier components

$$v_2'(x_2) = b_1 \cos\left(\frac{2\pi}{L/1}x_2\right) + b_2 \cos\left(\frac{2\pi}{L/2}x_2\right) + b_3 \cos\left(\frac{2\pi}{L/3}x_2\right) + b_4 \cos\left(\frac{2\pi}{L/4}x_2\right)$$

▶ On the previous slide, we showed a couple of different wave number (Fourier) modes.

$$v_1' = \sin(\kappa_c x_1)$$

▶ How large wave numbers (i.e. how short wavelengths) can we resolve in an LES?



$$\text{two cells} : \kappa_c 2\Delta x_1 = 2\pi \Rightarrow \kappa_c = 2\pi/(2\Delta x_1) = \pi/\Delta x_1$$

$$\text{four cells} : \kappa_c 4\Delta x_1 = 2\pi \Rightarrow \kappa_c = 2\pi/(4\Delta x_1) = \pi/(2\Delta x_1)$$

¶See Section 18.6, Subgrid model

▶Smagorinsky Subgrid model

$$
\begin{aligned}
\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} &= -\nu_{sgs}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) = -2\nu_{sgs}\bar{s}_{ij} \\
\nu_{sgs} &\propto v'\ell \propto \left(\Delta x_1 \frac{\partial \bar{v}_1}{\partial x_1}\right)\Delta x_1 \propto \Delta x_1^2 \left(\frac{\partial \bar{v}_1}{\partial x_1}\right) \propto (C_S\Delta)^2 \sqrt{2\bar{s}_{ij}\bar{s}_{ij}} = (C_S\Delta)^2 |\bar{s}| \\
\Delta &= (\Delta V_{IJK})^{1/3}
\end{aligned}
$$

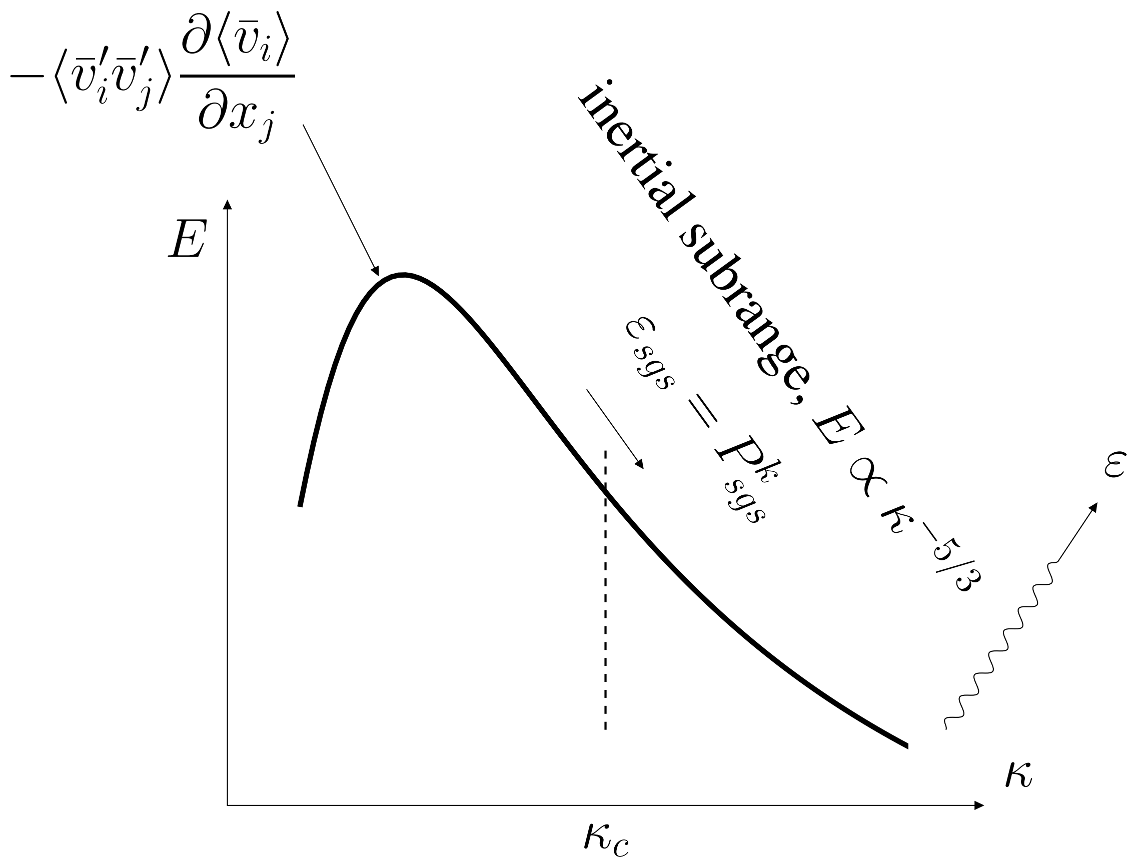
$|\bar{s}|$ stems from the production term in the k eq., $|\bar{s}^2| = 2\bar{s}_{ij}\bar{s}_{ij}$

¶See Section 18.21, One-equation k_{sgs} model

$$\frac{\partial k_{sgs}}{\partial t}+\frac{\partial}{\partial x_j}(\bar{v}_j k_{sgs}) = \frac{\partial}{\partial x_j}\left[(\nu+\nu_{sgs})\frac{\partial k_{sgs}}{\partial x_j}\right]+P_{k_{sgs}}-\varepsilon$$

$$\nu_{sgs}\propto \ell v' = c_k\Delta k_{sgs}^{1/2}, \quad P_{k_{sgs}} = 2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij}, \quad \varepsilon \propto \frac{v'^3}{\ell} = C_\varepsilon \frac{k_{sgs}^{3/2}}{\Delta}$$

¶See Section 18.8, Energy path



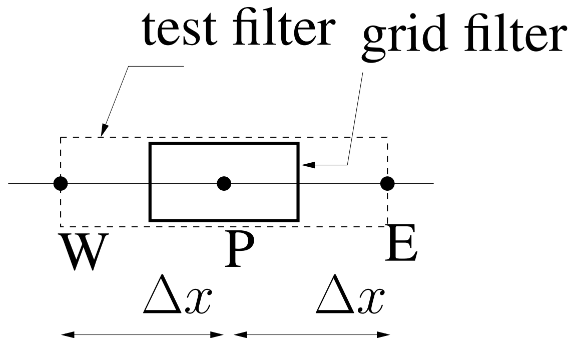
¶See Section 18.9, SGS kinetic energy

$$
\begin{aligned}
v_i &= \langle v_i \rangle + v_i', \quad v_i = \bar{v}_i + v_i'' = \langle \bar{v}_i \rangle + \bar{v}_i' + v_i'' \\
k &\equiv \frac{1}{2}\langle v_i' v_i' \rangle = \int_0^\infty E(\kappa) d\kappa, \quad k_{sgs} \equiv \frac{1}{2}\langle v_i'' v_i'' \rangle = \int_{\kappa_c}^\infty E(\kappa) d\kappa \\
\bar{k} &\equiv \frac{1}{2}\langle \bar{v}_i' \bar{v}_i' \rangle = \int_0^{\kappa_c} E(\kappa) d\kappa, \quad \bar{K} \equiv \frac{1}{2}\langle \bar{v}_i \rangle \langle \bar{v}_i \rangle
\end{aligned}
$$

On-line Lecture 8

¶See Section 18.11, The dynamic model

▶The dynamic model. C is computed. Test filter, $\widehat{\Delta} = 2\Delta$



Control volume for grid and test filter.

▶First, grid and test filter the Navier-Stokes (DNS)

$$
\begin{aligned}
\frac{\partial \widehat{\bar v}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\widehat{\overline{v_i v_j}}\right) &= -\frac{1}{\rho}\frac{\partial \widehat{\bar p}}{\partial x_i}+\nu\frac{\partial^2 \widehat{\bar v}_i}{\partial x_j \partial x_j}\\
\text{Left side} &: \frac{\partial}{\partial x_j}\left(\widehat{\overline{v_i v_j}}\right)-\frac{\partial}{\partial x_j}\left(\widehat{\overline{v_i v_j}}\right)+\frac{\partial}{\partial x_j}\left(\widehat{\bar v}_i \widehat{\bar v}_j\right)=\frac{\partial}{\partial x_j}\left(\widehat{\bar v}_i \widehat{\bar v}_j\right)\\
\text{Right side} &: -\frac{\partial}{\partial x_j}\left(\widehat{\overline{v_i v_j}}\right)+\frac{\partial}{\partial x_j}\left(\widehat{\bar v}_i \widehat{\bar v}_j\right)
\end{aligned}
$$

▶We get

$$
\frac{\partial \widehat{\bar v}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\widehat{\bar v}_i \widehat{\bar v}_j\right) = -\frac{1}{\rho}\frac{\partial \widehat{\bar p}}{\partial x_i}+\nu\frac{\partial^2 \widehat{\bar v}_i}{\partial x_j \partial x_j}-\frac{\partial T_{ij}}{\partial x_j}, \quad T_{ij}=\widehat{\overline{v_i v_j}}-\widehat{\bar v}_i \widehat{\bar v}_j \tag{37.1}
$$

$$\frac{\partial \widehat{\bar{v}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\widehat{\bar{v}}_i \widehat{\bar{v}}_j\right) = -\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \widehat{\bar{v}}_i}{\partial x_j \partial x_j} - \frac{\partial T_{ij}}{\partial x_j}, \quad T_{ij} = \widehat{\overline{v_i v_j}} - \widehat{\bar{v}}_i \widehat{\bar{v}}_j \tag{37.1}$$

►Second, we test filter the LES equations

$$\frac{\partial \widehat{\bar{v}}_i}{\partial t} + \frac{\partial \widehat{\bar{v}_i \bar{v}_j}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \widehat{\bar{v}}_i}{\partial x_j \partial x_j} - \frac{\partial \widehat{\tau}_{ij}}{\partial x_j}$$

$$\text{Left side} : \frac{\partial \widehat{\bar{v}_i \bar{v}_j}}{\partial x_j} - \frac{\partial \widehat{\bar{v}_i \bar{v}_j}}{\partial x_j} + \frac{\partial \widehat{\bar{v}}_i \widehat{\bar{v}}_j}{\partial x_j} = \frac{\partial \widehat{\bar{v}}_i \widehat{\bar{v}}_j}{\partial x_j}$$

$$\text{Right side} : \underbrace{-\frac{\partial \widehat{\bar{v}_i \bar{v}_j}}{\partial x_j} + \frac{\partial \widehat{\bar{v}}_i \widehat{\bar{v}}_j}{\partial x_j}}_{-\partial \mathcal{L}_{ij}/\partial x_j}$$

►We get

$$\frac{\partial \widehat{\bar{v}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\widehat{\bar{v}}_i \widehat{\bar{v}}_j\right) = -\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \widehat{\bar{v}}_i}{\partial x_j \partial x_j} - \frac{\partial \widehat{\tau}_{ij}}{\partial x_j} - \frac{\partial \mathcal{L}_{ij}}{\partial x_j} \tag{37.2}$$

Identification of Eqs. 37.1 and 37.2 gives

$$T_{ij} = \widehat{\overline{v_i v_j}} - \widehat{\bar{v}}_i \widehat{\bar{v}}_j + \widehat{\tau}_{ij} = \mathcal{L}_{ij} + \widehat{\tau}_{ij}, \quad \frac{1}{3}\delta_{ij} T_{kk} = \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + \frac{1}{3}\delta_{ij}\widehat{\tau}_{kk} \tag{37.3}$$

$$T_{ij} - \frac{1}{3}\delta_{ij}T_{kk} + \widehat{\tau}_{ij} - \frac{1}{3}\delta_{ij}\widehat{\tau}_{kk} = \mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} \tag{37.4}$$

▶Smagorinsky model for both grid and test level SGS stresses:

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2C\Delta^2|\bar{s}|\bar{s}_{ij} \tag{37.5}$$

$$T_{ij} - \frac{1}{3}\delta_{ij}T_{kk} = -2C\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} \tag{37.6}$$

where

$$\widehat{\bar{s}}_{ij} = \frac{1}{2}\left(\frac{\partial \widehat{\bar{v}}_i}{\partial x_j} + \frac{\partial \widehat{\bar{v}}_j}{\partial x_i}\right), \quad |\widehat{\bar{s}}| = \left(2\widehat{\bar{s}}_{ij}\widehat{\bar{s}}_{ij}\right)^{1/2}$$

▶Three equations, three unknowns!

▶Eqs. 37.5. 37.6 into Eq. 37.4 gives

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2\left(C\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \widehat{C\Delta^2|\bar{s}|\bar{s}_{ij}}\right)$$

▶We need to yank C out of the test filter; ▶If not, it's very difficult to solve for C. ▶We get

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C\left(\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \widehat{\Delta^2|\bar{s}|\bar{s}_{ij}}\right)$$

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2C\underbrace{\left(\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij} - \Delta^2\widehat{|\bar{s}|\bar{s}_{ij}}\right)}_{M_{ij}} = 0$$

▶Now we get

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij} = 0 \tag{37.7}$$

▶This cannot be satisfied for all i, j ▶Least-square problem:

$$Q = \left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij}\right)^2$$

▶Find a minimum of Q which best satisfies Eq. 37.7 for all i, j

$$\frac{\partial Q}{\partial C} = 4M_{ij}\left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij}\right) = 4M_{ij}\left(\mathcal{L}_{ij} + 2CM_{ij}\right) = 0 \tag{37.8}$$

since $\frac{1}{3}\delta_{ij}\mathcal{L}_{kk}M_{ij} = \frac{1}{3}\mathcal{L}_{kk}M_{ii} = 0$ since $\widehat{\bar{s}}_{ii} = \bar{s}_{ii} = 0$ thanks to continuity.

Eq. 37.8: Minimum or maximunm?

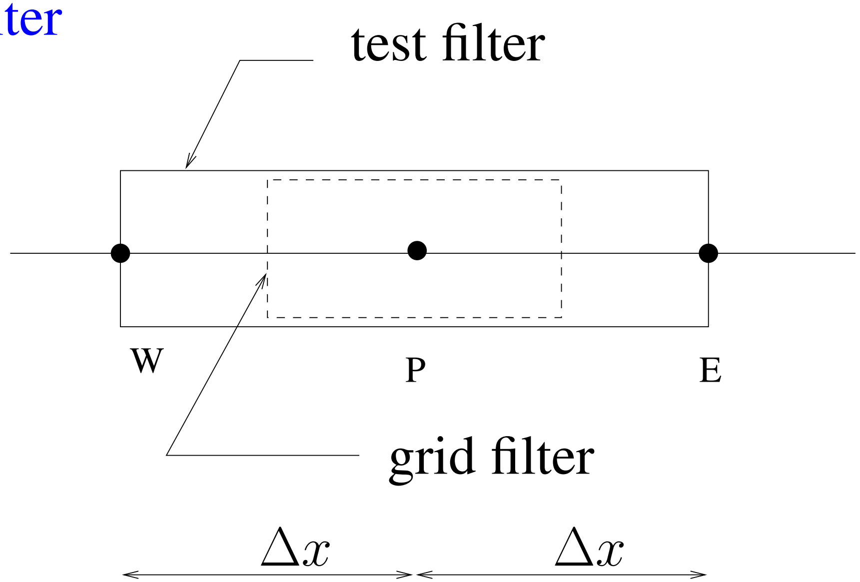
▶$\partial^2 Q/\partial C^2 = 8M_{ij}M_{ij} > 0$ ▶Hence, minimum (fortunately)

$$\frac{\partial Q}{\partial C} = 4M_{ij}\left(\mathcal{L}_{ij} + 2CM_{ij}\right) = 0$$

▶We get

$$C = -\frac{\mathcal{L}_{ij}M_{ij}}{2M_{ij}M_{ij}}, \qquad \text{stability problems: needs smoothing}$$

¶See Section 18.12, The test filter



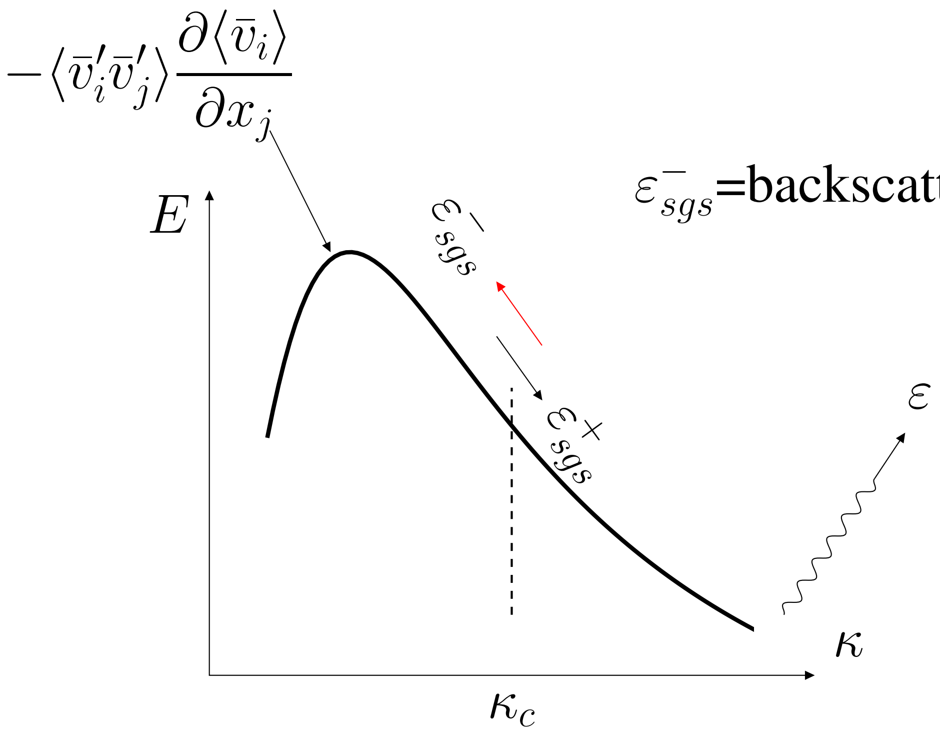
$\widehat{\bar{v}}_P$ is computed as ($\widehat{\Delta x} = 2\Delta x$)

$$\begin{aligned}
\widehat{\bar{v}}_P &= \frac{1}{2\Delta x}\int_W^E \bar{v}dx = \frac{1}{2\Delta x}\left(\int_W^P \bar{v}dx + \int_P^E \bar{v}dx\right) \\
&= \frac{1}{2\Delta x}\left(\bar{v}_w\Delta x + \bar{v}_e\Delta x\right) = \frac{1}{2}\left(\frac{\bar{v}_W + \bar{v}_P}{2} + \frac{\bar{v}_P + \bar{v}_E}{2}\right) = \frac{1}{4}\left(\bar{v}_W + 2\bar{v}_P + \bar{v}_E\right)
\end{aligned}$$

$$C = -\frac{\mathcal{L}_{ij} M_{ij}}{2 M_{ij} M_{ij}}, \qquad \nu_{sgs} = C \Delta^2 |\bar{s}|$$

▶Is C positive? ▶Do we want it to stay positive? ▶Limits on C?

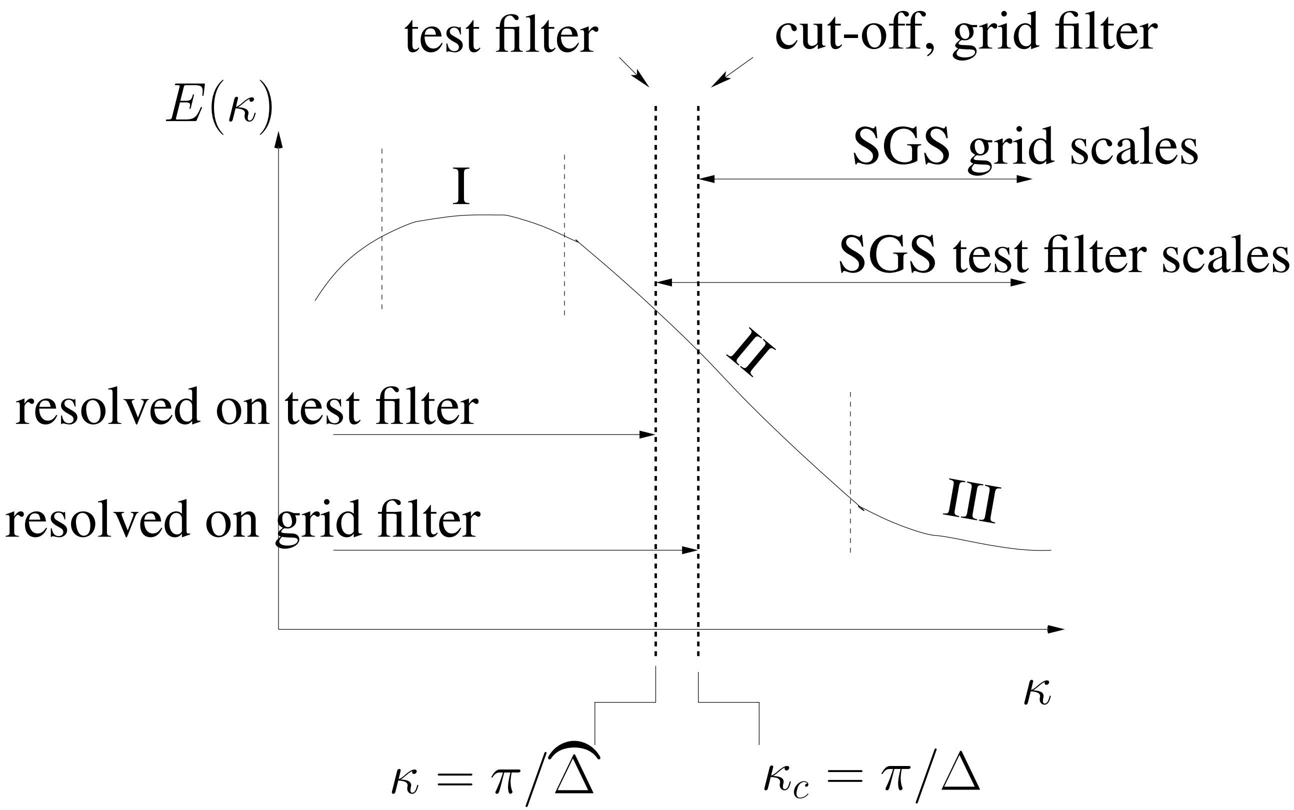
$$\nu_{tot} = \nu + \nu_{sgs} = \nu + C\Delta^2 |\bar{\bar{s}}| > 0 \quad \Rightarrow \nu_{sgs} > -\nu$$



ε^-_{sgs}=backscatter

$$\varepsilon_{sgs} = 2\nu_{sgs} \bar{s}_{ij} \bar{s}_{ij} = \varepsilon^+_{sgs} + \varepsilon^-_{sgs}$$

¶See Section 18.13, Stresses on grid, test and intermediate level

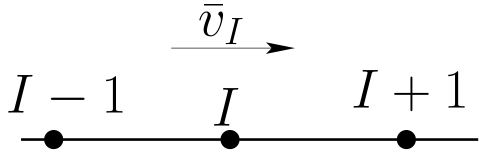


$$\tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j \qquad \textbf{stresses with } \ell < \Delta$$

$$T_{ij} = \widehat{\overline{v_i v_j}} - \widehat{\bar{v}}_i \widehat{\bar{v}}_j \qquad \textbf{stresses with } \ell < \widehat{\Delta}$$

$$\mathcal{L}_{ij} = T_{ij} - \widehat{\tau}_{ij} \qquad \textbf{stresses with } \Delta < \ell < \widehat{\Delta}$$

¶See Section 18.20, Numerical method



$$\bar v_I\left(\frac{\partial \bar v}{\partial x}\right)_{exact} = \bar v_I\left(\frac{\bar v_I-\bar v_{I-1}}{\Delta x}+\mathcal{O}\left(\Delta x\right)\right) \tag{37.9}$$

$$\bar v_{I-1} = \bar v_I - \Delta x\left(\frac{\partial \bar v}{\partial x}\right)_I + \frac{1}{2}(\Delta x)^2\left(\frac{\partial^2 \bar v}{\partial x^2}\right)_I + \mathcal{O}\left((\Delta x)^3\right) \tag{37.10}$$

►Insert Eq. 37.9 into Eq. 37.9

$$\bar v\left(\frac{\partial \bar v}{\partial x}\right)_{exact} = \bar v\frac{\partial \bar v}{\partial x} - \underbrace{\frac{1}{2}\Delta x \bar v\frac{\partial^2 \bar v}{\partial x^2}}_{\mathcal{O}(\Delta x)} + \bar v\mathcal{O}\left((\Delta x)^2\right)$$

►$\Delta x\bar v/2$ acts as an additional numerical viscosity

►The total diffusion now consists of

$$\text{diffusion term} = \frac{\partial}{\partial x}\left\{(\nu+\nu_{sgs}+\nu_{num})\frac{\partial \bar v}{\partial x}\right\}$$

►And the total dissipation

$$\varepsilon_{tot} = 2(\nu+\nu_{sgs}+\nu_{num})\bar s_{ij}\bar s_{ij}$$

On-line Lecture 9

¶See Section 18.15, Scale-similarity Models

$$\begin{aligned}\tau_{ij} &= \overline{v_i v_j} - \bar{v}_i \bar{v}_j = \overline{(\bar{v}_i + v''_i)(\bar{v}_j + v''_j)} - \bar{v}_i \bar{v}_j \\ &= \underbrace{(\overline{\bar{v}_i \bar{v}_j} - \bar{v}_i \bar{v}_j)}_{L_{ij}} + \underbrace{\left[\overline{\bar{v}_i v''_j + \bar{v}_j v''_i}\right]}_{C_{ij}} + \underbrace{\overline{v''_i v''_j}}_{R_{ij}}\end{aligned}$$

► C_{ij} denotes scale-similar stresses

$$v_i = \bar{v}_i + v''_j \quad \Rightarrow \overline{v''_j} = \bar{v}_i - \bar{\bar{v}}_i$$

► Scale-similarity model

$$C_{ij}^M = c_r (\bar{v}_i \bar{v}_j - \bar{\bar{v}}_i \bar{\bar{v}}_j), \quad R_{ij} = 0 \tag{38.1}$$

► The $C_{ij} = \overline{\bar{v}_i v''_j} + \overline{\bar{v}_j v''_i}$ and $L_{ij} = \overline{\bar{v}_i \bar{v}_j} - \bar{v}_i \bar{v}_j$ stresses are not Galilean invariant (but $C_{ij} + L_{ij}$ is).

► This is shown in an Appendix in the eBook.

¶See Section 18.16, The Bardina Model

► The Bardina model reads (since C_{ij}^M was not sufficiently dissipative, a Smagorinsky model is added)

$$C_{ij}^M = c_r(\overline{\bar{v}_i\bar{v}_j} - \bar{v}_i\bar{v}_j), \quad L_{ij} = \overline{\bar{v}_i\bar{v}_j} - \bar{v}_i\bar{v}_j, \quad R_{ij}^M = -2C_S^2\Delta^2|\bar{s}|\bar{s}_{ij}$$ This is called a mixed model

► The Bardina model is not Galielan invariant

► Germano, proposed redefined terms in the Bardina Model (which are Galielan invariant)

$$\begin{aligned}
\tau_{ij}^m &= \tau_{ij} = C_{ij}^m + L_{ij}^m + R_{ij}^m \\
L_{ij}^m &= c_r\left(\overline{\bar{v}_i\bar{v}_j} - \bar{\bar{v}}_i\bar{\bar{v}}_j\right) \\
C_{ij}^m &= 0 \\
R_{ij}^m &= R_{ij} = \overline{v_i''v_j''}
\end{aligned}$$

► The modified Leonard stresses is the same as the "unmodified" one plus the modeled cross term C_{ij}

¶See Section 18.22, Smagorinsky model derived from the k_{sgs} equation

- Small isotropic scales: production = dissipation (convection and diffusion are negligible)

$$P_{k_{sgs}} = 2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij} = \nu_{sgs}|\bar{s}|^2 = \varepsilon$$

▶Replace ε by ν_{sgs} and Δ.

$$\nu_{sgs} = \varepsilon^a (C_S\Delta)^b \quad \Rightarrow a = 1/3, b = 4/3 \quad \Rightarrow \nu_{sgs} = (C_S\Delta)^{4/3}\varepsilon^{1/3} \quad \Rightarrow \varepsilon = \nu_{sgs}^3 \Delta^{-4}/C_S$$

which gives

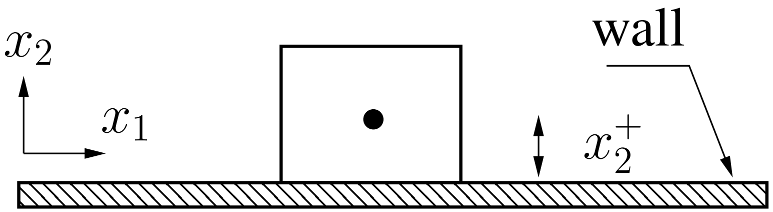
$$\nu_{sgs} = (C_s\Delta)^2|\bar{s}|$$

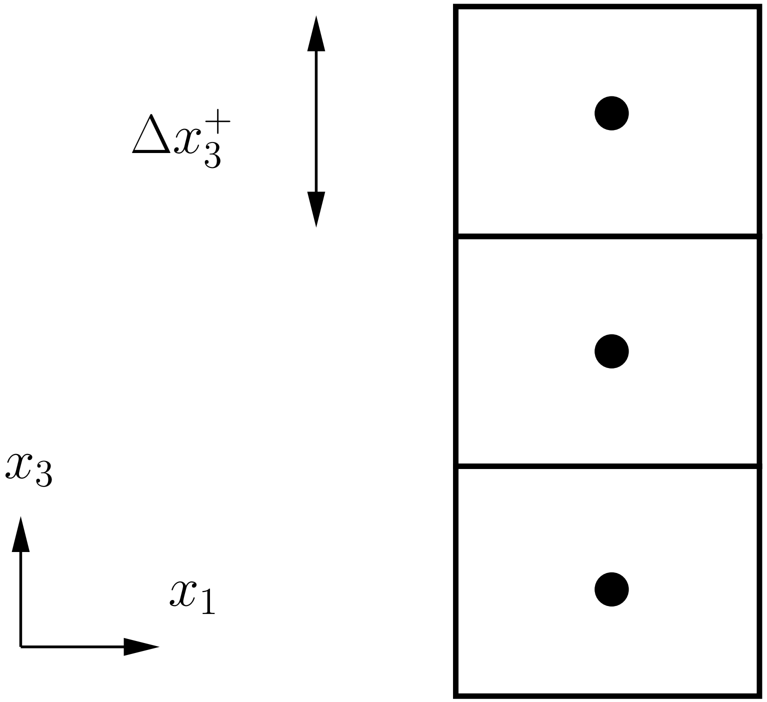
¶See Section 18.26, Resolution requirements

- In LES we resolve large scales.
- Near the wall, the “large” scales are not that large
- ⇒ very expensive to resolve these “large” scales.

$$\Delta x_1^+ \simeq 100, \quad \Delta x_{2,min}^+ \simeq 1, \quad \Delta x_3^+ \simeq 30$$

⇒ VERY expensive





▶There are many ways to estimate resolution (see Assignment 2a & 2b):

- Energy spectra: does they show a $-5/3$ range or not? NO GOOD
- Ratio between viscous and modelled turbulent viscosity (not recommended in [128, 129]). This quantity does not say much about how good the LES resolution is. It tells us how close the LES is to a DNS.
- Ratio between modeled and total shear stress (recommended in [128, 129]).
- Ratio between modeled and total turbulent kinetic energy (recommended in [205]).
- Ratio of integral lengthscale to cell size.
  - The integral lengthscale is computed from two-point correlations (they are explained below).
  - If the ratio is larger than, say, 16, the resolution is sufficient.
  - This is recommended in [128, 129].
- Ratio of boundary-layer thickness, δ to Δx and Δz. This is a measure of the resolution in the log-law region.

$$\delta/\Delta x_1 = 10 - 20, \quad \delta/\Delta x_3 = 20 - 40, \quad x_2^+ < 1$$

¶See Section 10.1, Two-point correlations

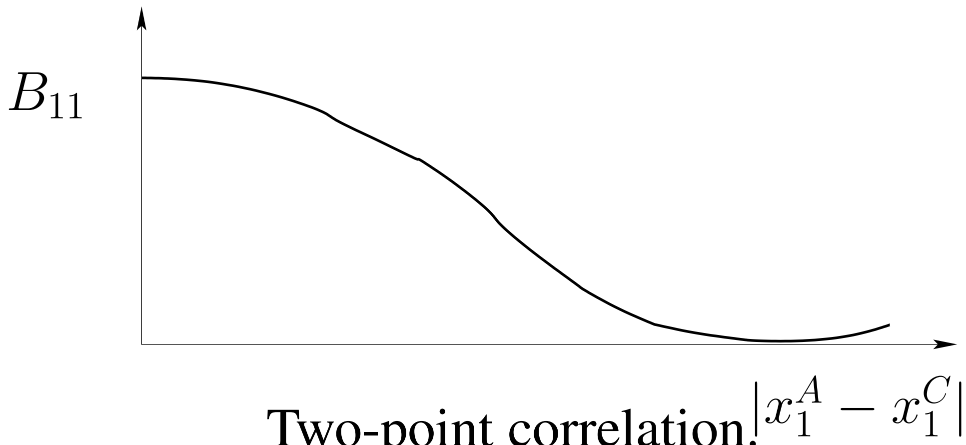
►The integral lengthscale is computed from two-point correlations which is defined as:

$$B_{11}(x_1^A, x_1^C) = \overline{v_1'(x_1^A) v_1'(x_1^C)}$$

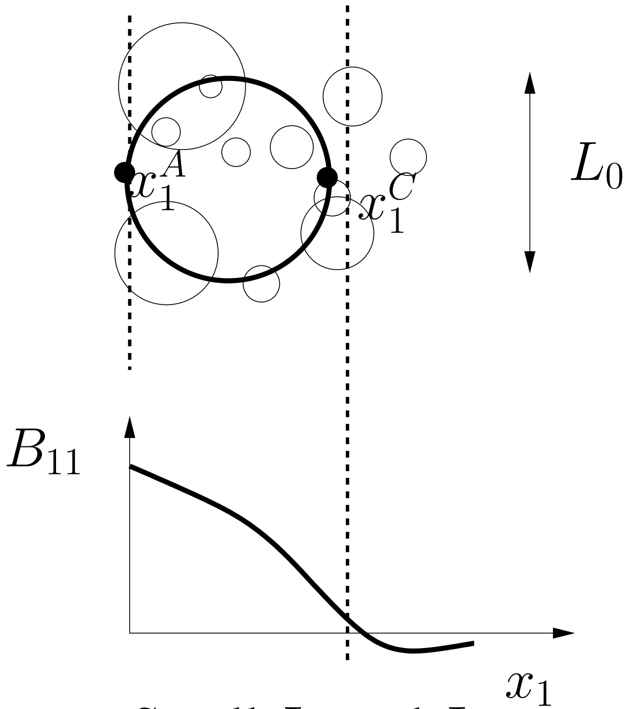
Often, expressed as

$$B_{11}(x_1^A, \hat{x}_1) = \overline{v_1'(x_1^A) v_1'(x_1^A + \hat{x}_1)}$$

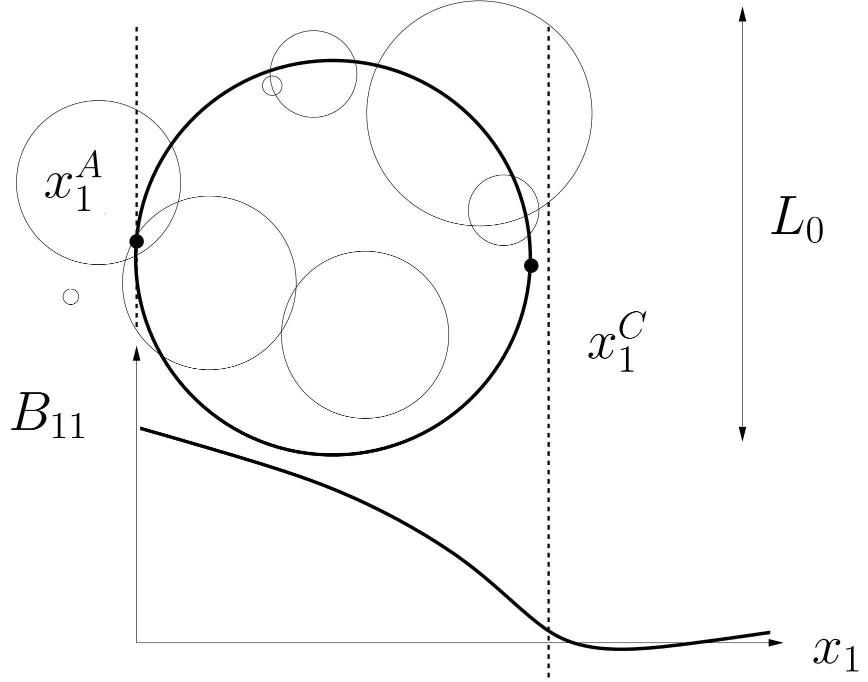
where $\hat{x}_1 = x_1^C - x_1^A$ is the separation distance between point A and C.



Two-point correlation.



Small L_0 and L_{int}



Large L_0 and L_{int}

$B_{11}(x_1^A, \hat{x}_1) = \overline{v_1'(x_1^A) v_1'(x_1^C)}$. Two-point corr, the largest eddies (thick lines), L_0.

- When we move point A and C closer to each other, B_{11} increases; when A=C, then $B_{11} = \overline{v'^2}(x_1^A)$
- When C moves further and further away from A, $\Rightarrow B_{11} \to 0$
- The normalized two-point correlation reads

$$B_{11}^{norm}(x_1^A, \hat{x}_1) = \frac{1}{v_{1,rms}(x_1^A) v_{1,rms}(x_1^A + \hat{x}_1)} \overline{v_1'(x_1^A) v_1'(x_1^A + \hat{x}_1)}$$

- Integral length scale is then computed as: $L_{int} = \int_0^\infty B_{11}^{norm}(\hat{x}_1) d\hat{x}_1$

¶See Section 10.2, Auto correlation

►Auto correlation is a "two-point correlation in time" which reads

$$B_{11}(t^A, \hat{t}\,) = \overline{v_1'(t^A)v_1'(t^A + \hat{t}\,)}$$

$\hat{t} = t^C - t^A$ is time separation between time A and C.

►In analogy to L_{int}, the *integral time scale*, T_{int}, is defined

$$T_{int} = \int_0^\infty B_{11}^{norm}(\hat{t})d\hat{t}$$

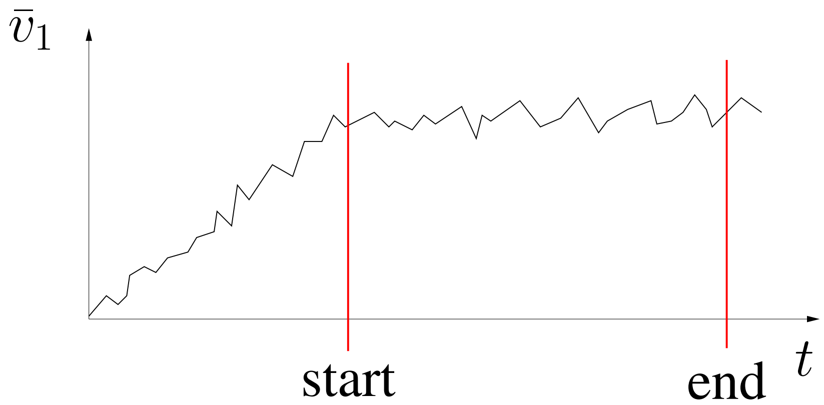
►Integral timescale is used an Assignment 2a for finding time samples that are *independent* (i.e. the time between the samples is at least one integral timescale).

¶See Section 18.20.1, RANS vs. LES

▶Numerical method: RANS vs. LES

	RANS	**LES**
Domain	2D or 3D	always 3D
Time domain	steady or unsteady	always unsteady
Space discretization	2nd order upwind	central differencing
Time discretization	1st order	2nd order (e.g. C-N)
Turbulence model	$\geq$ two-equations	zero- or one-equation

▶**Start** and **end** time averaging. $t_{end} - t_{start} \simeq 100H/\langle \bar{v} \rangle_{center}$



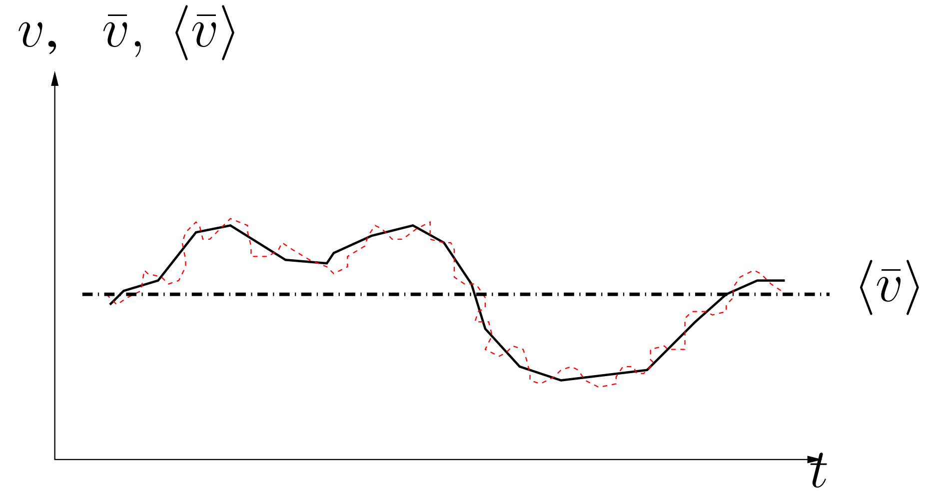
- Say that we want to store 3D inst. fields $\Rightarrow$ we can post-proc, e.g., two-point corr anywhere
- Then we want to store as few 3D fields as possible (otherwise our disk space will quickly be saturated)
- Answer: store only every T_{int} second: 100 indepedent samples gives a statistical error 0.01

¶See Section 19, URANS: Unsteady RANS

▶The usual Reynolds decomposition is employed   note that we now change notation (again!)

$$\bar{v}(t) = \frac{1}{2T}\int_{t-T}^{t+T} v(t)dt, \quad v = \bar{v} + v''$$

▶URANS eqns=RANS, but with the unsteady term retained



Decomposion in URANS. —: $\bar{v}$; - -: v; -·- $\langle\bar{v}\rangle$.

- Decomposition of velocities: $v = \bar{v} + v'' = \langle\bar{v}\rangle + \bar{v}' + v''$.
- In theory, T should be $\ll$ resolved time scale. This is called "scale separation".
- In practice, it is seldom satisfied.

▶RANS turbulence models are used in URANS

- We should choose a model with small dissipation (i.e. small ν_t) in order to not kill/dampen resolved turbulence.
- Reynolds-stress turbulence models best (but very expensive).
- The EARSM (Section 11.11) and non-linear eddy-viscosity models (Section 14) also seem to give less dissipation. Probably because the weaker connection between $\bar{s}_{ij}$ and $\overline{v_i'v_j'}$ which reduces P^k.
- Modelled dissipation (turbulence model) and numerical dissipation (discretization scheme) may be of equal importance

¶See Section 20, DES: Detached-Eddy-Simulations

▶DES=Detached Eddy Simulations:▶Use RANS near walls and LES away from walls

▶S-A one-equation model (RANS) reads

$$\frac{d\rho\tilde{\nu}_t}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu+\mu_t}{\sigma_{\tilde{\nu}_t}}\frac{\partial\tilde{\nu}_t}{\partial x_j}\right) + \text{cr. term} + P - C_{w1}\rho f_w\left(\frac{\tilde{\nu}_t}{d}\right)^2, \quad d = x_n$$

▶Replace d with $\tilde{d}$:

$$\left(\frac{\tilde{\nu}_t}{d}\right)^2 \Rightarrow$$

$$\tilde{d} = \min\{C_{DES}\Delta, d\}, \quad \Delta = \max\{\Delta x_1, \Delta x_3, \Delta x_3\} \tag{38.2}$$

▶This is the S-A DES one-equation model

¶See Section 20.1, DES based on two-equation models

▶ $k-\varepsilon$ RANS

$$C^k = D^k + P^k - \varepsilon, \quad C^\varepsilon = D^\varepsilon + P^\varepsilon - \Psi$$

▶ $k-\varepsilon$ DES I (modify ε_T)

$$C^k = D^k + P^k - \varepsilon \quad \Rightarrow \quad C^k = D^k + P^k - \varepsilon_T, \quad \varepsilon_T = \max\left(\varepsilon, C_\varepsilon \frac{k^{3/2}}{\Delta}\right), \quad \nu_t = C_\mu \frac{k^2}{\varepsilon}$$

▶ $\varepsilon_T \uparrow$ in LES region ▶ $\Rightarrow k \downarrow$ in LES region ▶ $\Rightarrow \nu_t \downarrow$ in LES region

▶ $k-\varepsilon$ DES II (modify ν_T and ε_T)

$$C^k = D^k + P^k - \varepsilon_T, \quad C^\varepsilon = D^\varepsilon + P^\varepsilon - \Psi^\varepsilon, \quad \ell_t = \min\left(C_\mu \frac{k^{3/2}}{\varepsilon}, C_{DES}\Delta\right), \quad \nu_T = k^{1/2}\ell_t$$

▶ $\varepsilon_T \uparrow$ in LES region ▶ $\Rightarrow k \downarrow$ in LES region

▶ $\ell_t \downarrow$ in LES region ▶ $\Rightarrow \nu_t \downarrow$ in LES region

On-line Lecture 10

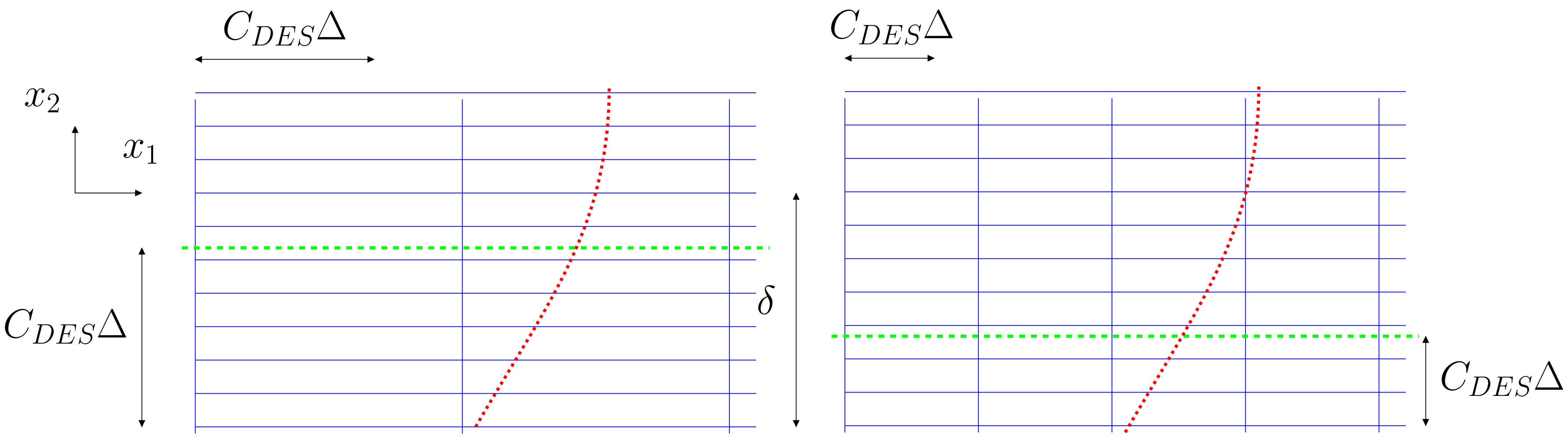
¶See Section 20.2, DES based on the $k-\omega$ SST model

$k-\omega$ SST DES (modify $\beta^* k\omega$)

$$C^k = D^k + P^k - F_{DES}\beta^* k\omega, \quad F_{DES} = \max\left\{\frac{L_t}{C_{DES}\Delta}, 1\right\} = \max\left\{\frac{k^{1/2}}{\beta^*\omega C_{DES}\Delta}, 1\right\}$$

- $L_t = \dfrac{k^{3/2}}{\varepsilon}$
- $\omega = \dfrac{\varepsilon}{\beta^* k}$
- $\Rightarrow L_t = \dfrac{k^{1/2}}{\beta^*\omega}$

¶See Section 20.3, DDES



Grid (solid lines) and a velocity profile (dotted line). RANS-LES interface: dashed line. $C_{DES} = 0.67$

- Consider the S-A DES (see Eq. 38.2). It may occur that the $\tilde{d}$ switches to LES in the boundary layer because Δx_1 is too small (Δx_3 is usually smaller than Δx_1). Recall: $\Delta = \max\{\Delta x_1, \Delta x_3, \Delta x_3\}$
- Hence boundary layer is treated in LES mode with too a coarse mesh $\Rightarrow$ poorly resolved LES $\Rightarrow$ inaccurate predictions.
- The left grid above is a good DES mesh because at the RANS-LES interface $\tilde{d} = \min(d, C_{DES}\Delta) = C_{DES}\Delta = C_{DES}\Delta x_1 \simeq \delta$ (see dashed line) $\Rightarrow$ the entire boundary layer is modeled by RANS.
- Right grid is a poor DES grid: $\tilde{d} = \min(d, C_{DES}\Delta) = C_{DES}\Delta x_1 \ll \delta$ (dashed line) $\Rightarrow$ the outer part of the boundary layer is in LES mode (and the LES resolution requirements, $\delta/\Delta x_1 > 10$, is not satisfied)

▶The solution is DDES (Delayed DES)

▶In DDES

$$F_{DES} = \max\left\{\frac{L_t}{C_{DES}\Delta}, 1\right\}$$

is replaced by

$$F_{DDES} = \max\left\{\frac{L_t}{C_{DES}\Delta}(1 - F_S), 1\right\}$$

where F_S ($F_S = 1$ in the boundary layer) is taken as F_1 or F_2 of the SST model.

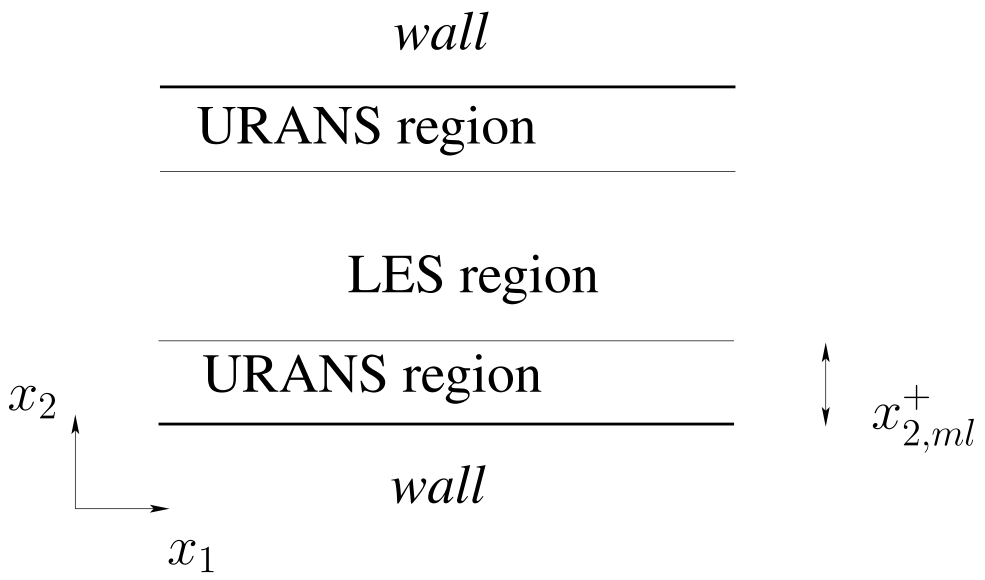
▶F_S is called the shielding function: ▶it protects the boundary layer from LES

¶See Section 21, Hybrid LES-RANS

▶DES: The entire boundary layer is modelled with URANS

Hybrid LES-RANS: Only the inner part of the log region is modelled with URANS.

▶Hybrid LES-RANS is also called WM-LES (WM=Wall-Modelled)



▶One-equation model in both URANS and LES region

$$\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j}(\bar{v}_j k_T) = \frac{\partial}{\partial x_j}\left[(\nu + \nu_T)\frac{\partial k_T}{\partial x_j}\right] + P_{k_T} - C_\varepsilon \frac{k_T^{3/2}}{\ell}$$

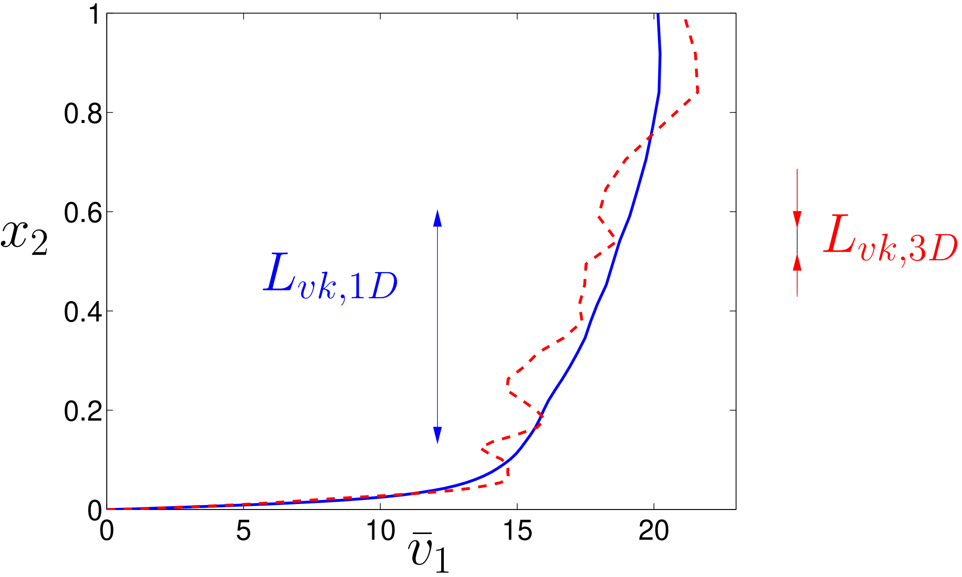
$$P_{k_T} = 2\nu_T \bar{s}_{ij}\bar{s}_{ij}, \quad \nu_T \propto k^{1/2}\ell$$

▶Inner region ($x_2 \leq x_{2,ml}$): $\ell \propto \kappa x_2$

▶outer region: $\ell = \Delta$

¶See Section 22, The SAS model

- ▶This is a method to improve URANS. ▶Why URANS? ▶It does not involve Δ
- ▶If the flow tries to go unsteady in URANS: ▶P^k increases
- ▶ $\Rightarrow$ ν_t increases ▶ $\Rightarrow$ the flow goes back to steady state.
- ▶The objective of SAS is to reduce ν_t when the equations want to go into unsteady, resolving turbulence mode (LES mode).



$$L_{vK,3D} = \kappa \frac{|\partial \bar{v}_1/\partial y|}{|\partial^2 \bar{v}_1/\partial x_2^2|}$$

$$L_{vK,1D} = \kappa \frac{|\partial \langle \bar{v}_1\rangle/\partial y|}{|\partial^2 \langle \bar{v}_1\rangle/\partial x_2^2|}$$

Solid line: $L_{vk,1D}$; dashed line: $L_{vk,3D}$

- ▶An additional source term, P_{SAS}, is used in the ω equation. $P_{SAS} \propto \dfrac{L_t}{L_{vK,3D}}$.
- ▶$L_{vK,3D}$ is used as detector

$$L_t \propto \frac{k^{1/2}}{\omega}, \quad L_{vK,3D} = \kappa \frac{|\bar{s}|}{|U''|}, \quad U'' = \left(\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j}\frac{\partial^2 \bar{v}_i}{\partial x_k \partial x_k}\right)^{0.5}$$

▶ k and ω equations

$$C^k = P^k + D^k - \beta^* k\omega$$

$$C^\omega = P^\omega + D^\omega - \Psi^\omega + P_{SAS}, \quad \nu_t = \frac{k}{\omega}, \quad P_{SAS} \propto \frac{L_t}{L_{vK,3D}}$$

▶ The von Kármán length scale is used to detect unsteadiness.

- $L_{vK,3D}$ detects fluctuations (i.e. it gets small)
- $\Rightarrow$ the P_{SAS} term increases
- $\Rightarrow$ ω increases
- $\Rightarrow$ k decreases
- $\Rightarrow$ ν_t decreases
- $\Rightarrow$ mom.eqns go into (or stay in) unsteady mode
- In URANS (without SAS), resolved fluctuations are damped.

On-line Lecture 11

¶See Section 23, The PANS Model

▶PANS: Partial-Averaging Navier-Stokes. It is a hybrid LES-RANS model based on the $k-\varepsilon$ model.

- $f_k = k/k_{tot}$ and $f_\varepsilon = \varepsilon/\varepsilon_{tot}$: ratio of modelled to total $k_{tot} = k + k_{res}, \quad \varepsilon_{tot} = \varepsilon + \varepsilon_{res}$.
- $f_\varepsilon < 1$ means that part of the dissipation is resolved.
  - This occurs only for DNS-like resolution.
  - Hence, in practice $f_\varepsilon = 1$
- $0 < f_k \leq 1$
  - DNS: $f_k \to 0$
  - RANS: $f_k = 1$
  - LES: it is in-between ▶$0 < f_k \lesssim 0.5$

▶Derivation of PANS k equation

- Multiply the RANS k_{RANS} equation ($k_{tot} = k_{RANS} = k + k_{res}$) by f_k

▶Left side, $f_k k_{tot} = k$ ($\bar{V}_j$ is the RANS velocity, f_k assumed constant)

$$f_k \left\{ \frac{\partial k_{tot}}{\partial t} + \bar{V}_j \frac{\partial k_{tot}}{\partial x_j} \right\} = \frac{\partial k}{\partial t} + \bar{V}_j \frac{\partial k}{\partial x_j} \simeq \frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j}$$

▶Right side, diffusion term

$$f_k \left\{ \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_{t,tot}}{\sigma_k} \right) \frac{\partial k_{tot}}{\partial x_j} \right] \right\} = \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_{t,tot}}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_{ku}} \right) \frac{\partial k}{\partial x_j} \right]$$

where $\sigma_{ku} = \sigma_k f_k^2 / f_\varepsilon, \quad \nu_t = c_\mu k^2/\varepsilon, \quad \nu_{t,tot} = c_\mu k_{tot}^2 / \varepsilon_{tot}$

▶Right side, production and dissipation term

▶$P^{k,tot}$ and ε_{tot} are replaced by P_k and ε, i.e.

$$f_k \left(P^{k,tot} - \varepsilon_{tot} \right) = P^k - \varepsilon \quad \Rightarrow \quad P^{k,tot} = \frac{1}{f_k}(P^k - \varepsilon) + \frac{\varepsilon}{f_\varepsilon}$$

$$P^{k,tot} = \frac{1}{f_k}(P^k - \varepsilon) + \frac{\varepsilon}{f_\varepsilon} \tag{40.1}$$

▶The k equation can now be written

$$\frac{\partial k}{\partial t} + \frac{\partial (k\bar{v}_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{ku}}\right)\frac{\partial k}{\partial x_j}\right] + P^k - \varepsilon$$

▶Derivation of ε equation

▶Left side (f_ε assumed constant, $f_\varepsilon \varepsilon_{tot} = \varepsilon$)

$$f_\varepsilon\left\{\frac{\partial \varepsilon_{tot}}{\partial t} + \bar{V}_j\frac{\partial \varepsilon_{tot}}{\partial x_j}\right\} = \frac{\partial \varepsilon}{\partial t} + \bar{V}_j\frac{\partial \varepsilon}{\partial x_j} \simeq \frac{\partial \varepsilon}{\partial t} + \bar{v}_j\frac{\partial \varepsilon}{\partial x_j}$$

▶Right side, diffusion term

$$f_\varepsilon\left\{\frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_{t,tot}}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon_{tot}}{\partial x_j}\right]\right\} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_{t,tot}}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{\varepsilon u}}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

▶Production and destruction terms ▶use Eq. 40.1, $k_{tot} = k/f_k$, $\varepsilon_{tot} = \varepsilon/f_\varepsilon$

$$P^{k,tot} = \frac{1}{f_k}(P^k - \varepsilon) + \frac{\varepsilon}{f_\varepsilon} \tag{40.1}$$

$$f_\varepsilon \left\{ C_{\varepsilon 1} P^{k,tot} \frac{\varepsilon_{tot}}{k_{tot}} - C_{\varepsilon 2} \frac{\varepsilon^2_{tot}}{k_{tot}} \right\} = f_\varepsilon C_{\varepsilon 1} \frac{\varepsilon_{tot}}{k_{tot}} \left(\frac{1}{f_k}(P^k - \varepsilon) + \frac{\varepsilon}{f_\varepsilon} \right) - f_\varepsilon C_{\varepsilon 2} \frac{\varepsilon^2_{tot}}{k_{tot}}$$

$$C_{\varepsilon 1} \frac{\varepsilon f_k}{k} \left(\frac{1}{f_k}(P^k - \varepsilon) + \frac{\varepsilon}{f_\varepsilon} \right) - C_{\varepsilon 2} \frac{\varepsilon^2 f_k}{f_\varepsilon k} = C_{\varepsilon 1} \frac{\varepsilon}{k} P^k - C_{\varepsilon 1} \frac{\varepsilon^2}{k} + C_{\varepsilon 1} \frac{\varepsilon^2 f_k}{k f_\varepsilon} - C_{\varepsilon 2} \frac{\varepsilon^2 f_k}{f_\varepsilon k} = C_{\varepsilon 1} \frac{\varepsilon}{k} P^k - C^*_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

where

$$C^*_{\varepsilon 2} = C_{\varepsilon 1} + \frac{f_k}{f_\varepsilon}(C_{\varepsilon 2} - C_{\varepsilon 1}) = 1.5 + \frac{f_k}{f_\varepsilon}(1.9 - 1.5)$$

▶The ε eqn can now be written

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial(\varepsilon \bar{v}_j)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_{\varepsilon u}} \right) \frac{\partial \varepsilon}{\partial x_j} \right] + C_{\varepsilon 1} P^k \frac{\varepsilon}{k} - C^*_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial(\varepsilon \bar{v}_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_{\varepsilon u}}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1} P^k \frac{\varepsilon}{k} - C^*_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$C^*_{\varepsilon 2} = C_{\varepsilon 1} + \frac{f_k}{f_\varepsilon}(C_{\varepsilon 2} - C_{\varepsilon 1}) = 1.5 + \frac{f_k}{f_\varepsilon}(1.9 - 1.5), \quad f_\varepsilon = 1, \quad \nu_t = c_\mu \frac{k^2}{\varepsilon}$$

▶When $f_k = 1$, the PANS eqns are in RANS mode

- When $f_k < 1$ (say, $f_k = 0.4$) then:

$C^*_{\varepsilon 2}\frac{\varepsilon^2}{k}$ is reduced

$\Rightarrow \varepsilon$ is increased

$\Rightarrow k$ is decreased

$\Rightarrow \nu_t$ is decreased (both small k and large ε)

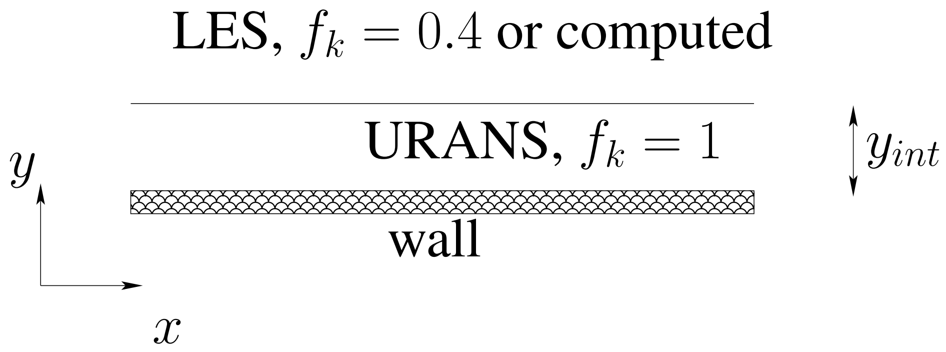
$\Rightarrow$ the momentum eqns go into LES mode.

¶See Section 23.2, Zonal PANS: different treatments of the RANS-LES interface

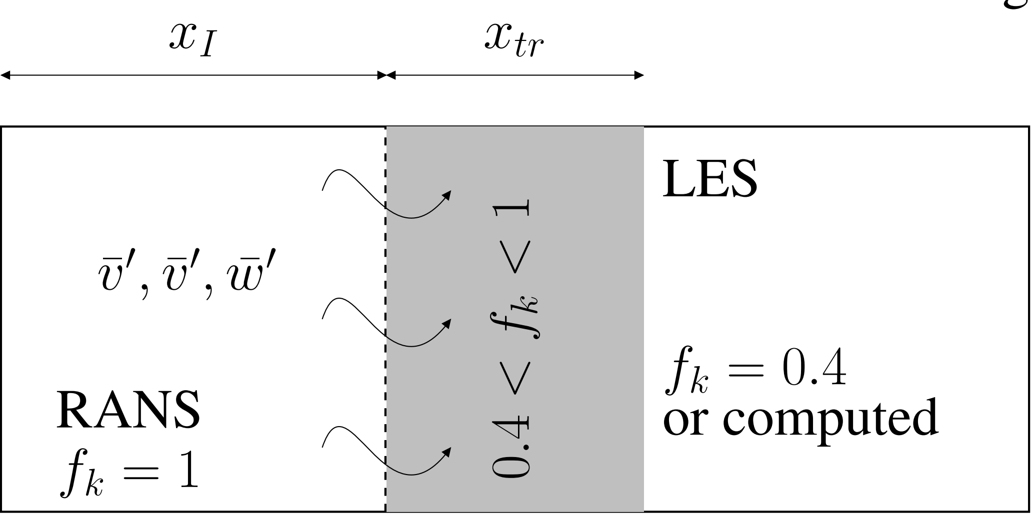
▶In the previous slides we assumed that f_k is constant. ($f_k = k/k_{tot}$, $k_{tot} = k_{res} + k$)

$$f_k \frac{dk_{tot}}{dt} = \frac{d(f_k k_{tot})}{dt} = \frac{dk}{dt} \tag{40.2}$$

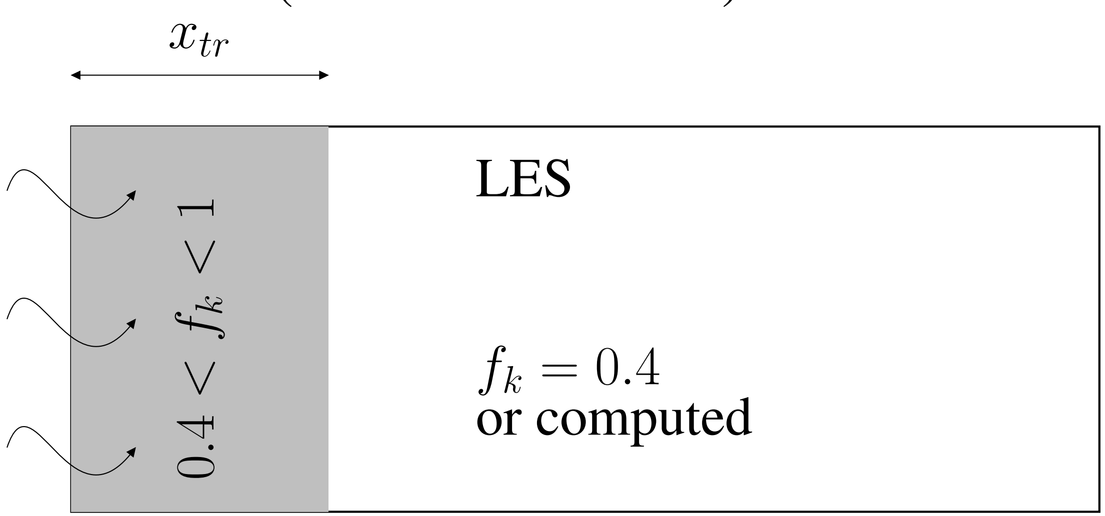
▶PANS as a hybrid RANS-LES model ($f_k = 1$ in RANS region, and $0 < f_k < 0.5$ in LES region)



The URANS and the LES regions near a wall (horizontal interface).



Embedded LES. RANS-LES interface at x_I.



RANS-LES interface at inlet.

►A gradient of f_k at RANS-LES interface since f_k varies in space; Eq. 40.2 replaced by

$$f_k\frac{dk_{tot}}{dt}=\frac{d(f_k k_{tot})}{dt}-k_{tot}\frac{df_k}{dt}=\frac{dk}{dt}-k_{tot}\frac{df_k}{dt} \tag{40.3}$$

►An extra term, $-k_{tot}df_k/dt$, appears on the left side. Right side: $k_{tot}\dfrac{df_k}{dt} \quad = k_{tot}\bar{v}_1\dfrac{\partial f_k}{\partial x_1}$

- Since $df_k/dt<0$, this is a sink term → reduction of k
- Since we add a sink term to the k equation, we must add a source term to the k_{res} equation
- This is done by adding a term to the mom. eq.

$$-(0.5+\langle k\rangle/\langle\bar{v}'_m\bar{v}'_m\rangle)\bar{v}'_i\frac{df_k}{dt}$$

$$-\langle k+0.5\bar{v}'_i\bar{v}'_i\rangle)\frac{df_k}{dt}$$

which agrees with $-k_{tot}\dfrac{df_k}{dt}$ on right side of k_{res} eq.

¶See Section 23.3, A new formulation of f_k for the PANS model

►How to compute f_k?

First, some old formulations.

$$1,2: \quad f_k = C_\mu^{-1/2}\left(\frac{\Delta}{L_t}\right)^{2/3}, L_t = \frac{k_{tot}^{3/2}}{\varepsilon} \text{ using } \Delta = \Delta_{min} \text{ or } \Delta = (\Delta V)^{1/3}.$$

$$3: \quad f_k = \frac{\Delta}{L_t}$$

►A formula derived an expression from the Kolmogorov energy spectrum

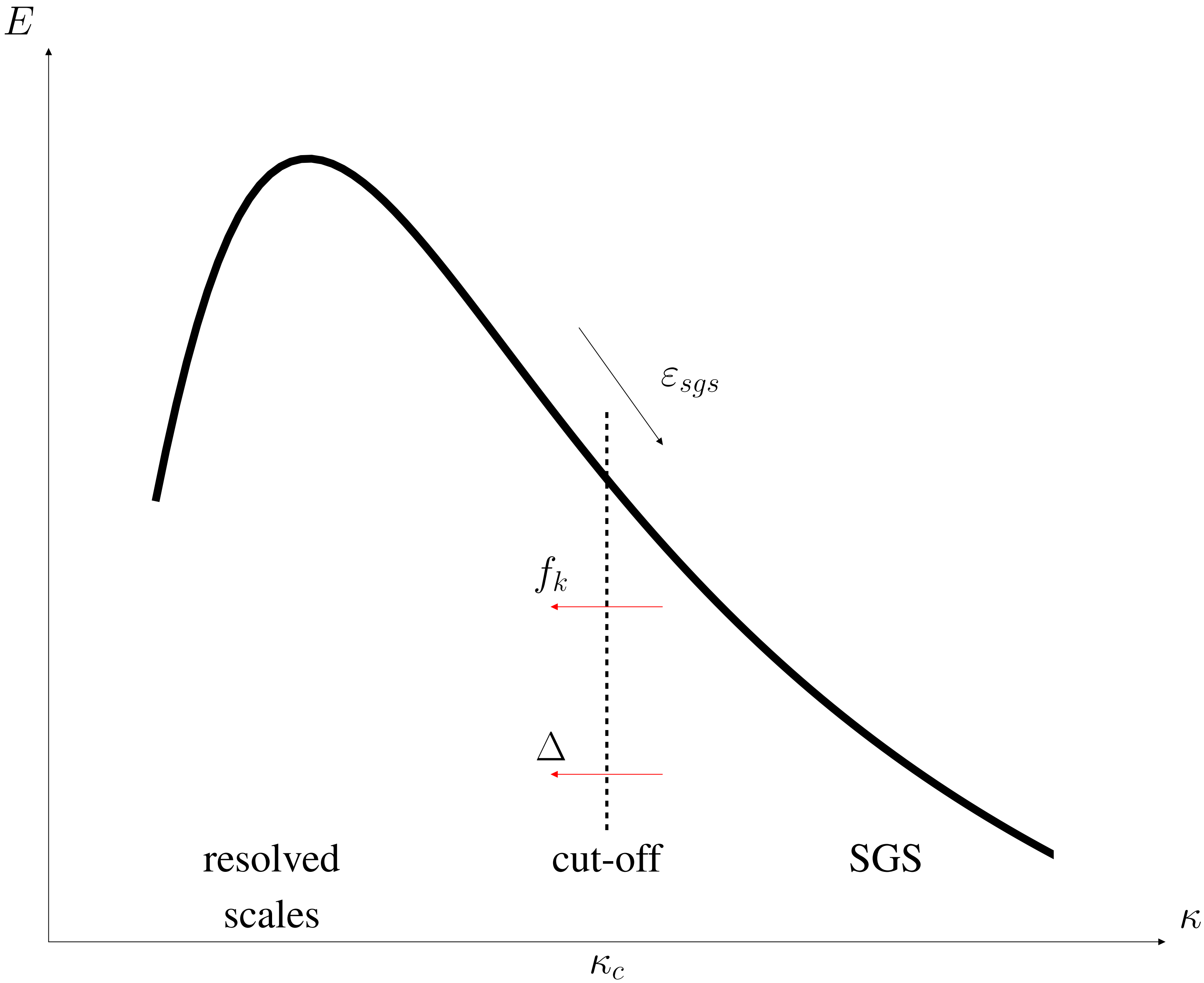
$$4: \quad f_k = 1 - \left[\frac{(\Lambda/\Delta)^{2/3}}{0.23 + (\Lambda/\Delta)^{2/3}}\right]^{9/2}$$

▶DES

$$\begin{aligned} C^k &= P^k + D^k - \psi\varepsilon \\ C^\varepsilon &= C_{\varepsilon 1}\frac{\varepsilon}{k}P^k + D^\varepsilon - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \\ \psi &= \max\left(1, \frac{k^{3/2}/\varepsilon}{C_{DES}\Delta_{max}}\right) \end{aligned}$$

▶PANS

$$\begin{aligned} C^k &= P^k + D^k - \varepsilon \\ C^\varepsilon &= C_{\varepsilon 1}\frac{\varepsilon}{k}P^k + D^\varepsilon - C^*_{\varepsilon 2}\frac{\varepsilon^2}{k} \\ C^*_{\varepsilon 2} &= C_{\varepsilon 1} + \frac{f_k}{f_\varepsilon}(C_{\varepsilon 2} - C_{\varepsilon 1}) \\ f_{k,obs} &= \frac{k_{model}}{k_{model} + k_{res}}, \quad f_\varepsilon \simeq 1 \end{aligned}$$



Spectrum of velocity fluctuations.

▶DES and PANS I

$$C^k - D^k \simeq C^\varepsilon - D^\varepsilon \simeq 0, \quad \gamma = \frac{P^k}{Sk}, \quad S = (2\bar{s}_{ij}\bar{s}_{ij})^{1/2}, \quad T = \frac{k}{\varepsilon}$$

$$\psi = \max\left(1, \frac{k^{3/2}/\varepsilon}{C_{DES}\Delta_{max}}\right), \quad C^*_{\varepsilon 2} = C_{\varepsilon 1} + \frac{f_k}{f_\varepsilon}(C_{\varepsilon 2} - C_{\varepsilon 1})$$

▶PANS

$$TP^\varepsilon - P^k = TC^*_{\varepsilon 2}\frac{\varepsilon^2}{k} - \varepsilon \Rightarrow$$

$$\gamma(C_{\varepsilon 1} - 1)Sk = (C^*_{\varepsilon 2} - 1)\varepsilon$$

▶Differentiation yields:

$$\frac{\delta\gamma}{\gamma} + \frac{\delta S}{S} + \frac{\delta k}{k} = \frac{\delta C^*_{\varepsilon 2}\varepsilon}{(C_{\varepsilon 1} - 1)\gamma Sk} = \frac{\delta C^*_{\varepsilon 2}}{C^*_{\varepsilon 2} - 1}$$

▶DES

$$TP^\varepsilon - P^k = TC_{\varepsilon 2}\frac{\varepsilon^2}{k} - \psi\varepsilon \Rightarrow$$

$$\gamma(C_{\varepsilon 1} - 1)Sk = (C_{\varepsilon 2} - \psi)\varepsilon$$

Differentiation yields:

$$\frac{\delta\gamma}{\gamma} + \frac{\delta S}{S} + \frac{\delta k}{k} = -\frac{\delta\psi\varepsilon}{(C_{\varepsilon 1} - 1)Sk\gamma} = -\frac{\delta\psi}{C_{\varepsilon 2} - \psi}$$

▶DES and PANS II

$$\frac{dC^*_{\varepsilon 2}}{C^*_{\varepsilon 2}-1}=\frac{-d\psi}{C_{\varepsilon 2}-\psi}$$

▶Integrate from RANS ($C_{\varepsilon 2}$ and $\psi = 1$) to LES ($C^*_{\varepsilon 2}$ and ψ) conditions

$$\int_{C_{\varepsilon 2}}^{C^*_{\varepsilon 2}} \frac{dC^*_{\varepsilon 2}}{C^*_{\varepsilon 2}-1} = \int_1^{\psi} -\frac{d\psi}{C_{\varepsilon 2}-\psi} \Rightarrow$$

$$\ln\left(\frac{C^*_{\varepsilon 2}-1}{C_{\varepsilon 2}-1}\right) = \ln\left(\frac{C_{\varepsilon 2}-\psi}{C_{\varepsilon 2}-1}\right)$$

▶By using the expression for $C^*_{\varepsilon 2}$ with requirement $0 < f_k \leq 1$ we get

$$f_k = \max\left[0, \min\left(1, 1-\frac{\psi-1}{C_{\varepsilon 2}-C_{\varepsilon 1}}\right)\right], \quad \psi = \max\left(1, \frac{k^{3/2}/\varepsilon}{C_{DES}\Delta_{max}}\right)$$

►Conclusions

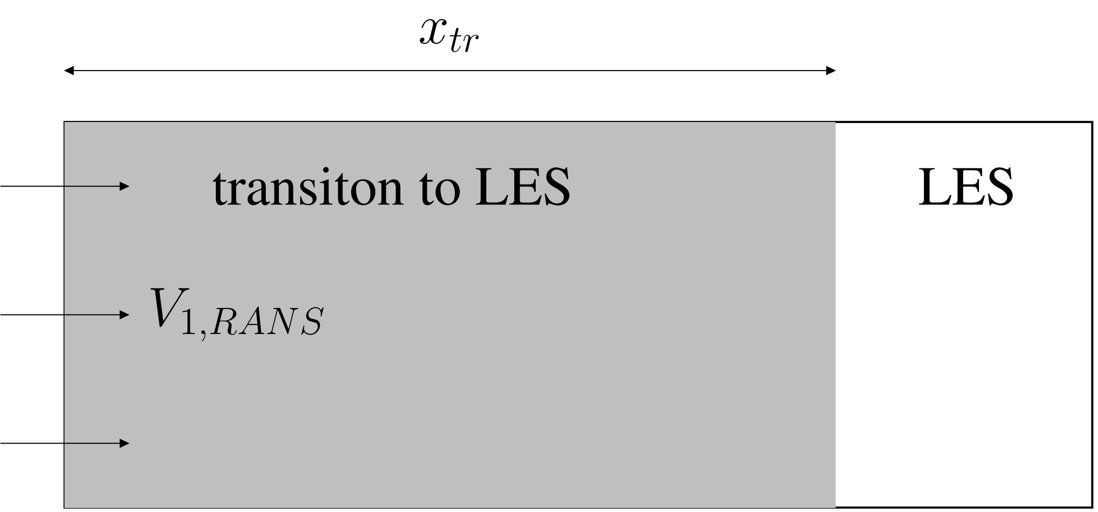
- It gives much better results than the old formulations of f_k
- It gives very similar results to the DES model
- Advantage of the new PANS model vs. the DES model
  - The PANS model is based on a rigorous derivation whereas DES is based on an ad-hoc modification of RANS models

On-line Lecture 12

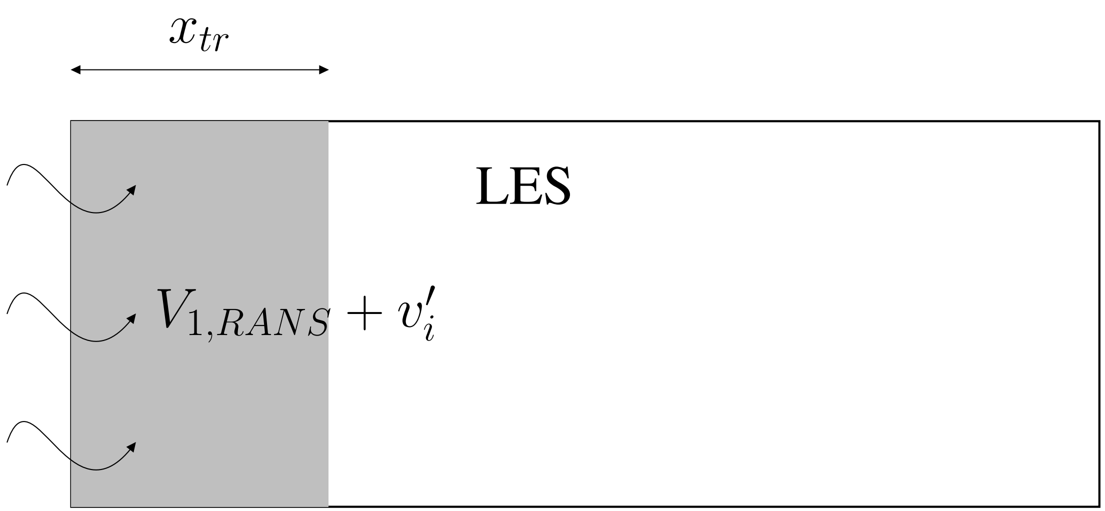
¶See Section 27.1, Synthesized turbulence

▶In LES, large-scale turbulence is resolved

▶Hence, turbulent fluctuations should be provided as inlet boundary conditions



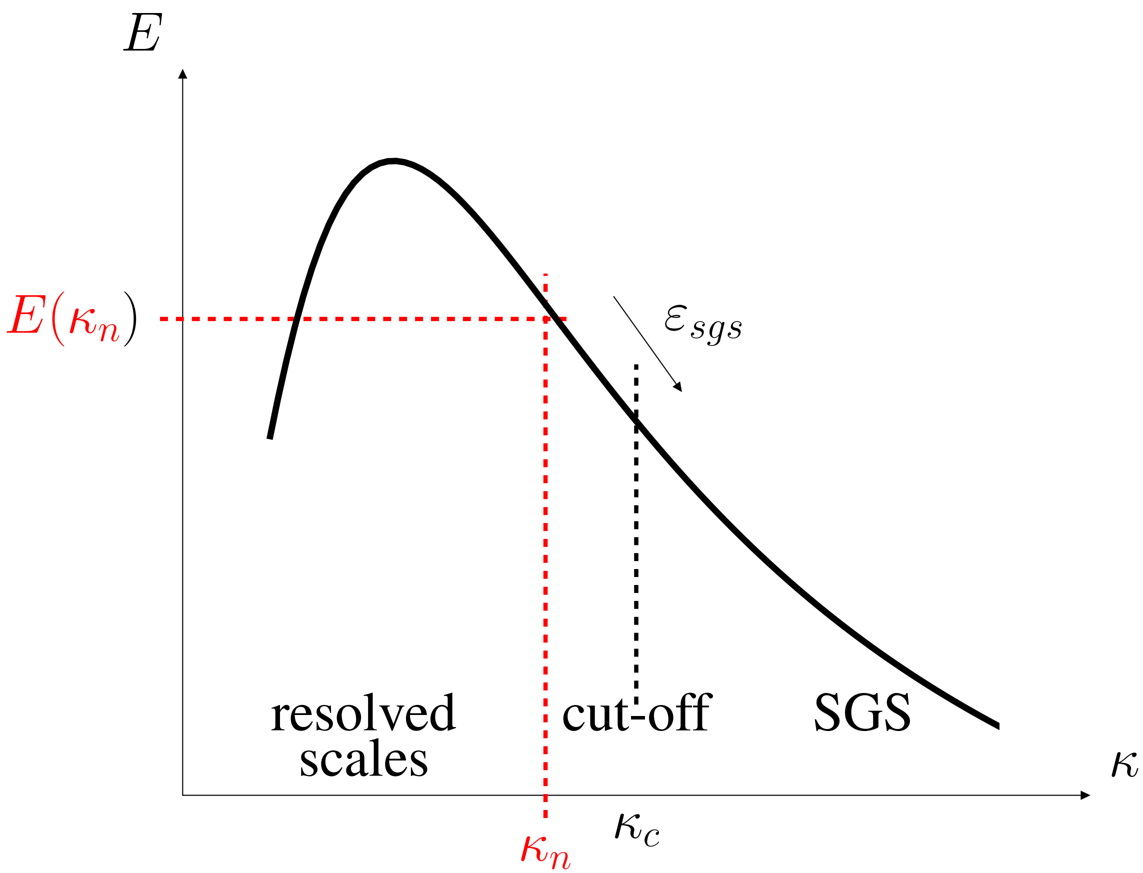
No inlet fluctuations, large x_{tr}.



Realistic, synthetic inlet fluctuations, small x_{tr}.

▶Synthetic fluctuations is one method. The fluctuating inlet velocity can be written as a Fourier serie

$$\mathbf{v}'(\mathbf{x}) = 2\sum_{n=1}^{N} \hat{u}^n \cos(\boldsymbol{\kappa}^n \cdot \mathbf{x} + \psi^n)\boldsymbol{\sigma}^n$$



Spectrum of velocity fluctuations.

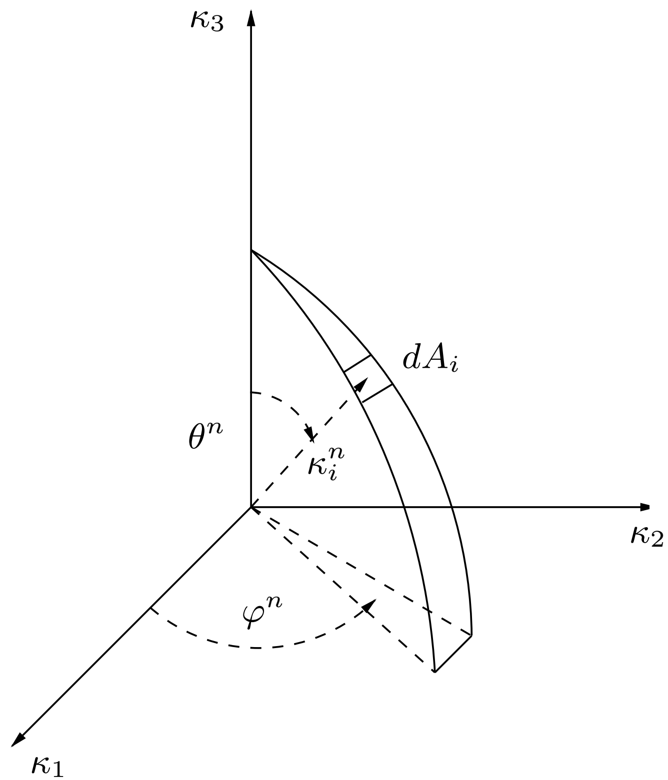
▶Usually we generate energy spectra from turbulent fluctuations. ▶Here we prescribe a spectrum and generate turbulent fluctuations. ▶$-5/3$ spectrum: ▶this gives the amplitude $\hat{u}^n$ for wavenumber κ_n

¶See Section 27.2, Random angles

$$\mathbf{v}'(\mathbf{x}) = 2\sum_{n=1}^{N} \hat{u}^n \cos(\boldsymbol{\kappa}^n \cdot \mathbf{x} + \psi^n)\boldsymbol{\sigma}^n, \qquad \text{Fourier serie}$$

$p(\varphi^n) = 1/(2\pi)$	$0 \leq \varphi^n \leq 2\pi$
$p(\psi^n) = 1/(2\pi)$	$0 \leq \psi^n \leq 2\pi$
$p(\theta^n) = 1/2\sin(\theta)$	$0 \leq \theta^n \leq \pi$
$p(\alpha^n) = 1/(2\pi)$	$0 \leq \alpha^n \leq 2\pi$

Probability distributions of the random variables. α^n is the angle for $\boldsymbol{\sigma}^n$.



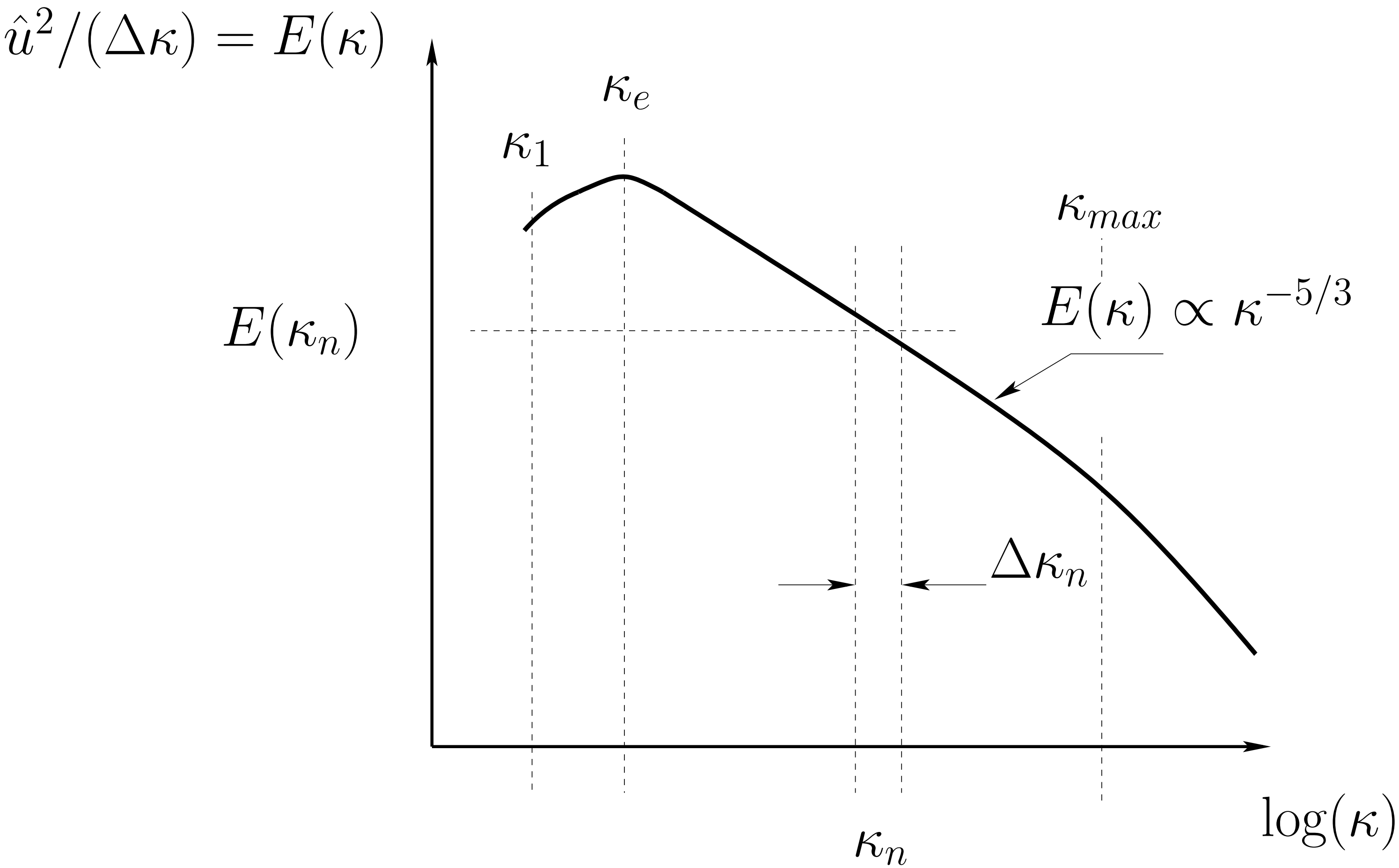
The probability of a wave in wave-space is the same for all dA_i on the shell of a sphere.

▶Randomize the angles according to the table. The figure above gives

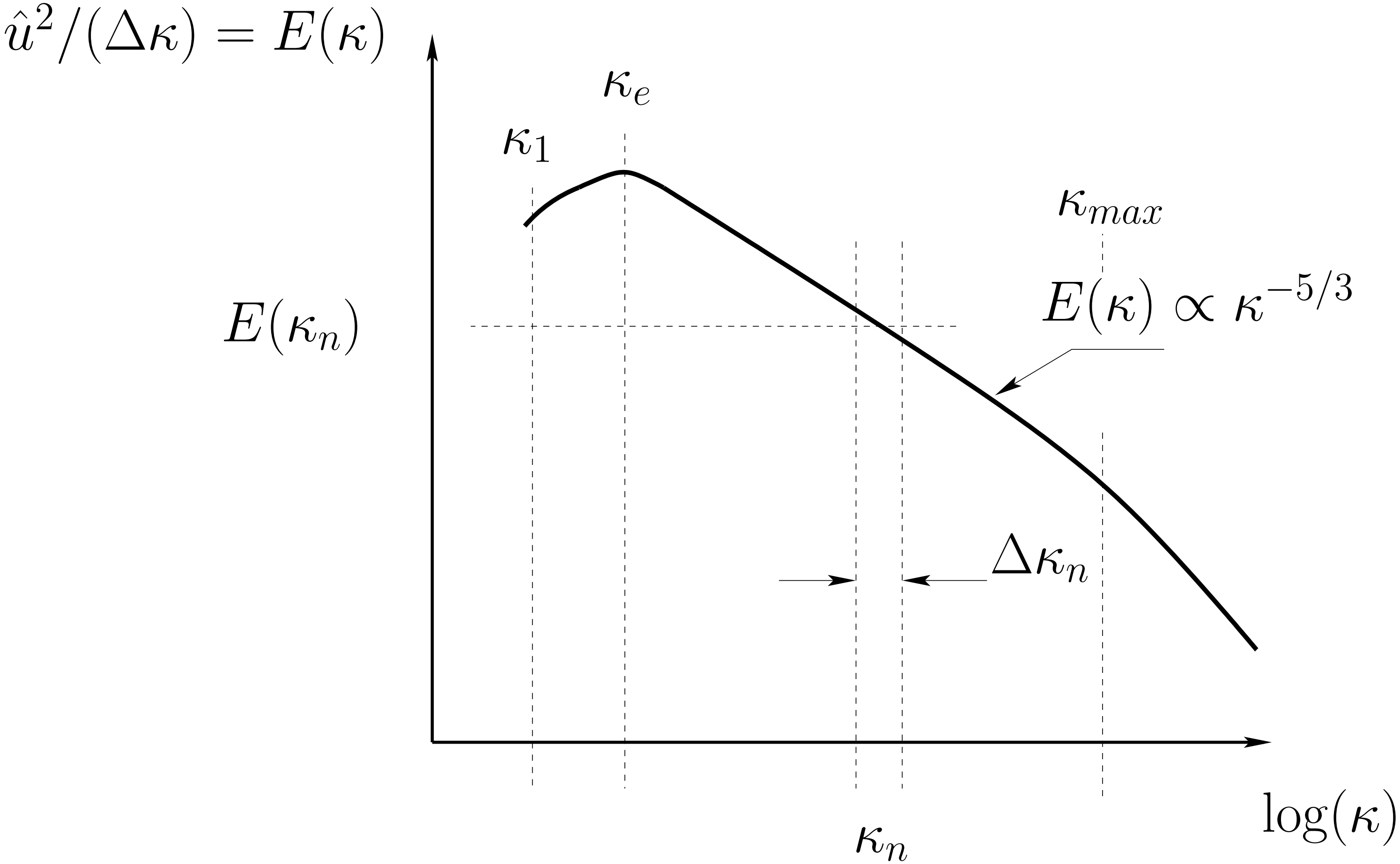
$$\begin{aligned}
\kappa_1^n &= \sin(\theta^n)\cos(\varphi^n) \\
\kappa_2^n &= \sin(\theta^n)\sin(\varphi^n) \\
\kappa_3^n &= \cos(\theta^n)
\end{aligned}$$

$$\mathbf{v}'(\mathbf{x}) = 2\sum_{n=1}^{N} \hat{u}^n \cos(\boldsymbol{\kappa}^n \cdot \mathbf{x} + \psi^n)\boldsymbol{\sigma}^n, \qquad \text{Fourier serie}$$

- Amplitude $\hat{u}^n$ related to energy spectrum: $\hat{u}^n = (E(\kappa)\Delta\kappa)^{1/2}$



- For each wavenumber κ_n the energy specrum above gives the amplitude $\hat{u}^n$



- Highest wave number: $\kappa_{max} = 2\pi/2\Delta$ from the cell size, $\Delta = \min(\Delta x_2)$
- Most energetic wave number: $\kappa_e \propto 1/L_t$: integral turbulent length scale. ► $\kappa_e = 0.75/L_t$
- Smallest wave number: $\kappa_{min} = \kappa_1 = \kappa_e/5$, $\Delta\kappa = (\kappa_{max} - \kappa_{min})/N$
- Number of wave numbers: N ► 150
- Now the fluctuations, $\mathbf{v}'(\mathbf{x})$, can be computed

$$v_1' = 2\sum_{n=1}^{N} \hat{u}^n \cos(\beta^n)\sigma_1$$

$$v_2' = 2\sum_{n=1}^{N} \hat{u}^n \cos(\beta^n)\sigma_2$$

$$v_3' = 2\sum_{n=1}^{N} \hat{u}^n \cos(\beta^n)\sigma_3$$

$$\beta^n = k_1^n x_1 + k_2^n x_2 + k_3^n x_3 + \psi^n$$

▶ Synthetic turbulent isotropic fluctuations at the inlet plane for all time steps.

▶ With a specified integral lengthscale

▶ BUT: ▶ no correlation between the timesteps ▶ i.e. white noise in time

¶See Section 27.8, Introducing time correlation

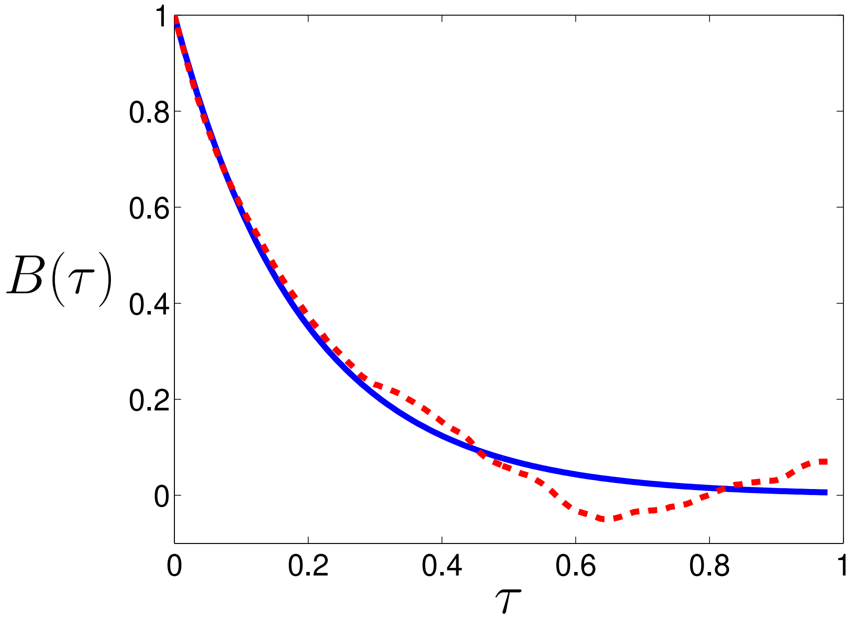
▶The synthetic fluctuations are not correlated in time. An asymmetric time filter is used

$$(\mathcal{V}_1')^m = a(\mathcal{V}_1')^{m-1} + b(v_1')^m$$

▶The coefficient a is related to the turbulent integral timescale, $\mathcal{T}$, as

$$a = \exp(-\Delta t/\mathcal{T}) \tag{41.1}$$

▶We want $\mathcal{V}'_{1,rms} = v'_{1,rms}$ ▶$b = (1-a^2)^{1/2}$ ensures that

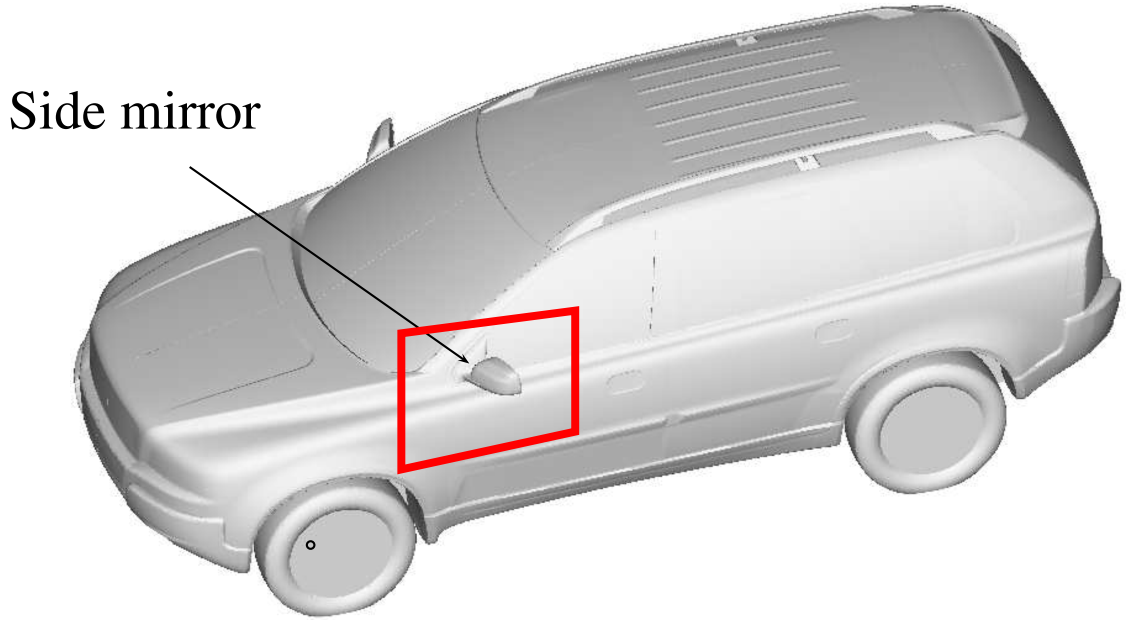


Auto correlation. —: Eq. 41.1; - - -: $B(\tau) = \langle \mathcal{V}_1'(t)\mathcal{V}_1'(t-\tau)\rangle_t$.

▶Finally, the turbulent synthetic fluctuations are superimposed to the inlet mean velocity.

¶See Section 23.2.1, The Interface Condition

▶Embedded LES and inlet b.c. for k and ε using PANS



Vehicle geometry (from [116]). Colored rectangle shows embedded LES region

▶An LES region (e.g. the side mirror, see figure above) is embedded in a steady RANS simulation.

▶LES is used around the mirror in order to compute aeroacoustic sources (wind noise)

▶Synthetic fluctuations are needed at the inlet region of LES.

▶Mean velocity, k and ε at the LES inlet region are taken from the RANS simulation

▶Summary

- Add synthetic fluctuations at inlet or embedded surfaces with prescribed integral length and timescale
- Use RANS values of k and ε
- The source terms in the k equation (see Eq. 40.3) quickly reduces k from RANS values to LES values
- No source terms are needed for ε because it is the same for RANS and LES
- For more detail, see Section 5 in [183]